$3^{\rm rd}$ Sheet of Exercise

$22^{\rm nd}$ February 2012

Notation. Along this sheet, we will follow the following notation. If X is an open subset of $\mathbb{R}^m$ with m a positive integer, then $C_0^\infty(X)$ is the space of smooth functions with compact support in X. Finally, $\mathcal{S}(\mathbb{R}^m)$ denotes the space of rapidly decreasing smooth functions.

Exercises.

1. The following problem appears in the method of steepest descent. Show that for $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda \geq 1$:

$$\int_{\mathbb{R}^n} e^{-\lambda x^2/2} u(x)\,\mathrm{d}x = \sum_{k=0}^{N-1} \frac{(2\pi)^{n/2}}{k!\lambda^{k+n/2}} \frac{1}{2^k} (\Delta^k u)(0) + S_N(u,\lambda),$$

   where

$$|S_N(u,\lambda)| \leq C_{n,N} \lambda^{-N-n/2} \sum_{|\alpha|=2N} \sup |\partial^\alpha u(x)|.$$

2. Set

$$F(\lambda) = \int_0^{+\infty} e^{-t}\, t^\lambda \,\mathrm{d}t$$

   with $\lambda \geq 1$.

   (a) Rewrite the integral by means of the change of variable $t = \lambda(1+s)$.

   (b) Use Exercise 1 and show

$$F(\lambda) = \left(\frac{\lambda}{e}\right)^\lambda \sqrt{2\pi\lambda}\left(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \dots\right).$$

   For $\lambda \in \mathbb{N}$, deduce Stirling's formula.

3. Calculate a_1 and a_2 from the Exercise 2.

4. Consider $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $u \in \mathcal{S}(\mathbb{R}^n)$. Set

$$Op(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, \mathrm{d}y\, \mathrm{d}\theta.$$

   Show that

$$\begin{aligned} [D_{x_j}, Op(a)] &= Op(D_{x_j} a) \\ [x_j, Op(a)] &= Op(-D_{\theta_j} a) \end{aligned}$$

   where in general we let $[M, N] = MN - NM$ denote the commutator of the operators M and N.

5. Set $S^0_{0,0} = \{a \in C^\infty(\mathbb{R}^{2n}) : \partial_x^\alpha \partial_\theta^\beta a \in L^\infty(\mathbb{R}^{2n}) \, \forall (\alpha, \beta) \in \mathbb{N}^{2n}\}$. Show that for $a \in \mathcal{S}(\mathbb{R}^{2n})$, $Op(a) : \mathcal{S}(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n)$ is continuous and the continuity is uniform for $a \in \mathcal{S}(\mathbb{R}^{2n}) \cap B$ if B is a bounded set in $S^0_{0,0}$. Here $Op(a)$ is as in Exercise 4.

6. Show the same result as in Exercise 5 for $x^\alpha \partial_x^\beta Op(a)$ and show that $Op(a) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is uniformly continuous for $a \in \mathcal{S}(\mathbb{R}^{2n}) \cap B$. Here B is as in Exercise 5.

Comments.

(i) Exercise 2 and Exercise 3 are proposed to find the asymptotic behaviour of $F(\lambda)$ as λ goes to $+\infty$. Note that $F(\lambda) = \Gamma(\lambda + 1)$.

(ii) Exercise 4, Exercise 5 and Exercise 6 correspond to Weyl quantization. We will keep on with in the next session(23rd February 2012).