Markov processes and backward Kolmogorov equation

1 Markov processes

Discrete or continuous time Markov process must satisfy the Chapman-Kolmogorov equation

$$\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x}\,,t\,|\boldsymbol{x}'\,,t') = \int_{\mathbb{S}} \mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x}\,,t\,|\boldsymbol{z}\,,s)\mathrm{p}_{\boldsymbol{\xi}}(\mathrm{d}^d\boldsymbol{z}\,,s\,|\boldsymbol{x}'\,,t') \tag{1.1}$$

for any $t' \leq s \leq t$. In (1.1) the notation

$$\int_{\mathbb{S}} \mathrm{p}_{\boldsymbol{\xi}}(\mathrm{d}^d\boldsymbol{z}\,,t\,|\boldsymbol{x}\,,s)\cdots = \begin{cases} \sum_{\boldsymbol{z}\in\mathbb{S}} \mathrm{P}_{\boldsymbol{\xi}}(\boldsymbol{z}\,,t\,|\boldsymbol{x}\,,s)\ldots & \text{if } \mathbb{S} \text{ is a countable state space} \\ \int_{\mathbb{S}} \mathrm{d}^d\boldsymbol{z}\,\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,,t\,|\boldsymbol{x}\,,s)\ldots & \text{if } \mathbb{S} \text{ is a continuous state space} \end{cases} \tag{1.2}$$

For continuous time Markov processes the Kolmogorov–Čentsov theorem gives us a criterion to assess the continuity of paths.

2 Markov processes and Ito lemma

Let us consider a Markov process $\boldsymbol{\xi}$ the transition probability $\mathrm{p}_{\boldsymbol{\xi}}$ whereof satisfies *for all* $\varepsilon > 0$ the following conditions.

i *Jump rate* condition: let $\mathrm{K}\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, we suppose

$$\lim_{t\downarrow t'} \frac{\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x}\,,t\,|\boldsymbol{x}'\,,t')}{t-t'} = \mathrm{K}_t(\boldsymbol{x}|\boldsymbol{x}') \tag{2.1}$$

for all $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^d$ such that

$$\parallel \boldsymbol{x} - \boldsymbol{x}' \parallel \geq \varepsilon \tag{2.2}$$

ii *Drift condition*: let

$$\boldsymbol{b}\colon \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{R}^d \tag{2.3}$$

we suppose

$$\lim_{t\downarrow t'} \frac{1}{t-t'} \int_{\|\boldsymbol{x}-\boldsymbol{x}'\| < \varepsilon} d^d z\,(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x}\,,t\,|\boldsymbol{x}'\,,t') = \boldsymbol{b}(\boldsymbol{x}'\,,t') \tag{2.4}$$

iii *Diffusivity condition*: let

$$\mathsf{G}\colon \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{R}^{d\times d} \tag{2.5}$$

positive definite, we suppose

$$\lim_{t\downarrow t'} \frac{1}{t-t'} \int_{\|\boldsymbol{x}-\boldsymbol{x}'\| < \varepsilon} d^d z\,(\boldsymbol{x}-\boldsymbol{x}') \otimes (\boldsymbol{x}-\boldsymbol{x})\,\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x}\,,t\,|\boldsymbol{x}'\,,t') = \mathsf{G}(\boldsymbol{x}'\,,t') \tag{2.6}$$

We will also assume, see discussion below, some regularity conditions on $\boldsymbol{b}$ and G. Some observations are in order.

i' We introduced the *Jump rate* condition to encompass cádlág processes. If we are instead interested in setting the focus on continuous processes we can replace it with the *continuity condition*:

$$\lim_{t \downarrow t'} \frac{1}{dt} \int_{\| \boldsymbol{x} - \boldsymbol{x}' \| > \varepsilon} d^d x \, \mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{x} \, , t \, | \boldsymbol{x}' \, , t') = 0 \tag{2.7}$$

- If for any $\boldsymbol{v} \in \mathbb{R}^d$

$$\mathrm{E} \left\{ \left(\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \right) \otimes \left(\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \right) \right\} : (\boldsymbol{v} \otimes \boldsymbol{v}) \; < \; \infty \tag{2.8}$$

  the continuity condition implies that the drift and diffusivity conditions simply mean that

$$\lim_{dt \downarrow 0} \mathrm{E}_{\boldsymbol{x},t} \left\{ \frac{\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t}{dt} \right\} = \boldsymbol{b}(\boldsymbol{x}, t) \tag{2.9a}$$

$$\lim_{dt \downarrow 0} \mathrm{E}_{\boldsymbol{x},t} \left\{ \frac{\left(\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \right) \otimes \left(\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \right)}{dt} \right\} = \mathsf{G}(\boldsymbol{x}, t) \tag{2.9b}$$

  where we used the abbreviation

$$\mathrm{E}_{\boldsymbol{x},t} \left\{ \cdot \right\} = \mathrm{E} \left\{ \cdot | \boldsymbol{\xi}_t = x \right\} \tag{2.10}$$

  In general the continuity condition implies that if there is one $\varepsilon_\star \; > \; 0$ such that the drift and diffusivity hold true, then the hold as well for any $\varepsilon \; > \; 0$.

- We saw that Kolmogorov's theorem guarantees that a stochastic process $\boldsymbol{\xi}_t$ has continuous realizations with probability one if for some $\alpha, \beta \, , K \, , h \; > \; 0$

$$\mathrm{E} \parallel \boldsymbol{\xi}_{t+h} - \boldsymbol{\xi}_t \parallel^{\beta} \leq K \, h^{1+\alpha} \tag{2.11}$$

  If ξ_t is such to satisfy (2.11) with $\beta \; > \; 2$ a *diffusion*: by Čebišev inequality it is possible to verify that it enjoys the continuity condition. Furthermore by Lyapunov inequality, moments of the time increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'}$, $t \geq t'$ up to second order must be finite. Hence (2.9) must also hold true.

- The drift and diffusivity conditions imply [1] that higher order truncated moments of the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'}$, $t \geq t'$ vanish. In particular for every $\boldsymbol{v} \in \mathbb{R}^d$ the inequality

$$\left| \mathrm{E}_{\boldsymbol{x},t'} \left\{ [(\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'}) \cdot \boldsymbol{v}]^3 \mathbb{1}(\parallel \boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'} \parallel \leq \varepsilon) \right\} \right| \leq \varepsilon \parallel \boldsymbol{v} \parallel \mathrm{E}_{\boldsymbol{x},t'} \left\{ [\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'}) \cdot \boldsymbol{v}]^2 \mathbb{1}(\parallel \boldsymbol{\xi}_t - \boldsymbol{\xi}_{t'} \parallel \leq \varepsilon) \right\} \tag{2.12}$$

  implies

$$\lim_{dt \downarrow} \frac{\left| \mathrm{E}_{\boldsymbol{x},t} \left\{ [(\boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t) \cdot \boldsymbol{v}]^3 \mathbb{1}(\parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel \leq \varepsilon) \right\} \right|}{dt} \leq \varepsilon \parallel \boldsymbol{v} \parallel \boldsymbol{v} \cdot \mathsf{G}(\boldsymbol{x}, t) \cdot \boldsymbol{v} \tag{2.13}$$

  We then see that owing to the arbitrariness of ε, we are entitled to take the limit of vanishing ε to prove the claim.

- If

$$\mathrm{E}_{\boldsymbol{x},t} \parallel \boldsymbol{\xi}_{t+h} - \boldsymbol{\xi}_t \parallel^{2+\varepsilon} \leq K\, h^{1+\alpha} \tag{2.14}$$

  holds for some $\alpha\,,\varepsilon\,,h\,,K\; > 0$ then it is readily seen that

$$\begin{aligned}
&\mathrm{E}_{\boldsymbol{x},t} \parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel^3 \\
&\quad\leq\; K\, dt\, \mathrm{E}_{\boldsymbol{x},t} \left\{ \parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel^2 \mathbb{1}(\parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel \leq dt) \right\} + \mathrm{E}_{\boldsymbol{x},t} \left\{ \parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel^3 \mathbb{1}(\parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel > dt) \right\} \\
&\quad\leq K\, dt^2 \left[\boldsymbol{b}(\boldsymbol{x},t) + O(dt) \right] + \mathrm{E}_{\boldsymbol{x},t} \left\{ \parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel^3 \mathbb{1}(\parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel > dt) \right\}
\end{aligned} \tag{2.15}$$

  Invoking the continuity condition we then find

$$\lim_{dt \downarrow 0} \mathrm{E}_{\boldsymbol{x},t} \parallel \boldsymbol{\xi}_{t+dt} - \boldsymbol{\xi}_t \parallel^3 = 0 \tag{2.16}$$

  and similarly for any moment of order strictly higher than two.

The connection of the continuity, drift and diffusivity conditions to the theory of stochastic differential equations is straightforward. Suppose the increment of $\boldsymbol{\xi}_t$ satisfies the *Ito*-stochastic differential equation

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t\,,t)\; dt + \mathsf{A}(\boldsymbol{\xi}_t,t) \cdot d\boldsymbol{w}_t \tag{2.17}$$

then existence and uniqueness of (2.17) can be proven if the drift $\boldsymbol{b}$ and diffusion fields A are Lipschitz-continuous (and satisfy some further asymptotic growth conditions). Up to an orthogonal transformation we can write

$$\mathsf{G} = \mathsf{A} \cdot \mathsf{A}^{\dagger} \tag{2.18}$$

It follows that

$$\mathrm{E}_{\boldsymbol{x},t} d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t\,,t)\; dt \tag{2.19a}$$

$$\mathrm{E}_{\boldsymbol{x},t} \left\{ d\boldsymbol{\xi}_t \otimes\, d\boldsymbol{\xi}_t \right\} = \mathsf{G}\,(\boldsymbol{\xi}_t\,,t)\; dt + o(dt) \tag{2.19b}$$

The Lipschitz condition can be imposed on A by requiring G either to be

- Lipschitz and *uniformly elliptic*:

$$c \sum_{i=1}^{d} x_i^2 \leq \sum_{ij=1}^{d} x_i \mathsf{G}^{ij} x_j \,\leq\, C \sum_{i=1}^{d} x_i^2$$

  for some $0 < c \leq\; C < \infty$ and all $\boldsymbol{x} \in\; \mathbb{R}^d$.

or

- if the G^{ij} is continuously twice differentiable with a bound on the second derivatives.

We are now ready to derive the law governing the evolution of the transition probability of the Markov process

Proposition 2.1. *If the conditions i, ii, iii hold true the transition probability of the Markov process satisfies with respect to the conditioning event the backward evolution equation*

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\mathrm{p}(\cdot|\boldsymbol{x},t) = \fint_{\mathbb{S}} d^d x_1 K_t(\boldsymbol{x}_1|\boldsymbol{x})\; \left[\mathrm{p}(\cdot|\boldsymbol{x},t) - \mathrm{p}(\cdot|\boldsymbol{x}_1,t)\right] \tag{2.20}$$

with

$$\mathfrak{L}_{\boldsymbol{x}} := \boldsymbol{b}\,(\boldsymbol{x},t) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2} \mathsf{G}\,(\boldsymbol{x},t) : \partial_{\boldsymbol{x}} \partial_{\boldsymbol{x}} \tag{2.21}$$

and $\fint_{\mathbb{S}}$ the principal value of the integral over $\mathbb{S}$.

Proof. Using the Chapman-Kolmogorov equation and the normalization condition we can write

$$\lim_{dt\downarrow 0} \frac{\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t)}{dt} = \lim_{dt\downarrow 0} \int_{\mathbb{S}} d^d z \, [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt)]\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,, t+dt\,|\boldsymbol{x}\,, t) \tag{2.22}$$

We can decompose the integral on the right hand side in two parts

$$\int_{\mathbb{S}} d^d z \, [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt)]\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,, t+dt\,|\boldsymbol{x}\,, t) = \left(\int_V d^d z + \int_{\bar{V}} d^d z\right) [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt)]\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,, t+dt\,|\boldsymbol{x}\,, t) \tag{2.23}$$

with

$$V = \{\boldsymbol{z} \in \mathbb{S}\,| \, \parallel \boldsymbol{x} - \boldsymbol{z} \parallel \leq \varepsilon\} \tag{2.24}$$

and $\bar{V} = \mathbb{S}/V$. Then on the one hand, we can perform the integral over V by expanding the integrand in a Taylor series centered at $\boldsymbol{z} = \boldsymbol{x}$

$$\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt) = \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) + \left\{(\boldsymbol{z}-\boldsymbol{x})\cdot\partial_{\boldsymbol{x}} + \frac{1}{2}(\boldsymbol{z}-\boldsymbol{x})\otimes(\boldsymbol{z}-\boldsymbol{x}) : \partial_{\boldsymbol{x}}\partial_{\boldsymbol{x}}\right\} \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) + o(\parallel \boldsymbol{z}-\boldsymbol{x} \parallel^2) \tag{2.25}$$

and thus obtain

$$\lim_{\varepsilon\downarrow 0}\lim_{dt\downarrow 0} \frac{1}{dt}\int_V d^d z \, [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt)]\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,, t+dt\,|\boldsymbol{x}\,, t) = -\mathfrak{L}_{\boldsymbol{x}}\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t) \tag{2.26}$$

On the other hand, we can use i to write

$$\lim_{\varepsilon\downarrow 0}\lim_{dt\downarrow 0} \int_{\bar{V}} d^d z \, [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t+dt) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t+dt)]\mathrm{p}_{\boldsymbol{\xi}}(\boldsymbol{z}\,, t+dt\,|\boldsymbol{x}\,, t) = \fint_{\mathbb{S}} d^d z \, [\mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{x}\,, t) - \mathrm{p}_{\boldsymbol{\xi}}(\cdot\,|\boldsymbol{z}\,, t)]K_t(\boldsymbol{z}\,|\boldsymbol{x}) \tag{2.27}$$

which yields the claim. □

3 Backward Kolmogorov equation for diffusion processes

Let $\boldsymbol{\phi}$ be the flow expressing the fundamental solution of (2.17). In other words for any initial condition

$$\boldsymbol{\xi}_{t_o} = \boldsymbol{\xi}_o \tag{3.1}$$

for any $t \geq t_o$ we suppose we can write the diffusion unique solution of (2.17) as

$$\boldsymbol{\xi}_t = \boldsymbol{\phi}_t(\boldsymbol{\xi}_o, t_o) \tag{3.2}$$

Proposition 3.1. *Let*

$$g\colon \mathbb{R}^d \mapsto \mathbb{R} \tag{3.3}$$

smooth and integrable over $\mathbb{R}^d$. Then for any $t \leq T$

$$f(\boldsymbol{x}, t) = \mathrm{E}_{\boldsymbol{x},t} g(\boldsymbol{\xi}_T) \tag{3.4}$$

satisfies the backward Kolmogorov equation

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}}) f = 0 \tag{3.5a}$$

$$f(\boldsymbol{x}, T) = g(\boldsymbol{x}) \tag{3.5b}$$

Proof. We already proved that the transition probability density of a diffusion satisfies (3.5a). Consistence with the Chapman-Kolmogorov equation also requires

$$\lim_{t \uparrow t_1} \mathrm{p}(\boldsymbol{x}_1, t_1 \,|\, \boldsymbol{x}, t) = \delta^{(d)}(\boldsymbol{x}_1 - \boldsymbol{x}) \tag{3.6}$$

Hence by linearity

$$\mathrm{E}_{\boldsymbol{x},t} g(\boldsymbol{\xi}_T) = \int_{\mathbb{S}} d^d x_1 \, g(\boldsymbol{x}_1) \, \mathrm{p}(\boldsymbol{x}_1, t_1 \,|\, \boldsymbol{x}, t) \tag{3.7}$$

satisfies (3.5). This proves the claim. It is, however, instructive to derive the result using explicitly the properties of the flow $\boldsymbol{\phi}_t$. Let us therefore write

$$\mathrm{E}_{(\boldsymbol{x},t+dt)} \, g(\boldsymbol{\xi}_T) - \mathrm{E}_{(\boldsymbol{x},t)} \, g(\boldsymbol{\xi}_T) = \mathrm{E} \left\{ g \circ \boldsymbol{\phi}_T(\boldsymbol{x}, t + dt) - g \circ \boldsymbol{\phi}_T(\boldsymbol{x}, t) \right\} \tag{3.8}$$

Since the flow is the fundamental solution of a stochastic differential equation, it must enjoy the property

$$\boldsymbol{\phi}_T(\boldsymbol{x}, t) = \boldsymbol{\phi}_T(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t), t + dt) \tag{3.9}$$

Hence we can couch (3.8) into the form

$$\begin{aligned} &\mathrm{E}_{(\boldsymbol{x},t+dt)} g(\boldsymbol{\xi}_T) - \mathrm{E}_{(\boldsymbol{x},t)} g(\boldsymbol{\xi}_T) = \\ &\quad \mathrm{E} \left\{ g \circ \boldsymbol{\phi}_T(\boldsymbol{x}, t + dt) - \, g \circ \boldsymbol{\phi}_T(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t), t + dt) \right\} \end{aligned}$$

If we now define

$$\tilde{g} := g \circ \boldsymbol{\phi}_T \tag{3.10}$$

the chain of equalities

$$f(\boldsymbol{x}, t + dt) - f(\boldsymbol{x}, t) = \mathrm{E}_{(\boldsymbol{x},t+dt)} \, f(\boldsymbol{\xi}_t) - \mathrm{E}_{(\boldsymbol{x},t)} \, f(\boldsymbol{\xi}_t) = \mathrm{E} \, \left\{ \tilde{g}(\boldsymbol{x}, t + dt) - \tilde{g}(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t), t + dt) \right\} \tag{3.11}$$

hold true. We have therefore

$$\begin{aligned} \partial_t f(\boldsymbol{x}, t) &:= \lim_{dt \downarrow 0} \frac{f(\boldsymbol{x}, t + dt) - f(\boldsymbol{x}, t)}{dt} \\ &= \lim_{dt \downarrow 0} \frac{\mathrm{E} \, \left\{ \tilde{g}(\boldsymbol{x}, t + dt) - \tilde{g}(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t + dt), t + dt) \right\}}{dt} = \lim_{dt \downarrow 0} \frac{f(\boldsymbol{x}, t) - \mathrm{E} \, \tilde{g}(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t), t)}{dt} \end{aligned} \tag{3.12}$$

since by definition

$$\mathrm{E}\tilde{g}(\boldsymbol{x}, t) = \mathrm{E}g \circ \boldsymbol{\phi}_T(\boldsymbol{x}, t) \equiv \mathrm{E}_{\boldsymbol{x},t} g(\boldsymbol{\xi}_T) \tag{3.13}$$

We have therefore proved the proposition if we invoke Ito lemma and write

$$f(\boldsymbol{x}, t) - \mathrm{E} \, \tilde{g}(\boldsymbol{\phi}_{t+dt}(\boldsymbol{x}, t), t) = -\mathrm{E}_{\boldsymbol{x},t} \int_t^{t+dt} dt_1 \, \mathfrak{L}_{\boldsymbol{\xi}_{t_1}} \tilde{g}(\boldsymbol{\phi}_{t_1}(\boldsymbol{x}, t), t) \tag{3.14}$$

□

An important consequence of the above proposition is that any integrable function f solution of (3.5) is on average a conserved quantity of the diffusion process (2.17)

$$\mathrm{E}f(\boldsymbol{\xi}_t, t) = \mathrm{E}_{\boldsymbol{x},t} g(\boldsymbol{\xi}_T) = \mathrm{E}g(\boldsymbol{\xi}_T) \tag{3.15}$$

3.1 Example

Consider

$$\left(\partial_t + \frac{\alpha x}{\tau}\partial_x + \frac{\kappa}{2\,\tau}\partial_x^2\right)\, f = 0 \tag{3.16a}$$

$$f(x,T) = x^2 \tag{3.16b}$$

with $\tau\,,\kappa\,>\,0$. The corresponding stochastic differential equation is

$$d\xi_t = \frac{\alpha\,\xi_t}{\tau}dt + \sqrt{\frac{\kappa}{\tau}}dw_t \tag{3.17}$$

and

$$f(x,t) = \mathrm{E}_{x,t}\xi_T^2 = \mathrm{E}\left\{e^{\alpha\frac{T-t}{\tau}}x + \sqrt{\frac{\kappa}{\tau}}\int_t^T dw_{t_1}e^{\alpha\frac{T-t_1}{\tau}}\right\}^2 \tag{3.18}$$

Since

$$\mathrm{E}\int_t^T dw_{t_1}\int_t^T dw_{t_2}e^{\alpha\frac{2\,T-t_1-t_2}{\tau}} = \int_t^T dt_1\,e^{2\,\alpha\frac{T-t_1}{\tau}} = \tau\,\frac{e^{2\,\alpha\frac{T-t}{\tau}}-1}{2\,\alpha} \tag{3.19}$$

the solution of the partial differential equation is

$$f(x,t) = e^{2\,\alpha\frac{T-t}{\tau}}x^2 + \kappa\frac{e^{2\,\alpha\frac{T-t}{\tau}}-1}{2\,\alpha} \tag{3.20}$$

References

[1] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*, volume 13 of *Springer series in synergetics*. Springer, 2 edition, 1994.