1 Introduction

An example of concrete application of the Karhunen-Loève representation can be found in [3].

2 Summary of properties of the Brownian motion

- Covariance

$$\mathrm{E} w_t \, w_s = t \wedge s := R(t, s) \tag{2.1}$$

- Stationary increments: for any $t, s \in \mathbb{R}_+$

$$\mathrm{E}\,(w_t - w_s)^2 = |t - s|$$

- Characteristic function: for $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_m$

$$\mathrm{E}\, e^{\imath \sum_{k=1}^{m-1} \lambda_k (w_{t_{k+1}} - w_{t_k})} = \prod_{k=1}^{m-1} e^{\imath \lambda_k (w_{t_{k+1}} - w_{t_k})} = \prod_{k=1}^{m-1} e^{-\frac{\lambda_k^2}{2}(t_{k+1} - t_k)}$$

3 Karhunen-Loève representation of the Wiener process

The Karhunen-Loève representation provides us with a convenient *deterministic* functional orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$, which will allow us to write the Brownian motion

$$w_t : \Omega \times [0, T] \to \mathbb{R}$$

as a series

$$w_t = \sum_{n=0}^{\infty} c_n \psi_n(t)$$

Randomness is encoded in the coefficients

$$c_n = \int_0^T \frac{dt}{T} \, \psi_n(t) w_t$$

which need to form a sequence $\{c_n\}_{n=0}^{\infty}$ of independent Gaussian random variable with zero average and variance determined by the Karhunen-Loève representation.

Let us consider the space of the Lebesgue square integrable $\mathbb{L}^2([0, T])$ real functions on $[0, T]$. This space is an Hilbert space with respect to the scalar product:

$$\langle f, g \rangle := \int_0^T \frac{dt}{T} \, f(t) g(t) \qquad f, g \in \mathbb{L}^2([0, T])$$

Proposition 3.1. *The covariance R of the Brownian motion (2.1) defines the* kernel *of an operator mapping $\mathbb{L}^2([0, T])$ into itself:*

$$g(t) = \int_0^T \frac{dt}{T} \, R(t\,, s) \, f(s) := R[f](t)$$

i.e.

$$f \in \mathbb{L}^2([0, T]) \to g \equiv R[f] \in \mathbb{L}^2([0, T])$$

Proof. The following chain of equalities holds by definition

$$||g||_2^2 := \langle\, R[f]\,, R[f]\,\rangle = \int_0^T \frac{dt}{T} \left[\int_0^T \frac{ds}{T}\, R(t,s)\, f(s)\right] \left[\int_0^T \frac{ds'}{T}\, R(t,s')\, f(s')\right]$$

where

$$\int_0^T \frac{ds}{T}\, R(t,s)\, f(s) = \int_0^t \frac{ds}{T}\, s\, f(s) + t \int_t^T \frac{ds}{T}\, f(s)$$

For the first integral we have the bound

$$\left|\int_0^t \frac{ds}{T}\, s\, f(s)\right| \leq \frac{t}{T} \int_0^t ds\, |f(s)| \tag{3.1}$$

using Lyapunov inequality we can then write

$$|R[f](t)| \leq\; T \int_0^T \frac{ds}{T} |f(s)| \;\leq\; T \left(\int_0^T \frac{ds}{T} |f(s)|^2\right)^{1/2}$$

We conclude

$$||g||_2^2 \leq \sigma^4\, T^2 ||f||_2^2$$

□

The as an operator kernel $R(t,s)$ enjoys two further properties

- $R(t,s)$ is positive definite: i.e. for any collection of $\{c_i\}_{i=1}^n \in \mathbb{C}$ and sampling of the arguments $\{t_i\}_{i=1}^n$ we have

  $$\sum_{ij=1}^{n} c_i R(t_i, s_i) c_j^* \geq\; 0$$

  Namely

  $$\sum_{ij=1}^{n} c_i R(t_i, t_j) c_j^* = \sum_{i,j=1}^{n} \mathrm{E}\, w_{t_i} w_{t_j} c_i\, c_j^* = \mathrm{E} \left|\sum_{i=1}^{n} c_i w_{t_i}\right|^2 \geq 0$$

- The kernel is symmetric:

  $$R(t,s) = R(s,t)$$

These properties of the kernel allows us to invoke

Theorem 3.1 (Mercer). *Let R be a continuous symmetric non-negative definite kernel. Then there is an orthonormal basis $\{\psi_n\}_{n=0}^\infty$ of $\mathbb{L}^2([0,T])$ consisting of eigenfunctions of of the operator*

$$R[f] = \int_0^T \frac{ds}{T} R(t,s)\, f(s)$$

such that the corresponding sequence of eigenvalues $\{r_n\}_{n=0}^\infty$ is non-negative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $[0,T]$ and R admit the representation

$$R(t,s) = \sum_{n=0}^{\infty} r_n\, \psi_n(t)\, \psi_n(s)$$

Proof. The theorem is proved in chapter III of [1]. It can also be regarded as a consequence of the Hilbert-Schmidt theorem [2] □

In other words the operator specified by R is diagonalizable with discrete spectrum. The idea underlying the Karhunen-Loève representation is to use the orthonormal basis of eigenvectors to write the Brownian motion as the $\mathbb{L}^2$-convergent series:

$$w_t = \sum_{n=0}^{\infty} c_n \, \psi_n(t) \tag{3.2}$$

Namely, if the $\{c_n\}_{n=0}^{\infty}$ are Gaussian independent random variables satisfying

$$\mathrm{E}\, c_n = 0 \qquad \& \qquad \mathrm{E}\, c_n^2 = r_n$$

it follows immediately that (3.2) satisfies all the requirements to be a Brownian motion:

i Covariance:

$$\mathrm{E}\, w_t \, w_s = \sum_{n=0}^{\infty} r_n \, \psi_n(t) \psi_n(s) = R(t, s)$$

ii Independent increments: for $t_1 \leq t_2 \leq t_3 \leq t_4$

$$\mathrm{E}\,(w_{t_4} - w_{t_3})(w_{t_2} - w_{t_1}) = \sum_n r_n [t_4 \wedge t_2 - t_4 \wedge t_1 - t_3 \wedge t_2 + t_3 \wedge t_1] = 0$$

iii Gaussian structure of the characteristic function :

$$\mathrm{E}\, e^{\imath \sum_{k=1}^{m} \sum_{n=0}^{\infty} c_n \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]} = e^{-\frac{1}{2} \sum_n r_n \{\sum_{k=1}^{m} \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]\}^2}$$

We use the identity

$$\sum_{k=1}^{m} \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})] = \\ \sum_{k=1}^{m-1} (\lambda_k - \lambda_{k+1}) \psi_n(t_k) + \lambda_m \, \psi_n(t_m) - \lambda_1 \, \psi_n(t_0) := \sum_{k=0}^{m} \Delta_k \, \psi_n(t_k)$$

Thus

$$\mathrm{E}\, e^{\imath \sum_{k=1}^{m} \sum_{n=0}^{\infty} c_n \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]} = e^{-\frac{1}{2} \sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2}$$

Now we observe that

$$\sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2 = \\ \sum_{k\,,l=1}^{m} \sum_n r_n \Delta_k \, \Delta_l \, \psi_n(t_k) \psi_n(t_l) = \sum_{k\,,l=1}^{m} \Delta_k \, \Delta_l R(t_k \,, t_l) = \mathrm{E} \left(\sum_{l=0}^{m} \Delta_l \, w_{t_l} \right)^2$$

Finally using the definition of the Δ_l coefficients

$$\sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2 = \mathrm{E} \left[\sum_{k=1}^{m} \lambda_k (w_{t_k} - w_{t_{k-1}}) \right]^2 = \sum_{k=1}^{m} \mathrm{E}\, \lambda_k^2 (w_{t_k} - w_{t_{k-1}})^2$$

3.1 Explicit construction of the Karhunen-Loève basis

We need to solve:

$$\int_0^T \frac{ds}{T}\, R(t,s)\,\psi_n(s) = r_n\,\psi_n(t)$$

or equivalently

$$\int_0^t \frac{ds}{T} ds\; s\,\psi_n(s) + t\int_t^T \frac{ds}{T}\,\psi_n(s) = r_n\,\psi_n(t)$$

The first derivative is

$$\int_t^T \frac{ds}{T}\,\psi_n(s) = r_n\,\dot{\psi}_n(t) \qquad \Rightarrow \qquad \dot{\psi}_n(T) = 0$$

and the second derivative

$$-\frac{1}{T}\psi_n(t) = r_n\,\ddot{\psi}_n(t) \qquad \Rightarrow \qquad \ddot{\psi}_n(t) + \frac{1}{T\,r_n}\psi_n(t) = 0$$

Thus the problem is equivalent to solving the differential equation

$$\ddot{\psi}_n(t) + \frac{1}{T\,r_n}\psi_n(t) = 0$$

with boundary conditions

$$\psi_n(0) = \dot{\psi}_n(T) = 0$$

The boundary condition in zero yields (the factor $\sqrt{2}$ comes from unit normalization)

$$\psi_n(t) = \sqrt{2}\sin\left(\frac{t}{\sqrt{T\,r_n}}\right)$$

Imposing the condition for $t = T$ gives the "*quantization*" condition for the eigenvalues:

$$\sqrt{\frac{T}{r_n}} = (2\,n+1)\,\frac{\pi}{2} \qquad n = 0, 1, \ldots$$

The conclusion is that the explicit Karhunen-Loève representation of the covariance of the Wiener process is

$$R(t,s) = \sum_{n=0}^{\infty} \frac{T\,\psi_n(t)\psi_n(s)}{\left(n+\frac{1}{2}\right)^2\,\pi^2}$$

References

[1] R. Courant and D. Hilbert. *Methods of mathematical physics Vol I.* John Wiley & Sons, Inc., 1989.

[2] A. N. Kolmogorov and S. V. Fomin. *Elements of the Theory of Functions and Functional Analysis.* Dover books on mathematics. Courier Dover Publications, 1999.

[3] T. P. Sapsis and P. F. Lermusiaux. Dynamically orthogonal field equations for continuous stochastic dynamical systems. *Physica D: Nonlinear Phenomena*, 238:23472360, 2009.