1 Introduction

The content of these notes is also covered by chapter 3 section C of [1].

2 Kolmogorov–Čentsov theorem

Theorem 2.1 (*Kolmogorov-Čentsov*)**.** *If $\xi_t(\omega)$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ satisfying*

$$\mathrm{E}|\xi_t - \xi_s|^{\beta} \leq C\,|t-s|^{1+\alpha}$$

for some positive constants α, β and C, then if necessary , $\xi_t(\omega)$ can be modified for each t on a set of measure zero, to obtain an equivalent version $\tilde{\xi}_t(\omega)$ that is almost surely continuous with exponent γ for every $\gamma \in [0, \alpha/\beta]$ and some $\delta > 0$:

$$\mathrm{P}\left(\omega\,;\ \sup_{\substack{0<t-s<h(\omega)\\ t,s\in[0\,,T]}} \frac{|\tilde{\xi}_t(\omega) - \tilde{\xi}_s(\omega)|}{|t-s|^{\gamma}} \leq \delta\right) = 1$$

Proof. Let

$$\mathcal{T}_n = \left\{\frac{j\,T}{2^n}\right\}_{j=0}^{2^n}$$

The countable union $\mathcal{T} := \cup_{n=0}^{\infty}\mathcal{T}_n$ is a *countable dense* subset of $[0\,,T]$. By linear interpolation we can construct a sequence of approximations to the original process $\xi(t)$ coinciding with it on dyadic rationals e.g.

- the $\mathcal{T}_n$-based interpolation is

$$\xi_n(t) = \xi\left(\frac{j}{2^n}T\right) + \left(t - \frac{j\,T}{2^n}\right)\frac{\xi\left(\frac{j+1}{2^n}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^n}} \qquad \& \qquad t \in [\frac{j}{2^n}T\,, \frac{j+1}{2^n}T] \equiv \mathcal{I}_j^n$$

- The $\mathcal{T}_{n+1}$-based interpolation is

$$\xi_{n+1}(t) = \xi\left(\frac{j}{2^n}T\right) + \left(t - \frac{j\,T}{2^n}\right)\frac{\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^{n+1}}} \qquad \& \qquad t \in [\frac{j\,T}{2^n}\,, \frac{(2\,j+1)\,T}{2^{n+1}}] \equiv \mathcal{I}_{2\,j}^{n+1}$$

so that $t \in \mathcal{T}_n$ we have

$$\xi(t) = \xi_n(t) = \xi_{n+1}(t) \qquad\qquad t \in \mathcal{T}_n$$

and for $t \in \mathcal{I}_j^n \cap \mathcal{I}_{2\,j}^{n+1}$

$$|\xi_{n+1}(t) - \xi_n(t)| = \left(t - \frac{j\,T}{2^n}\right)\left|\frac{\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^{n+1}}} - \frac{\xi\left(\frac{j+1}{2^n}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^n}}\right|$$

which can be rewritten as

$$|\xi_{n+1}(t) - \xi_n(t)| = \frac{2^{n+1}}{T}\left(t - \frac{j\,T}{2^n}\right)\left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \frac{\xi\left(\frac{j\,T}{2^n}\right) + \xi\left(\frac{j+1}{2^n}T\right)}{2}\right|$$

so that by the triangular inequality:

$$\begin{aligned}|\xi_{n+1}(t)-\xi_n(t)| \le\ & \frac{\left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{jT}{2^n}\right)\right|+\left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{j+1}{2^n}T\right)\right|}{2}\\ &\le \left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{(j+1)T}{2^n}\right)\right|\vee\left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{jT}{2^n}\right)\right|\end{aligned}$$

The inequality entails that

$$\sup_{0\le t\le T}|\xi_{n+1}(t)-\xi_n(t)|\le \sup_{0\le j<2^n-1}\left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{(j+1)T}{2^n}\right)\right|\vee\left|\xi\left(\frac{2j+1}{2^{n+1}}T\right)-\xi\left(\frac{jT}{2^n}\right)\right|$$

The following inequalities hold true:

- if the occurrence of the event A implies the occurrence of the event B we must have $A\subseteq B$ and therefore $P(A)\le P(B)$ which translates for us into

$$\begin{aligned}&\mathrm{P}\left(\sup_{0\le t\le 1}|\xi_{n+1}(t)-\xi_n(t)|\ge 2^{-n\gamma}\right)\\ &\quad\le \mathrm{P}\left(\sup_{0\le j\le 2^{n+1}-1}\left|\xi\left(\frac{(j+1)T}{2^{n+1}}\right)-\xi_n\left(\frac{jT}{2^{n+1}}\right)\right|\ge 2^{-n\gamma}\right)\end{aligned}$$

- as $\mathrm{P}(\cup_k A_k)\le\sum_k \mathrm{P}(A_k)$:

$$\begin{aligned}&\mathrm{P}\left(\sup_{0\le j\le 2^{n+1}-1}\left|\xi\left(\frac{(j+1)T}{2^{n+1}}\right)-\xi_n\left(\frac{jT}{2^{n+1}}\right)\right|\ge 2^{-n\gamma}\right)\\ &\quad\le 2^{n+1}\sup_{0\le j\le 2^{n+1}-1}\mathrm{P}\left(\left|\xi\left(\frac{(j+1)T}{2^{n+1}}\right)-\xi_n\left(\frac{jT}{2^{n+1}}\right)\right|\ge 2^{-n\gamma}\right)\end{aligned}$$

- By Čebyšev inequality and using the theorem's hypothesis

$$\begin{aligned}&\mathrm{P}\left(\left|\xi\left(\frac{(j+1)T}{2^{n+1}}\right)-\xi_n\left(\frac{jT}{2^{n+1}}\right)\right|\ge 2^{-n\gamma}\right)\\ &\quad\le\frac{\mathrm{E}|\xi\left(\frac{(j+1)T}{2^{n+1}}\right)-\xi_n\left(\frac{jT}{2^{n+1}}\right)|^\beta}{2^{-n\beta\gamma}}\le 2^{n\beta\gamma}C\left(\frac{T}{2^{n+1}}\right)^{\alpha+1}\end{aligned}$$

Gleaning the above information together we obtain

$$\mathrm{P}\left(\sup_{0\le t\le T}|\xi_n(t)-\xi_{n+1}(t)|\right)\le C\,T^{\alpha+1}\left(\frac{1}{2}\right)^{n(\alpha-\gamma\beta)}$$

If

$$\gamma>\frac{\alpha}{\beta}$$

the inequality entitle us to apply Borel-Cantelli lemma and conclude

$$\sup_{0\le t\le T}|\xi_n(t)-\xi_{n+1}(t)|\to 0\qquad a.s.$$

and consequently

$$\lim_{n\uparrow\infty}\xi_n(t)=\tilde{\xi}(t)$$

The limit $\xi\tilde{(t)}$ will be continuous on $[0,T]$ and will coincide with $\xi(t)$ on $\mathcal{T}$ thereby establishing our result. □

Observation

The set $\mathcal{T}$ is the set of dyadic rationals. An a posteriori verification that such set is indeed countable is that its complementary set $\mathcal{T}^c$ has full Lebesgue measure. Namely for any fixed n the elements of T_n are equally spaced by intervals of size 2^{-n}:

$$|\mathcal{T}_n^c| = 1 \tag{2.1}$$

Passing to the limit $n \uparrow \infty$ does not affect the result thus

$$|\mathcal{T}^c| = 1 \tag{2.2}$$

3 Summary of notions of convergence

There are free notion of convergence

- *Convergence in probability*: $\{\xi_i\}_{i=1}^{\infty}$ converges to ξ in probability if for every positive ϵ

$$\lim_{n\uparrow\infty} \mathrm{P}(|\xi_n - \xi| < \epsilon) = 0$$

- *Mean square convergence*:$\{\xi_i\}_{i=1}^{\infty}$ converges to ξ in mean square if for every positive ϵ

$$\lim_{n\uparrow\infty} \mathrm{E}(\xi_n - \xi)^2 = 0 \qquad \xi_n \overset{n\uparrow\infty}{\to} \xi \quad m.s.$$

- *Almost sure convergence*: $\{\xi_i\}_{i=1}^{\infty}$ converges to ξ almost surely (i.e. $P = 1$) if the event

$$\left\{\xi_n \overset{n\uparrow\infty}{\to} \xi\right\} \equiv \left\{\omega \in \Omega \mid \xi_n(\omega) \overset{n\uparrow\infty}{\to} \xi\right\}$$

  has probability one.

Convergence in probability is the weakest notion

$$\begin{aligned} &\xi_n \overset{m.s.}{\to} \xi \quad \Rightarrow \xi_n \overset{P}{\to} \xi \qquad \text{by Čebyšev} \\ &\xi_n \overset{a.s.}{\to} \xi \quad \Rightarrow \quad \xi_n \overset{P}{\to} \xi \end{aligned}$$

The second implication follows in general from the fact that for telescopic events

$$\mathrm{P}\left(\lim_{n\uparrow\infty} A_n\right) = \lim_{n\uparrow\infty} \mathrm{P}\left(A_n\right) \tag{3.1}$$

and the chain of inequalities

$$\mathrm{P}\left(\limsup_{n\uparrow\infty\, k\geq n} A_k\right) \equiv \mathrm{P}\left(\lim_{n\uparrow\infty} \cup_{k=n}^{\infty} A_k\right) = \lim_{n\uparrow\infty} \mathrm{P}\left(\cup_{k=n}^{\infty} A_k\right) \geq \limsup_{n\uparrow\infty\, k\geq n} \mathrm{P}\left(A_k\right) \tag{3.2}$$

Almost sure convergence *does not* imply mean square convergence:

$$\xi_n \overset{a.s.}{\to} \xi \quad \not\Rightarrow \quad \xi_n \overset{m.s.}{\to} \xi$$

For example let $\{\xi_n\}_{n=0}^{\infty}$ independent uniformly distributed such that

$$\xi_n = \begin{cases} n & \omega \in [0\,, 1/n] \\ 0 & \end{cases}$$

then

$$\xi_n \overset{a.s.}{\to} 0$$

but

$$\mathrm{E}\xi_n^2 = n$$

Conversely, mean square convergence *does not* almost sure convergence:

$$\xi_n \overset{m.s.}{\to} \xi \quad \not\Rightarrow \quad \xi_n \overset{a.s.}{\to} \xi$$

References

[1] L. C. Evans. An Introduction to Stochastic Differential Equations. Lecture notes, UC Berkeley, 2006.