Stochastic calculus with Hermite polynomials and stochastic differential equations

1 Introduction

The proof of the existence and uniqueness theorem for stochastic differential equations can be found in chapter 5 of [1].

2 Stochastic calculus with Hermite polynomials

This section expands example D.3 of chapter 4 of [1].

Proposition 2.1. *the transition probability of the Wiener process admits the expansion*

$$\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sum_{n=0}^{\infty} \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t)$$

where the h_n's are Hermite polynomials

$$h_n\,(x\,,t) = \frac{(-t)^n}{\Gamma(n+1)}\,e^{\frac{x^2}{2\,t}}\,\frac{d^n}{dx^n} e^{-\frac{x^2}{2\,t}} \tag{2.1}$$

Proof. Let us postulate for the n-th order of the Taylor expansion the form

$$\frac{y^n}{\Gamma(n+1)}\,\frac{d^n}{dz^n}\bigg|_{z=0} \frac{e^{-\frac{(x-z)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} := \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t)$$

we can then calculate the explicit form of the polynomial h_n. Namely

$$\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x\,,t) = \\ \frac{y^n}{\Gamma(n+1)}\,\frac{d^n}{dz^n}\bigg|_{z=0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{d^d p}{(2\,\pi)^d}\,e^{\imath\,p x - \frac{t\,p^2}{2}}\,\frac{(-\imath\,p\,z)^k}{\Gamma(k+1)} = \frac{y^n}{\Gamma(n+1)} \int_{\mathbb{R}} \frac{d^d p}{(2\,\pi)^d}\,e^{\imath\,p x - \frac{t\,p^2}{2}}\,(-\imath\,p)^n$$

Observing that powers of p are generated by taking derivatives with respect to x we get into

$$\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x\,,t) = \frac{(-y)^n}{\Gamma(n+1)}\,\frac{d^n}{dx^n} \int_{\mathbb{R}} e^{\imath\,p x - \frac{t\,p^2}{2}}$$

Performing the integral and contrasting the left to the right hand side yields the claim. □

The polynomials h_n defined by (2.1) are called the Hermite polynomials. It is readily checked that they enjoy the scaling property

$$h_n(\lambda\, x\,,\lambda^2\, t) = \lambda^n h_n(x\,,t) \qquad \Rightarrow \qquad (x\,\partial_x + 2\,t\partial_t)\,h_n(x\,,t) = n\,h_n(x\,,t)$$

Furthermore

Proposition 2.2. *The expected value of an Hermite polynomial having for argument a Wiener process starting at x is*

$$\mathrm{E}\,h_n\,(w_t + x\,,t) = \frac{x^n}{\Gamma(n+1)} = h_n(x,0) \tag{2.2}$$

Proof.

$$\mathrm{E}\,h_n\,(w_t + x\,,t) := \int_{\mathbb{R}} dy\,h_n(y,t)\,\frac{e^{-\frac{(y-x)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \frac{(-t)^n}{\Gamma(n+1)}\int_{\mathbb{R}} dy\,\frac{e^{-\frac{x^2}{2\,t}+\frac{x\,y}{t}}}{\sqrt{2\,\pi\,t}}\,\frac{d^n}{dy^n}e^{-\frac{y^2}{2\,t}}$$

integrating by parts yields the claim. □

The reason why the expectation value is preserved is that the differential of Hermite along realizations of the Wiener process takes the form.

Proposition 2.3. *The differential of Hermite polynomials of the Wiener process is*

$$dh_n(w_t,t) = dw_t\partial_{w_t}h_n(w_t,t)$$

Proof. By Ito lemma we have

$$dh_n(w_t,t) = dt\left(\partial_t + \frac{1}{2}\partial^2_{w_t}\right)h_n(w_t,t) + dw_t\partial_{w_t}h_n(w_t,t)$$

In order to prove the claim we need to show that

$$\left(\partial_t + \frac{1}{2}\partial^2_{w_t}\right)h_n(w_t,t) = 0$$

Such result can be achieved by direct calculation. It is instructive to proceed in a slightly indirect way. For any $t > 0$

$$0 = \left(\partial_t - \frac{1}{2}\partial^2_x\right)\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}}$$

$$\sum_{n=0}^{\infty}\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n\frac{1}{t}\left(-\,n\,+t\,\partial_t - \frac{t}{2}\partial^2_x + x\,\partial_x\right)h_n = -\sum_{n=0}^{\infty}\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n\left(\partial_t + \frac{1}{2}\partial^2_x\right)h_n$$

which implies

$$\left(\partial_t + \frac{1}{2}\partial^2_x\right)h_n = 0$$

as each of these multiply positive definite terms of different order in y. □

We have therefore a probabilistic interpretation of the statistical conservation law

$$h_n(w_t + x,t) = h_n(x,0) + \int_0^t dw_t\,\partial_{w_t}h_n(w_t,t)$$

From the property of the Ito integral

$$\mathrm{E}\,h_n(w_t + x,t) = h_n(x,0) = \frac{x^n}{\Gamma(n+1)}$$

2.1 Recursion relation and multiple integrals over the Wiener process

Proposition 2.4. *Stochastic integrals over Hermite polynomials satisfy the simple recursion relation*

$$\int_0^t dw_s \, h_n(w_s, s) = h_{n+1}(w_s, s)$$

Proof. Consider the exponential process:

$$\xi_t = e^{\lambda w_t - \frac{\lambda^2 t}{2}} \tag{2.3}$$

Applying to it Ito lemma yields

$$d\xi_t = \lambda \, \xi_t \, dt + \lambda \, dw_t \tag{2.4}$$

or equivalently

$$e^{\lambda w_t - \frac{\lambda^2 t}{2}} = 1 + \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}}$$

If we differentiate an arbitrary number of times with respect to λ we then obtain

$$\left.\frac{d^n}{d\lambda^n}\right|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = \left.\frac{d^n}{d\lambda^n}\right|_{\lambda=0} \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}}$$

Contrasting the left-hand side with the definition of Hermite polynomials we conclude

$$\left.\frac{d^n}{d\lambda^n}\right|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = t^n \left.\frac{d^n}{dz^n}\right|_{z=0} e^{\frac{z}{t} w_t - \frac{z^2}{2t}} = \Gamma(n+1) \, h_n(w_t \, , t)$$

The right hand side is

$$\left.\frac{d^n}{d\lambda^n}\right|_{\lambda=0} \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}} = n \left.\frac{d^{n-1}}{d\lambda^{n-1}}\right|_{\lambda=0} \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}} = \Gamma(n+1) \int_0^t dw_s \, h_{n-1}(w_s \, , s)$$

We have therefore proved that

$$h_n(w_t \, , t) = \int_0^t dw_s \, h_{n-1}(w_s \, , s)$$

□

An important consequence is the following. Since

$$h_0(w_t, t) = 1$$

we have that

$$\int_0^t dw_s = \int_0^t dw_s \, h_0(w_s \, , s) = h_1(w_t, t)$$

and

$$\int_0^t dw_{s_1} \int_0^{s_1} dw_{s_0} = h_2(w_t, t)$$

or in full generality

$$\int_0^t dw_{s_1} \prod_{i=1}^{n-1} \int_0^{s_{i-1}} dw_{s_{i-1}} = h_n(w_t, t)$$

3 Existence and uniqueness theorem

Theorem 3.1 (*Existence and uniqueness*)**.** *Suppose that for some* $T \in \mathbb{R}_+$

$$\boldsymbol{b} : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$$

and

$$\mathsf{A} : \mathbb{R}^{d \times d} \times [0, T] \to \mathbb{R}^{d \times m}$$

are continuous and satisfy the following conditions in the Euclidean norm

$$||\boldsymbol{b}\,(\boldsymbol{x}, t) - \boldsymbol{b}\,(\boldsymbol{y}, t)\,|| < C\,||\boldsymbol{x} - \boldsymbol{y}|| \qquad \& \qquad ||\mathsf{A}\,(\boldsymbol{x}, t) - \mathsf{A}\,(\boldsymbol{y}, t)\,|| < C\,||\boldsymbol{x} - \boldsymbol{y}||$$

and

$$||\boldsymbol{b}\,(\boldsymbol{x}, t)\,|| < C\,(1 + ||\boldsymbol{x}||) \qquad \& \qquad ||\mathsf{A}\,(\boldsymbol{x}, t)\,|| < C\,(1 + ||\boldsymbol{x}||)$$

for all $0 \leq t \leq T$ *and some positive constant* C*. Let also* $\boldsymbol{\xi}_o$

$$\boldsymbol{\xi}_o : \Omega \to \mathcal{R}^d$$

a random variable such that

$$\mathrm{E}||\boldsymbol{\xi}_o||^2 < \infty$$

Furthermore $\boldsymbol{\xi}_o$ *is independent of the* σ*-algebra* $\mathcal{W}$ *generated by a given* m*-dimensional Wiener process for* $t \geq 0$*. Then, there exists a unique solution of*

$$d\boldsymbol{\xi}_t = \boldsymbol{b}(\boldsymbol{\xi}_t, t)\,dt + \mathsf{A}(\boldsymbol{\xi}_t, t) \cdot d\boldsymbol{w}_t \tag{3.1a}$$

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_o \tag{3.1b}$$

Uniqueness here means that any square integrable $\boldsymbol{\xi}_t$ *and* $\tilde{\boldsymbol{\xi}}_t$ *with continuous paths, satisfying (3.1a), (3.1b) then for all* $0 \leq t \leq T$

$$\boldsymbol{\xi}_t = \tilde{\boldsymbol{\xi}}_t \qquad a.s.$$

3.1 Example: absence of Lipschitz continuity

Consider the ordinary differential equation:

$$\dot{\xi} = C\xi^{1/3}$$

The field

$$f = C\,x^{1/3}$$

is *not* differentiable in zero therefore not Lipschitz continuous there. As a consequence the equation has multiple solutions

$$\xi_t = \begin{cases} 0 & t < t_o \\ \tilde{C}\,t^{3/2} & t \geq t_o \end{cases}$$

for arbitrary t_o.

4 Solution by iteration

If $\boldsymbol{b}$ and A are smooth

$$
\begin{aligned}
\boldsymbol{\xi}_t &= \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}(\boldsymbol{\xi}_s, s) + \int_0^t \mathsf{A}(\boldsymbol{\xi}_s, s) \cdot d\boldsymbol{w}_s \\
&= \boldsymbol{\xi}_o + \boldsymbol{b}(\boldsymbol{\xi}_o, 0)\, t + \mathsf{A}(\boldsymbol{\xi}_o, 0) \cdot d\boldsymbol{w}_t + \int_0^t ds \int_0^s d\boldsymbol{b}(\boldsymbol{\xi}_u, u) + \int_0^t \int_0^s d\mathsf{A}(\boldsymbol{x}_u, u) \cdot d\boldsymbol{w}_s \\
&= \boldsymbol{\xi}_o + \boldsymbol{b}(\boldsymbol{\xi}_o, 0)\, t + \mathsf{A}(\boldsymbol{\xi}_o, 0) \cdot d\boldsymbol{w}_t + \int_0^t d\boldsymbol{b}(\boldsymbol{\xi}_s, s)\,(t - s) + \int_0^t d\mathsf{A}(\boldsymbol{x}_s, s)[\boldsymbol{w}_t - \boldsymbol{w}_s]
\end{aligned} \tag{4.1}
$$

We then apply Ito lemma to $\boldsymbol{b}$ and A and iterate. In such a way the solution is constructed as a power series in t and w_t.

Example 4.1 ($1d$-*linear case*)**.** Consider the Ito SDE

$$
d\xi_t = \frac{\xi_t}{\tau} dt + \sigma\, \xi_t dw_t \tag{4.2}
$$

we can remove the drift by setting

$$
\xi_t = \tilde{\xi}_t\, e^{\frac{t}{\tau}}
$$

The new process $\tilde{\xi}_t$ is related to the original by a function independent of the Wiener process. Hence, Ito calculus lemma

$$
d(\tilde{\xi}_t \eta_t) = (d\tilde{\xi}_t)\eta_t + \tilde{\xi}_t d\eta_t + \langle d\tilde{\xi}_t\, , d\eta_t \rangle
$$

($\langle \bullet\, , \bullet \rangle$ is the quadratic co-variation) reduces for

$$
\eta_t = e^{\frac{t}{\tau}}
$$

to the standard Leibniz rule. We find

$$
d(\tilde{\xi}_t e^{\frac{t}{\tau}}) = (d\tilde{\xi}_t) e^{\frac{t}{\tau}} + \tilde{\xi}_t \frac{e^{\frac{t}{\tau}}}{\tau}
$$

The new Ito stochastic differential equation is

$$
d\tilde{\xi}_t = \sigma\, \tilde{\xi}_t dw_t
$$

If we apply the recursion equations (4.1) we get into

$$
\begin{aligned}
\tilde{\xi}_t &= \tilde{\xi}_o + \sigma\, \tilde{\xi}_o\, w_t + \sigma \int_0^t dw_s \int_0^s d\tilde{\xi}_{s_1} \\
&= \tilde{\xi}_o + \sigma\, \tilde{\xi}_o\, w_t + \sigma^2 \tilde{\xi}_o \int_0^t dw_s \int_0^s dw_{s_1} + \sigma^2 \int_0^t dw_s \int_0^s dw_{s_1} \int_0^{s_2} d\tilde{\xi}_{s_2}
\end{aligned}
$$

Repeating for an arbitrary number of steps

$$
\tilde{\xi}_t = \tilde{\xi}_o + \tilde{\xi}_o \sum_{i=1}^{\infty} \sigma^i \int_0^t dw_{s_1} \prod_{j=1}^{i-1} \int_0^{s_j} dw_{s_j} \tag{4.3}
$$

We already know that

$$\int_0^t dw_{s_1} \prod_{j=1}^{i-1} \int_0^{s_j} dw_{s_j} = h_i(w_t, t)$$

with h_i the Hermite polynomial

$$h_i(x,t) = \frac{t^n}{\Gamma(i+1)} \left.\frac{d^n}{dz^n}\right|_{z=0} e^{\frac{z\,x}{t} - \frac{z^2}{2\,t}} = \frac{1}{\Gamma(i+1)} \left.\frac{d^n}{d\lambda^n}\right|_{z=0} e^{\lambda\,x - \frac{\lambda^2\,t}{2}}$$

Upon inserting in (4.3), we get into

$$\tilde{\xi}_t = \tilde{\xi}_o \left\{ 1 + \sum_{i=1}^{\infty} \frac{\sigma^i h_i(w_t, t)}{\Gamma(i+1)} \right\} = \xi_o\, e^{\sigma w_t - \frac{\sigma^2\,t}{2}}$$

and consequently

$$\xi_t = \xi_o\, e^{\frac{t}{\tau} + \sigma w_t - \frac{\sigma^2\,t}{2}}$$

The same result is straightforwardly obtained by converting (4.2) to *Stratonovich form*

$$d\xi_t = \left(1 - \frac{\sigma^2\,\tau}{2}\right) \xi_t \frac{dt}{\tau} + \sigma\,\xi_t\, dw_t$$

and by integrating it according to the usual rules of calculus

$$\xi_t = \xi_o\, e^{\left(1 - \frac{\sigma^2\,\tau}{2}\right)\frac{t}{\tau} + \sigma\, w_t}$$

References

[1] L. C. Evans. An introduction to stochastic differential equations. UC Berkeley, Lecture Notes, 2006.