From Bernoulli variables to Random Walk

1 Introduction

A short but nice introduction to σ-algebras and their probabilistic interpretation is given in [2] §1.1 and §1.3 . The solution of the master equation of the random walk is discussed in §3.8.2 of [1].

2 Probability space for iterated coin tossing

2.1 Single coin toss

For a single coin tossing with outcomes *head* H or *tail* T we already seen that

$$\Omega = \{H\,, T\} \qquad \& \qquad \mathcal{F} = \{\emptyset, H, T, \Omega\} \tag{2.1}$$

Note that $\Omega = H \cup T$ i.e. $\{H\,, T\}$ are the atoms of Ω.

2.2 Double coin toss

Let us consider the case of an iterated coin tossing. The space of events is

$$\Omega = \{HH\,, HT\,, TH\,, TT\} \tag{2.2}$$

We can construct several σ-algebras on Ω depending on the probability model we wish to consider. The minimal σ-algebra on Ω is

$$\mathcal{F}_{min} = \sigma_{min}(\Omega) = \{\emptyset, \Omega\} \tag{2.3}$$

If in a double coin tossing we can observe only the outcome of the first tossing the σ-algebra attuned to describe such situation is

$$\mathcal{F}_1 = \sigma_1(\Omega) = \{\emptyset, HH \cup HT, TH \cup TT\,, \Omega\} \tag{2.4}$$

Finally the *maximal* σ-algebra on Ω is

$$\begin{aligned}\mathcal{F}_{max} = \sigma_{max}(\Omega) = \{&\emptyset, HT\,, HT\,, TH\,, TT\,,\\ &HH \cup HT\,, TH \cup TT\,, HH \cup TH\,, HT \cup TT\,, HH \cup TT\,, HT \cup TH\,,\\ &HH \cup HT \cup TH\,, HT \cup TH \cup TT\,, HH \cup HT \cup TT\,, HH \cup TH \cup TT\,, \Omega\}\end{aligned} \tag{2.5}$$

In the maximal σ-algebra we can attribute a probability to events such as

- Probability to get an head in a double coin tossing: $P\,(HH \cup HT \cup TH)$
- Probability to get the same result if we toss the coin twice: $P\,(HH \cup TT)$

and so on.

3 Random walk

Consider a sequence $\{\xi_i\}_{i=1}^{N}$ of i.i.d. *Benoulli variables*:

$$\forall\, i \qquad \xi_i \overset{d}{=} \xi = \{-x\,, x\} \tag{3.1}$$

with

$$\mathrm{P}\left(\xi = x\right) = p \,<\, 1 \tag{3.2}$$

We interpret the sum of this variable as giving the displacement from zero of a random walker

$$\Xi_n = \sum_{i=1}^{n} \xi_i \tag{3.3}$$

and suppose that of the n steps n_r were to the right and n_l to the left:

$$n = n_l + n_r \tag{3.4}$$

The displacement from the origin in units of x is then given by

$$m = n_r - n_l \tag{3.5}$$

We can solve for n_l, n_r and obtain

$$n_r = \frac{n+m}{2} \qquad \& \qquad n_l = \frac{n-m}{2}$$

The probability of an *individual sequence* of samples of Bernoulli variables such that $\Xi_n(\omega) = m\,x$ is

$$\mathrm{P}(\omega) = p^{\frac{n+m}{2}}(1-p)^{\frac{n-m}{2}}$$

In order to evaluate the total probability $P(R_n = m\,x)$ we must *count* all possible sequences of samples such that (3.4), (3.5) are verified. This number is equal to the number of ways we can extract n_r out of n *indistinguishable object* (this means that the extraction order does not matter):

$$C_n^{n_r} = \frac{n!}{n_r!(n-n_r)!} \equiv \frac{n!}{n_r!n_l!}$$

The conclusion is

$$\mathrm{P}(\Xi_n = m\,x) = \frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} p^{\frac{n+m}{2}}(1-p)^{\frac{n-m}{2}} \tag{3.6}$$

Using binomial formula it is straightforward to check the normalization condition for $m = -n, -(n-1), \ldots, n-1, n$.

4 Master equation for Random Walk

In the previous section we derived the probability distribution of the random walk by counting the path leading to the same displacement in a fixed time horizon of n steps. The feasibility of such calculation strictly depends on the specific nature of the process. A more general approach is to derive the evolution law that *locally* i.e. over one time step (3.6) must obey. Suppose we are given the displacement probabilities $\{\mathrm{P}_n(k)\}_{k=-n}^{n}$ at time n (by definition the displacement satisfies $|m| \,\leq\, n$). The probability of finding the walker at time $n+1$ in m comprises two terms i the probability of being at time n in $m-1$ times the probability of making a step *forward*

ii the probability of being at time n in $m+1$ times the probability of making a step *backward*

more explicitly

$$\mathrm{P}_{n+1}(m) = p\,\mathrm{P}_n(m-1) + (1-p)\,\mathrm{P}_n(m+1) \tag{4.1}$$

Note that (4.1) is equvalent to say

$$\mathrm{P}(\Xi_{n+1} = m \,|\, \Xi_n = k) = p\,\delta_{k\,m-1} + (1-p)\,\delta_{k\,m+1} \tag{4.2}$$

Namely let us observe that

$$\mathrm{P}_{n+1}(m) \equiv \mathrm{P}(\Xi_{n+1} = m) = \\ \sum_{k\in\mathbb{Z}} \mathrm{P}(\Xi_{n+1} = m\,, \Xi_n = k) = \sum_{k\in\mathbb{Z}} \frac{\mathrm{P}(\Xi_{n+1} = m\,, \Xi_n = k)}{\mathrm{P}(\Xi_n = k)} \mathrm{P}(\Xi_n = k) \tag{4.3}$$

By definition of conditional probability, we have

$$\mathrm{P}(\Xi_{n+1} = m \,|\, \Xi_n = k) = \frac{\mathrm{P}(\Xi_{n+1} = m\,, \Xi_n = k)}{\mathrm{P}(\Xi_n = k)} \tag{4.4}$$

hence

$$\mathrm{P}_{n+1}(m) = \sum_{k\in\mathbb{Z}} \mathrm{P}(\Xi_{n+1} = m \,|\, \Xi_n = k)\mathrm{P}_n(k) \tag{4.5}$$

The contents of (4.1) is that the evolution of the probability density of the walker displacement is specified by

- the current state of the walker (i.e. "*the present*")
- the *transition probabilities* $\mathrm{P}(\Xi_{n+1} = m \,|\, \Xi_n = k)$ over the time unit.

The simplifying feature of (4.1) is that the transition rates are *spatially homogeneous*. This means that by they do not depend neither on the initial state nor on the final state. This *translational invariance* suggests that the natural tool to solve (4.1) is the Fourier transform. In other words, it is expedient to reformulate (4.1) as an equation for the characteristic function of the walker position process:

$$\mathrm{G}_n(t) := \mathrm{E}\, e^{\imath\, t\, \Xi_n} = \sum_{k=-n}^{n} e^{\imath\, t\, k\, x} \mathrm{P}_n(k) \tag{4.6}$$

As (4.6) involves only a finite sum, we have

$$\mathrm{G}_n(t) = \mathrm{G}_n\left(t + \frac{2\,\pi}{x}\right) \tag{4.7}$$

so that the Fourier anti-transform is specified by the integral

$$\mathrm{P}_n(m) = \int_0^{\frac{2\,\pi}{x}} \frac{dt}{2\,\pi}\, e^{-\imath\, t\, m\, x}\, \mathrm{G}_n(t) = \\ \sum_{k=-n}^{n} \mathrm{P}_n(k) \int_0^{\frac{2\,\pi}{x}} \frac{dt}{2\,\pi}\, e^{-\imath\, t\, x\,(m-k)} = \sum_{k=-n}^{n} \mathrm{P}_n(k) \delta_{m\,k} \tag{4.8}$$

Taking the Fourier transform of both sides of (4.1) yields

$$\mathrm{G}_{n+1}(t) = [p\, e^{\imath\, t\, x} + (1-p)\, e^{-\imath\, t\, x}]\, \mathrm{G}_n(t) \tag{4.9}$$

For every fixed t, (4.9) defines a linear map in n which we can solve by iteration

$$\mathrm{G}_n(t) \;=\; [p\, e^{\imath\, t\, x} + (1-p)\, e^{-\imath\, t\, x}]^n\, \mathrm{G}_0(t) \tag{4.10}$$

If we posit that the walker starts her path from the origin at time $n = 0$ then

$$\mathrm{P}_0(k) = \delta_{0\,k} \qquad \Rightarrow \qquad \mathrm{G}_0(t) = 1 \tag{4.11}$$

Thus the Fourier transform of the solution of (4.1) is

$$\mathrm{G}_n(t) \;=\; [p\, e^{\imath\, t\, x} + (1-p)\, e^{-\imath\, t\, x}]^n \tag{4.12}$$

From this latter expression we recover (3.6) using the binomial formula in the Fourier anti-transform. Namely, a straightforward calculation yields

$$\mathrm{P}_n(m) = \int_0^{\frac{2\,\pi}{x}} \frac{dt}{2\,\pi}\, e^{-\imath\, t\, m\, x}\, \mathrm{G}_n(t) = \\ \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^k \int_0^{\frac{2\,\pi}{x}} \frac{dt}{2\,\pi}\, e^{-\imath\, t\, x\, m} e^{\imath\, t\, x\, k} e^{-\imath\, t\, x\,(n-k)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^k \delta_{k\, \frac{n+m}{2}} \tag{4.13}$$

whence the claim is proved.

Appendix

A Summary of limit theorems for Bernoulli variables

The characteristic function of a Bernoulli random variable

$$\xi \colon \Omega \mapsto \{-x, x\} \qquad \& \qquad \mathrm{P}(\xi = x) = p \tag{A.1}$$

is

$$\mathrm{E}\, e^{\imath t \xi} = \cos(xt) + \imath\, (2\, p - 1)\, \sin(xt)$$

Differentiating the characteristic function for $t = 0$ we obtain all the positive moments

$$\mathrm{E}\, \xi^n = \begin{cases} x^n & \text{if } n = 2\,k \\ x^n\,(2\,p-1) & \text{if } n = 2\,k+1 \end{cases}$$

In particular

$$\mathrm{E}\, \xi = (2\,p - 1)\, x \qquad \& \qquad \mathrm{E}\,(\xi - \mathrm{E}\,\xi)^2 = 4\,p\,(1-p)\,x^2$$

and

$$\mathrm{E}(\xi - \mathrm{E}\,\xi)^4 = 16\,p\;(1-p)\;[1 - 3\;(1-p)\;p]\;x^4$$

If we introduce the random variable

$$S_n := \frac{\sum_{i=1}^{n} \xi_i}{n}$$

then

$$\mathrm{E}\, S_n = \mathrm{E}\,\xi \qquad \& \qquad \mathrm{E}(S_n - \mathrm{E}\,\xi)^2 = \frac{4\,p\,(1-p)\,x^2}{n}$$

- By Čebyšev theorem we have immediately

$$P(|S_n - \mathrm{E}\,\xi| \,\geq\, \varepsilon) \leq \frac{\mathrm{E}\,(S_n - \mathrm{E}\,\xi)^2}{\varepsilon^2} = \frac{4\,p\,(1-p)\,x^2}{n\,\varepsilon^2} \overset{n\uparrow\infty}{\to} 0 \tag{A.2}$$

This is *law of large numbers* for i.i.d. Bernoulli variables.

- Upon setting

$$\tilde{\xi}_i = \xi_i - \mathrm{E}\,\xi \tag{A.3}$$

we can write

$$\begin{aligned}\mathrm{E}\,(S_n - \mathrm{E}\,\xi)^4 &= \sum_{i=1} \frac{\mathrm{E}\,\tilde{\xi}_i^4}{n^4} + 3\sum_{ijkl} \delta_{ij}\delta_{kl}(1-\delta_{jk})\frac{(\mathrm{E}\,\tilde{\xi}_i^2)^2}{n^4} \\ &= \frac{16\,p\,(1-p)\,x^2}{n^3}\,[1 - 3\,\,(1-p)\,\,p] + 3\,n\,(n-1)\frac{16\,p^2\,(1-p)^2}{n^4} \overset{n\gg 1}{\leq} \frac{C\,x^2}{n^2}\end{aligned}$$

We obtain therefore the estimate

$$P(|S_n - \prec \xi\,| \,\geq\, \varepsilon) \leq \frac{\mathrm{E}\,(S_n - \mathrm{E}\,\xi\,)^4}{\varepsilon^4} \leq \frac{C\,x^4}{n^2\,\varepsilon^4} \overset{n\uparrow\infty}{\to} 0$$

As in the general case, this last upper bound bound tends to zero sufficiently fast at infinity to permit the application of the the Borel-Cantelli lemma. We can therefore prove that

$$S_n \overset{n\uparrow\infty}{\to} \mathrm{E}\,\xi \qquad a.s.$$

i.e. the validity of the *strong law of large numbers* for Bernoulli schemes.

- Consider now the characteristic function

$$\mathrm{E}\,e^{\imath S_n\,t} = \mathrm{E}\,e^{\imath \xi\,t\,n} = \left[\cos\left(\frac{x\,t}{n}\right) + \imath\,(2\,p-1)\,\sin\left(\frac{x\,t}{n}\right)\right]^n$$

If we hold t fixed and take $n \gg 1$ we can expand the right hand side in Taylor series

$$\ln \mathrm{E}\,e^{\imath S_n\,t} = \imath\,(2\,p-1)\,x\,t - \frac{x^2\,t^2}{2}\frac{4\,p\,(p-1)}{n} + o\left(\frac{1}{n}\right)$$

We can couch the result into the form

$$\ln \mathrm{E}\,e^{\imath S_n\,t} \overset{n\gg 1}{\cong} \imath\,\mathrm{E}\,\xi\;t - \frac{t^2}{2}\frac{\mathrm{E}\,(\xi - \mathrm{E}\,\xi\,)^2}{n} + o\left(\frac{1}{n}\right) \tag{A.4}$$

which also suggests

$$\mathrm{E}\,e^{\imath S_n\,t} = e^{\imath\,\mathrm{E}\,\xi\;t - \frac{t^2}{2}\frac{\mathrm{E}\,(\xi-\mathrm{E}\,\xi\,)^2}{n}} + o\left(\frac{1}{n}\right) \tag{A.5}$$

Hence we recovered the content of the central limit theorem in the case of Bernoulli variables:

$$\mathrm{P}_{S_n}(x) \overset{n\gg 1}{\approx} \sqrt{\frac{n}{2\,\pi\,\mathrm{E}\,(\xi - \mathrm{E}\,\xi\,)^2}}\,e^{-\frac{n\,(x-\mathrm{E}\,\xi)^2}{2\,\mathrm{E}\,(\xi-\mathrm{E}\,\xi)^2}} \tag{A.6}$$

References

[1] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*, volume 13 of *Springer series in synergetics*. Springer, 2 edition, 1994.

[2] R. van Handel. Stochastic calculus and stochastic control. Lecture Notes, Caltech, 2007.