7^{th} Sheet of Exercise

18^{th} April 2012

Exercises.

1. Let $A \in L^{2m}_{\mathrm{cl}}(X)$ be properly supported, $m \geq 0$. Assume that the principal symbol $a_{2m}(x, \xi)$ satisfies

   $$\operatorname{Re} a_{2m} \geq \frac{1}{C_0}|\xi|^{2m}, \quad C_0 > 0.$$

   Let $K \subset\subset X$ and $u \in C_0^\infty(X) \cap \mathcal{E}'(K)$. Show that $\operatorname{Re}(Au|u) = (Bu|u)$ where $B = 1/2(A + A^*)$.

2. Show that there exists $C \in L^m_{\mathrm{cl}}$ elliptic and $R \in L^{-\infty}$ such that $B = C^*C + R$.

3. Show that there exists a constant $C(K)$ such that for every $u \in C_0^\infty(X) \cap \mathcal{E}(K)$

   $$\operatorname{Re}(Au|u) \geq \frac{1}{C(K)} \|u\|^2_{H^m} - C(K) \|u\|^2_{H^0}.$$

4. Let $v = \sum_1^n a_j(x)\partial_{x_j}$ be a C^∞ vector field, defined in a neighbourhood of $0 \in \mathbb{R}^n$, and suppose $v(0) = 0$. We introduce the linearized vector field of v at 0 as $\sum\sum \partial_{x_k} a_j(0) x_k \partial_{x_j}$, and we consider $A = (\partial_{x_k} a_j(0))$ as an element of $\operatorname{End}(T_0\mathbb{R}^n)$ and let $\sigma(A) \subset \mathbb{C}$ be the set of eigenvalues of A.

   If $\lambda \in \sigma(A)$ implies $\operatorname{Re}\lambda < 0$, show that there exists $C > 0$ such that if $|x| \leq 1/C$ and $t \geq 0$ then $|\exp(tv)(x)| \leq Ce^{-t/C}|x|$.

5. Suppose that $n = 2$ and that A has one strictly positive eigenvalue and one strictly negative eigenvalue. Study the shape of the integral curves near $(0, 0)$.

6. Also with $n = 2$, study the integral curves of $x_1\partial_{x_2} - x_2\partial_{x_1} + w$ in the following case:

   - $w = 0$.
   - $w = -|x|^2(x_1\partial_{x_1} + x_2\partial_{x_2})$.
   - $w = |x|^2(x_1\partial_{x_1} + x_2\partial_{x_2})$.

Comments.

1. The result of Exercise 3 is called the Gårding inequality.