Elements of Itô calculus

1 Introduction

The purely analytic introduction to Itô calculus given here is based on [1] (in French, english translation available as an appendix in [4] [1]). Another good reference is [3] in particular chapter I.

2 Functions of finite linear variation

Let $I = [a, b] \in \mathbb{R}$ and

$$f : I \mapsto \mathbb{R} \tag{2.1}$$

a deterministic function.

Definition 2.1. *We call the variation of f the quantity*

$$V^{(1)}_{[f]}(I) = \sup_{\mathsf{p}} \sum_{t_i \in \mathsf{p}} |f(t_{i+1}) - f(t_i)| \tag{2.2}$$

whether the sup is taken over the partitions $\{\mathsf{p}\}$ (see appendix A)) of I

By triangular inequality we also have

$$V^{(1)}_{[f]}(I) = \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_i \in \mathsf{p}} |f(t_{i+1}) - f(t_i)| \tag{2.3}$$

for p the mesh of the partition.

Proposition 2.1. *If $f \in C^1(I)$ and*

$$\int_I dt \left| \frac{df}{dt} \right| (t) < \infty \tag{2.4}$$

then

$$V^{(1)}_{[f]}(I) = \int_I dt \left| \frac{df}{dt} \right| (t) \tag{2.5}$$

Proof. Let

$$\dot{f} = \frac{df}{dt} \tag{2.6}$$

[1] you can download the paper from `http://wiki.helsinki.fi/download/attachments/79560764/Foellmer.pdf`

the chain of equalities holds

$$
\begin{aligned}
V^{(1)}_{[f]}(I) &= \lim_{|\mathsf{p}|\downarrow 0} \sum_{t_i \in \mathsf{p}} |f(t_{i+1}) - f(t_i)| = \lim_{|\mathsf{p}|\downarrow 0} \sum_{t_i \in \mathsf{p}} |\dot{f}(t_i)|(t_{i+1} - t_i) \\
&= \lim_{|\mathsf{p}|\downarrow 0} \sum_{t_i \in \mathsf{p}} \left| \int_{t_i}^{t_{i+1}} dt\, \dot{f}(t) \right| = \sup_{\mathsf{p}} \sum_{t_i \in \mathsf{p}} \left| \dot{f}(t_i) \right| (t_{i+1} - t_i) = \int_I dt \left| \frac{df}{dt} \right| (t)
\end{aligned}
\tag{2.7}
$$

thus proving the claim. □

Thus, trajectories solutions of ordinary differential equations

$$
\dot{x}_t = v(x_t) \tag{2.8}
$$

with v sufficiently smooth as customary in applications are functions of finite variation. A larger class of functions includes those with discontinuities.

Definition 2.2. *A function* $f\colon I \mapsto \mathbb{R}$ **continuous from the right**

$$
\lim_{t \downarrow t_o} f(t) = f(t_o) \tag{2.9}
$$

with **limit from the left**

$$
\lim_{t \uparrow t_o} f(t) = f(t_o-) \tag{2.10}
$$

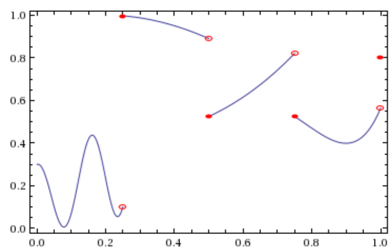
is called a **CADLAG** *function (French: Continue à Droite Limite à Gauche, less used English acronym CORLOL: Continuous On the Right Limit On the Left).*

The difference

$$
J_{[f]}(t) = f(t) - f(t-) \tag{2.11}
$$

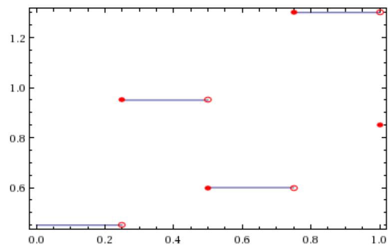
is called a **jump**. A theorem from analysis guarantees that a function defined on an interval $[a, b]$ can have no more than countably many jumps (see e.g. [3] and refs therein).

a CADLAG function:
circles denote the limit
dots continuity at jump locations.



If a cadlag function varies only at jump locations $\mathcal{J} \in I$ we have

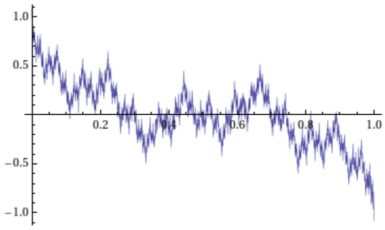
$$
\begin{aligned}
f(t) &= \textstyle\sum_{s \in \mathcal{J} \cap \{s : s \leq t\}} J_{[f]}(s) \\
V^{(1)}_{[f]}(I) &= \textstyle\sum_{s \in \mathcal{J}} |J_{[f]}(s)|
\end{aligned}
\tag{2.12}
$$



3 Quadratic (co)-variation

A classical example of a continuous function no-where differentiable is the *Weierstrass function* $W_t \colon \mathbb{R} \mapsto \mathbb{R}$

$$W_t := \sum_{n=0}^{\infty} a^n \cos(b^n t) \qquad \text{for} \qquad \begin{array}{l} 0 < a < 1 \\ b \in 2\mathbb{N}+1 \\ a\, b > 1 \end{array} \tag{3.1}$$



G.H. Hardy [2] proved that the Weierstrass is everywhere continuous and non-differentiable. One can get an intuition of the reason observing that

1. a sequence of continuous functions (i.e. approximations by finite sum) uniformly converging admits as a limit a continuous function;
2. differentiating individual addends in the series one obtains

$$\frac{dW_t}{dt}\text{"} = \text{"} - \sum_{n=0}^{\infty} a^n\, b^n \sin(b^n t) \tag{3.2}$$

   which is a diverging series.

Hardy also showed that

$$|W_{t+h} - W_t| \leq C\, h^{\alpha} \qquad \alpha = -\frac{\ln a}{\ln b} \tag{3.3}$$

A further consequence is that the Weierstrass function is not of finite variation. It makes sense to consider functions of *finite second variation*:

Definition 3.1 (*Quadratic (co-)variation*)**.** *Let $\xi_t \colon I \mapsto \mathbb{R}$ and $\chi_t \colon I \mapsto \mathbb{R}$ the limit*

$$V^{(2)}_{[\xi\,,\chi]}(I) = \lim_{|\mathsf{p}_{(n)}|\downarrow 0} \sum_{t_k \in \mathsf{p}_n} (\xi_{t_k} - \xi_{t_{k-1}})(\chi_{t_k} - \chi_{t_{k-1}})$$

is called the **quadratic co-variation** *of the processes. In particular*

$$V^{(2)}_{[\xi\,,\xi]}(I) = \lim_{|\mathsf{p}_{(n)}|\downarrow 0} \sum_{t_k \in \mathsf{p}_n} (\xi_{t_k} - \xi_{t_{k-1}})^2$$

is called the **quadratic variation** *of ξ_t.*

Finiteness of the quadratic variation is possible only if the (first) variation of a function diverges. Namely

$$\sum_k [f(t_k) - f(t_{k-1})]^2 \leq \max_k |f(t_k) - f(t_{k-1})| \sum_k |f(t_k) - f(t_{k-1})|$$

so that for a differentiable function

$$\sum_k [f(t_k) - f(t_{k-1})]^2 \leq \max_k |f(t_k) - f(t_{k-1})| \int_0^t dt\, |f'(t)| \overset{\max_k (t_k - t_{k-1}) \to 0}{\rightarrow} 0$$

In the case of the Wiener process the finiteness of the right hand side implies

$$\sum_k |f(t_k) - f(t_{k-1})| \overset{\max_k (t_k - t_{k-1}) \to 0}{\uparrow} \infty$$

4 Differential calculus for functions of finite quadratic variation

Suppose $x_t \colon I \mapsto \mathbb{R}$ is a CADLAG function of finite quadratic variation. Then we have

$$V^{(2)}_{[x\,,x]}(I) = V^{(2,c)}_{[x\,,x]}(I) + \sum_{s \in \mathcal{J}} J^2_{[x]}(s) \tag{4.1}$$

where $V^c_{[x\,,x]}(I)$ is the quadratic variation of the continuous part of x and $\mathcal{J}$ denotes the set of the jump locations over I. The following proposition shows that the jump component of the quadratic variation defines the atomic part of the measure defined by $V_{[x\,,x]}(I)$. In other words, $V^c_{[x\,,x]}(I)$ defines a measure absolutely continuous with respect to the Lebsegue measure

$$dV^{(2,c)}_{[x\,,x]}([0,t)) = g(x_t)\,dt \tag{4.2}$$

for some positive definite

$$g \colon \mathbb{R} \mapsto \mathbb{R}_+ \tag{4.3}$$

whilst

$$d \sum_{s \in \mathcal{J}} J^2_{[x]}(s) = \sum_{s \in \mathcal{J}} dt\,(x_s - x_{s^-})^2\,\delta(t-s) \tag{4.4}$$

for some $r_i \in \mathbb{R}_+$

Proposition 4.1. *Let $\{\mathsf{p}_n\}_{n=0}^{\infty}$ a sequence of partitions of the interval I. For any continuous function*

$$f \colon I \mapsto \mathbb{R} \tag{4.5}$$

the limit

$$\lim_{n \uparrow \infty} \sum_{t \geq t_i \in \mathsf{p}_n} f(x_{t_i})\,(x_{t_{i+1}} - x_{t_i})^2 = \int_{(0,t)} dV^{(2)}_{[x\,,x]}((0,s])\,f(x_{s^-}) \tag{4.6}$$

exists

Proof. Let C be the countable set of points in I where x_t performs jumps of size strictly larger than $O(\varepsilon^2)$ for any arbitrary $\varepsilon > 0$. Let also z_t be the distribution function of finite-size jumps in I

$$z_t = \sum_{s \in C \cap (0,t]} (x_s - x_{s^-}) \tag{4.7}$$

we have

$$\lim_{n \uparrow \infty} \sum_{t \in \mathsf{p}_n} f(x_{t_i})\,(z_{t_{i+1}} - z_{t_i})^2 = \sum_{t \in C \cap (0,s]} f(x_{t^-})\,(x_t - x_{t^-})^2 \tag{4.8}$$

Let now y be the discrete measure such that

$$\sum_{t_i \in \mathsf{p}_n \cap (0,t]} (x_{t_{i+1}} - x_{t_i})^2 = \sum_{t_i \in \mathsf{p}_n \cap (0,t]} (y_{t_{i+1}} + z_{t_{i+1}} - y_{t_i} + z_{t_i})^2 \tag{4.9}$$

i.e. the discrete approximant of the absolute continuous part of the measure defined by the quadratic variation of x_t. Then

$$\sum_{t_i\in\mathsf{p}_n\cap(0,t]}(x_{t_{i+1}}-x_{t_i})^2 = \\ \sum_{t_i\in\mathsf{p}_n\cap(0,t]}(y_{t_{i+1}}-y_{t_i})^2+\sum_{t_i\in\mathsf{p}_n\cap(0,t]}(z_{t_{i+1}}-z_{t_i})^2+2\sum_{t_i\in\mathsf{p}_n\cap(0,t]}(z_{t_{i+1}}-z_{t_i})(y_{t_{i+1}}-y_{t_i}) \tag{4.10}$$

By definition of y the third term of the right hand side converges to zero and the measure associated to y weakly converges to a measure the atomic part thereof only comprises jumps of size less equal $O(\varepsilon^2)$. It follows that

$$\limsup_{n}\left|\lim_{n\uparrow\infty}\sum_{t\in\mathsf{p}_n}f(x_{t_i})\,(y_{t_{i+1}}-y_{t_i})^2-\int_{(0,t)}dV^{(2,c)}_{[x\,,x]}((0,s])\,f(x_{s}-)\right|\leq O(\varepsilon^2) \tag{4.11}$$

which proves the claim ☐

Föllmer [1] proved the following theorem, which we reproduce here in abridged form.

Theorem 4.1. *Let $x_t\colon\,[0,T]\mapsto\mathbb{R}$ be a CADLAG function of finite quadratic variation and $F\in C^2(\mathbb{R})$. Then for any $t\,\in\,[0,T]$ Itô's formula*

$$F(x_t)-F(x_0)=\int_0^t dx_s\,(\partial_x F)(x_s-) \\ +\frac{1}{2}\int_0^t d\,V^{(2,c)}_{[x\,,x]}([0,s])\,(\partial_x^2 F)(x_s)+\sum_{s\in\mathcal{J}}\left[F(x_s)-F(x_s-)-(\partial_x F)(x_s-)J_{[x]}(s)\right] \tag{4.12}$$

holds with

$$\int_0^t dx_s\,(\partial_x F)(x_s-)=\lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})\,(\partial_x F)(x_{t_k}) \tag{4.13}$$

holds true and the series in (4.13) is absolutely convergent.

Proof. Since x_t is not of finite variation the difficulty of the proof is to prove (4.13). The strategy of the proof is to prove the absolute convergence of the left hand side and of the other terms on the right hand side. As far as the left hand side is concerned, by hypothesis x_t is continuous to the right so guaranteeing the convergence

$$F(x_t)-F(x_0)=\lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}[F(x_{t_{k+1}})-F(x_{t_k})] \tag{4.14}$$

since the addends give rise to alternating sums. Let us now distinguish two cases

1. Suppose now that x_t is continuous. Then by Taylor's formula

$$F(x_{t_{k+1}})-F(x_{t_k})=(x_{t_{k+1}}-x_{t_k})(\partial_x F)(x_{t_k})+\frac{(x_{t_{k+1}}-x_{t_k})^2}{2}(\partial_x^2 F)(x_{\tilde{t}_k}) \tag{4.15}$$

   for some $\tilde{t}_k\in\,(t_k,t_{k+1})$. We can rewrite the equality as

$$F(x_{t_{k+1}})-F(x_{t_k})=(x_{t_{k+1}}-x_{t_k})(\partial_x F)(x_{t_k}) \\ +\frac{(x_{t_{k+1}}-x_{t_k})^2}{2}(\partial_x F)(x_{t_k})+\frac{(x_{t_{k+1}}-x_{t_k})^2}{2}[(\partial_x^2 F)(x_{\tilde{t}_k})-(\partial_x^2 F)(x_{t_k})] \tag{4.16}$$

We then have

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})^2[(\partial_x^2F)(x_{\tilde{t}_k})-(\partial_x^2F)(x_{t_k})]$$
$$\leq \lim_{|\mathsf{p}_n|\downarrow 0}\max_{t_k\in\mathsf{p}_n}[(\partial_x^2F)(x_{\tilde{t}_k})-(\partial_x^2F)(x_{t_k})]\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})^2\to 0 \tag{4.17a}$$

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})^2\,(\partial_x^2F)(x_{t_k})\leq \lim_{|\mathsf{p}_n|\downarrow 0}\max_{t_k\in\mathsf{p}_n}(\partial_x^2F)(x_{t_k})\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})^2\,<\,\infty \tag{4.17b}$$

where (4.17a) holds because of the continuity of x_t and F and (4.17b) since F is continuous over a finite closed interval. Gleaning the information provided (4.14) and (4.17a)-(4.17b) we conclude that the claim (4.13) must hold true.

2. Let us now turn to the general case of a CADLAG function. Let $\varepsilon\,>\,0$ We divide the jumps of x_t on $[0,t]$ into two classes:

   (a) $C_1\equiv C_1(\varepsilon,t)$ with jumps of finite size;
   (b) $C_2\equiv C_2(\varepsilon,t)$ such that $\sum_{s\in C_2}J_{[x]}^2(s)\leq\varepsilon^2$.

   We then write

$$\sum_{t_k\in\mathsf{p}_n}\big[F(x_{t_{k+1}})-F(x_{t_k})\big]=\sum_1\big[F(x_{t_{k+1}})-F(x_{t_k})\big]+\sum_2\big[F(x_{t_{k+1}})-F(x_{t_k})\big] \tag{4.18}$$

   where $\sum_1$ indicates the summation over those $t_k\in\mathsf{p}_n$ for which the interval $]t_k,t_{k+1}]$ contains a jump of class C_1. Clearly we have

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_1\big[F(x_{t_{k+1}})-F(x_{t_k})\big]=\sum_{s\in\mathcal{J}}[F(x_s)-F(x_s-)] \tag{4.19}$$

   On the other hand we can apply Taylor's formula to write

$$\sum_2\big[F(x_{t_{k+1}})-F(x_{t_k})\big]=\sum_{t_k\in\mathsf{p}_n}\left[(x_{t_{k+1}}-x_{t_k})(\partial_xF)(x_{t_k})+\frac{(x_{t_{k+1}}-x_{t_k})^2}{2}(\partial_x^2F)(x_{t_k})\right]$$
$$-\sum_1\left[(x_{t_{k+1}}-x_{t_k})(\partial_xF)(x_{t_k})+\frac{(x_{t_{k+1}}-x_{t_k})^2}{2}(\partial_x^2F)(x_{t_k})\right]+\frac{1}{2}\sum_2(x_{t_{k+1}}-x_{t_k})^2R(\tilde{t}_k,t_k) \tag{4.20}$$

   with

$$R(\tilde{t}_k,t_k):=\big[(\partial_x^2F)(x_{\tilde{t}_k})-(\partial_x^2F)(x_{t_k})\big] \tag{4.21}$$

   Furthermore

$$\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})^2(\partial_x^2F)(x_{t_k})=\left(\sum_1+\sum_2\right)(x_{t_{k+1}}-x_{t_k})^2(\partial_x^2F)(x_{t_k}) \tag{4.22}$$

   we have

$$\sum_2\big[F(x_{t_{k+1}})-F(x_{t_k})\big]=\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}}-x_{t_k})(\partial_xF)(x_{t_k})$$
$$+\sum_2(x_{t_{k+1}}-x_{t_k})^2\,\big[(\partial_x^2F)(x_{t_k})+R(\tilde{t}_k,t_k)\big]-\sum_1(x_{t_{k+1}}-x_{t_k})(\partial_xF)(x_{t_k}) \tag{4.23}$$

   Let us analyze the terms occurring in (4.20) separately.

(a) Since $\sum_2$ does not contain jumps of finite size

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_2 (x_{t_{k+1}} - x_{t_k})^2 R(\tilde{t}_k, t_k) = 0 \tag{4.24}$$

for the same reasons put forward in the continuous x_t case. Similarly

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_2 (x_{t_{k+1}} - x_{t_k})^2 (\partial_x^2 F)(x_{t_k}) = \int_0^t d\,V^{(2,c)}_{[x\,,x]}([0,s])\,(\partial_x^2 F)(x_s) \tag{4.25}$$

(b) The limit of the sum including jumps is dominated by these latter ones

$$\lim_{|\mathsf{p}_n|\downarrow 0}\sum_1 (x_{t_{k+1}} - x_{t_k})(\partial_x F)(x_{t_k}) = \sum_{s\in\mathcal{J}} J_{[x]}(s)(\partial_x F)(x_s-) \tag{4.26}$$

We have then

$$\begin{aligned}F(x_t) - F(x_o) &= \lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}[F(x_{t_{k+1}}) - F(x_{t_k})] = \lim_{|\mathsf{p}_n|\downarrow 0}\sum_{t_k\in\mathsf{p}_n}(x_{t_{k+1}} - x_{t_k})(\partial_x F)(x_{t_k})\\ &+\frac{1}{2}\int_0^t d\,V^{c}_{[x\,,x]}([0,s])\,(\partial_x^2 F)(x_s) + \sum_{s\in\mathcal{J}}\big[F(x_s) - F(x_s-) - (\partial_x F)(x_s-)J_{[x]}(s)\big]\end{aligned} \tag{4.27}$$

where Taylor's formula also guarantees that

$$\sum_{s\in\mathcal{J}}\big[F(x_s) - F(x_s-) - (\partial_x F)(x_s-)J_{[x]}(s)\big] \le C\sum_{s\in\mathcal{J}} J^2_{[x]}(s) \tag{4.28}$$

Thus the last two term on the right hand side of (4.27) denote a finite limit of the approximating sums. This observation yields the claim and concludes the proof.

□

Remark 4.1. Observation if x_t is CADLAG of finite (first) variation we have

$$F(x_t) - F(x_0) = \int_0^t \mathrm{d}x_s\,(\partial_x F)(x_s-) + \sum_{s\in\mathcal{J}}[F(x_s) - F(x_s-) - (\partial_x F)(x_s-)J_{[x]}(s)] \tag{4.29}$$

Let us consider for example

$$x_t = x_0 + \begin{cases} c_1\,t & t\in[0,1/2)\\ c_2\,t + c_3 & t\in[1/2,1]\end{cases} \tag{4.30}$$

and suppose F is the identity map:

$$F(x) = x \qquad \Rightarrow \qquad \partial_x F(x) = 1$$

Then we have

$$x_t - x_0 = \int_0^t \mathrm{d}x_s\; + [x_{1/2} - x_{1/2^-} - J_{[x]}(1/2)]\,1_{[1/2,1]}(t) = \int_0^t \mathrm{d}x_s$$

as by definition

$$J_{[x]}(s) \equiv\; x_s - x_{s^-}$$

which is non-vanishing only when a jump occurs. Namely we see that

1. if $t < 1/2$ we have trivially

$$x_t - x_0 = \int_0^t \mathrm{d}(c_1\, s)\, 1 = c_1\, t$$

2. $t > 1/2$

$$\begin{aligned}\int_0^t \mathrm{d}x_s &= \int_0^{1/2^-} \mathrm{d}s\, c_1 + (x_{1/2} - x_{1/2^-}) + \int_{1/2^-}^t \mathrm{d}s\, c_2 \\ &= \frac{c_1}{2} + \frac{c_2 - c_1}{2} + c_3 + c_2 \left(t - \frac{1}{2}\right) = c_2\, t + c_3\end{aligned}$$

Gleaning all the terms together

$$x_t - x_0 = c_2\, t + c_3$$

Had we considered a general smooth function F given (4.30) we would have instead gotten for $t > 1/2$ into

$$\begin{aligned}\int_0^t \mathrm{d}x_s\, (\partial_x F)(x_{s^-}) = \int_0^{1/2} \mathrm{d}s\, \dot{x}_s (\partial_x F)(x_{s^-}) + (x_{1/2} - x_{1/2^-})(\partial_x F)(x_{1/2^-}) \\ + \int_{1/2}^t \mathrm{d}s\, \dot{x}_s (\partial_x F)(x_s) = F(x_t) - F(x_{1/2}) + (x_{1/2} - x_{1/2^-})\, (\partial_x F)(x_{1/2^-}) + F(x_{1/2^-}) - F(x_0)\end{aligned}$$

and

$$F(x_{1/2}) - F(x_{1/2^-}) - (\partial_x F)(x_{1/2^-}) J_{[x]}(1/2) = F(x_{1/2}) - F(x_{1/2^-}) - (\partial_x F)(x_{1/2^-})\, (x_{1/2} - x_{1/2^-})$$

It is readily see that the sum of the two terms equals $F(x_t) - F(x_0)$.

Appendix

A Partitions

Definition A.1 (*Partition*)**.** *If $I = [x_-\,, x_+] \subset \mathbb{R}$ is an interval a partition p (subdivision) of I is a finite sequence $\{x_k\}_{k=1}^n$ of points in I such that*

$$x_- = x_1 < \ldots < x_n = x_+$$

Definition A.2 (*Mesh of a partition*)**.** *The mesh size of a partition p of an interval $I = [x_-\,, x_+]$ is*

$$|\mathsf{p}| = \max_{1 \leq k \leq n} |x_{k+1} - x_k|$$

Definition A.3 (*Refinement of a partition*)**.** *The refinement of a partition p of the interval I is another partition p' that contains all the points from p and* some additional points*, again sorted by order of magnitude.*

References

[1] H. Föllmer. Calculditô sans probabilités. In *Séminaire de probabilités XV (Strasbourg)*, volume 850 of *Spinger Lecture Notes in Mathematics*, pages 143–150. Springer, 1981.

[2] G. H. Hardy. Weierstrass's non-differentiable function. *Transactions of the American Mathematical Society*, 17(3):301–325, July 1916.

[3] F. C. Klebaner. *Introduction to stochastic calculus with applications*. Imperial College Press, 2 edition, 2005.

[4] D. Sondermann. *Introduction to stochastic calculus for finance: a new didactic approach*, volume 579 of *Lecture Notes in Economics and Mathematical Systems*. Springer, 2006.