2$^{\text{nd}}$ Sheet of Exercise

15$^{\text{th}}$ February 2012

Notation. Along this sheet, we will follow the following notation. If X is an open subset of $\mathbb{R}^m$ with m a positive integer, then $C^\infty(X)$ denotes the space of smooth functions in X and $C^\infty(X;\mathbb{R})$ denotes the space of real-valued smooth functions in X. Additionally, $C_0^\infty(X)$ is the subspace of $C^\infty(X)$ such that its elements have compact support in X. If M is an arbitrary subset of $\mathbb{R}^m$, $C_0^\infty(M)$ is the subspace of elements $C_0^\infty(\mathbb{R}^m)$ such that its elements have compact support in M. We also use the notations $\mathcal{D}'(X)$ for the space of distributions in X and $\mathcal{E}'(X)$ for the subspace of $\mathcal{D}'(X)$ such that its elements have compact support in X. Finally, $\mathcal{S}(\mathbb{R}^m)$ denotes the space of rapidly decreasing smooth functions.

Exercises.

1. Let X_1 and X_2 be two open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Consider a continuous linear map $\mathcal{K}$ from $C_0^\infty(X_2)$ to $\mathcal{D}'(X_1)$. Prove that for any compact set $K_j \subset X_j$ the bilinear map

$$(\psi, \phi) \in C_0^\infty(K_1) \times C_0^\infty(K_2) \longmapsto \langle \mathcal{K}\phi, \psi \rangle$$

   is continuous.

2. Let ψ_j belong to $C_0^\infty(\mathbb{R}^{n_j})$ with $\psi_j \geq 0$, $\operatorname{supp}\psi_j \subset \{x_j \in \mathbb{R}^{n_j} : |x_j| \leq 1\}$ and $\int_{\mathbb{R}^{n_j}} \psi_j \, \mathrm{d}x_j = 1$. Let Y_j be a open subset in $\mathbb{R}^{n_j}$ compactly contained in X_j and set, for any $(x_1, x_2) \in Y_1 \times Y_2$ and $\varepsilon > 0$ small enough,

$$K_\varepsilon(x_1, x_2) = \varepsilon^{-n_1-n_2} \langle \mathcal{K}\psi_2((x_2 - \bullet)/\varepsilon), \psi_1((x_1 - \bullet)/\varepsilon) \rangle.$$

   Prove that:

   (a) There exists a positive integer μ such that $|K_\varepsilon(x_1, x_2)| \leq C\varepsilon^{-\mu}$ if $x_j \in Y_j$.

(b) For a fixed small δ and ε going to 0

$$K_\varepsilon = \sum_{j=0}^{\mu} (\varepsilon - \delta)^j K_\delta^{(j)}/j! + (\varepsilon - \delta)^{\mu+1} \int_0^1 K_{\delta+t(\varepsilon-\delta)}^{(\mu+1)} (1-t)^\mu/\mu! \, \mathrm{d}t.$$

3. Show that there exists $K_0 \in \mathcal{D}'(Y_1 \times Y_2)$ such that K_ε converges to K_0 in $\mathcal{D}'(Y_1 \times Y_2)$.

4. Prove that K_0 (from Exercise 3) satisfies

$$K_0(\psi \otimes \phi) = \langle \mathcal{K}\phi, \psi \rangle \qquad \psi \in C_0^\infty(X_1),\ \phi \in C_0^\infty(X_2).$$

5. (a) Do the following functions converge in $\mathcal{D}'(\mathbb{R})$ as λ goes to $+\infty$?

$$u_\lambda(x) = \lambda^N e^{-i\lambda x}, \quad v_\lambda(x) = \lambda^{1/2} e^{-i\lambda x^2/2}, \quad w_\lambda(x) = \lambda^{1/2} e^{i\lambda x^2/2}.$$

(b) Let f belong to $C^\infty(\mathbb{R}; \mathbb{R})$ with $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Then answer the same question as in (a) for

$$u_\lambda(x) = \lambda^N e^{-i\lambda f(x)}, \qquad v_\lambda(x) = \lambda^{1/2} e^{-i\lambda (f(x))^2/2}.$$

6. Let χ belong to $\mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$.

(a) Show the existence of a limit as ε goes to 0 of

$$\int_{\mathbb{R}} e^{-i\lambda(y+y^3/3)} \chi(\varepsilon y) \, \mathrm{d}y, \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

Show that the limit $I = I(\lambda)$ belongs to $C^\infty(\mathbb{R} \setminus \{0\})$ (as a function of λ).
Hint: chose a suitable differential operator L such that $L(e^{-i\lambda(y+y^3/3)}) = e^{-i\lambda(y+y^3/3)}$.

(b) Show that for every N, there exists a constant $C_N > 0$ such that $|I(\lambda)| \leq C_N |\lambda|^{-N}$ if $|\lambda| > 1$.

Comments.

(i) Exercises from 1 to 4 complete the proof for the Schwartz Kernel Theorem.