Distributed delays

- $n_\varphi(t) := \int_0^\infty \varphi(\tau) n(t-\tau) d\tau$

  is a weighted average of past population densities.

- $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\int_0^\infty \varphi(\tau) d\tau = 1$

  is a probability density of delay times $\tau > 0$.

- Note that models with a fixed delay $T > 0$ are recovered by puting $\varphi(\tau) = \delta(\tau - T)$, where δ is the Dirac delta distribution.

- Consider the DDE

(*) $\frac{dn}{dt} = f(n, n_T)$

for some cont. diff. $f : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$.
Suppose $\bar{n}$ is an equilibrium, i.e,

$0 = f(\bar{n}, \bar{n})$.

Linearization of (*) near $\bar{n}$ gives

(**) $$\frac{du}{dt} = au + bu_\varphi$$

where $u := n - \bar{n}$, $u_\varphi := n_\varphi - \bar{n}$,
$a := \partial_1 f(\bar{n},\bar{n})$ and $b := \partial_2 f(\bar{n},\bar{n})$.

Put $u(t) = e^{\lambda t}$ to get the characteristic equation:

$$\lambda = a + b\, M_\varphi(\lambda)$$

where

$$M_\varphi(\lambda) := \int_0^\infty \varphi(\tau) e^{-\lambda\tau}\, d\tau$$

is the moment generating function of φ

Examples

(a) $\varphi(\tau) = \delta(\tau - T) \Rightarrow M_\varphi(\lambda) = e^{-\lambda T}$

(b) $\varphi(\tau) = \alpha e^{-\alpha\tau} \Rightarrow M_\varphi(\lambda) = \dfrac{\alpha}{\alpha+\lambda}$

(c) $\varphi(\tau) = \begin{cases} \dfrac{1}{T_1 - T_2} & \text{if } \tau \in [T_2, T_1] \\ 0 & \text{elsewhere} \end{cases} \Rightarrow M_\varphi(\lambda) = \dfrac{e^{-\lambda T_2} - e^{-\lambda T_1}}{(T_1 - T_2)\lambda}$

etc.

Stability boundaries in the (a, b)-plane: (i.e, $\underline{\mathrm{Re}\,\lambda = 0}$)

Real eigenvalues

Put $\boxed{\lambda = 0}$ in the characteristic equation $\Rightarrow$ $\boxed{0 = a + b}$

Complex eigenvalues

Put $\boxed{\lambda = i\omega}$ in the characteristic equation and separate real and imaginary parts $\Rightarrow$

$$\begin{cases} 0 = a + b \int_0^\infty \varphi(z) \cos(\omega z)\, dz \\ \omega = a - b \int_0^\infty \varphi(z) \sin(\omega z)\, dz \end{cases}$$

Solving for a and b gives:

$$(***) \quad \begin{cases} a = \omega \cdot \dfrac{\overline{\cos(\omega T)}}{\overline{\sin(\omega T)}} \\ b = -\omega \cdot \dfrac{1}{\overline{\sin(\omega T)}} \end{cases}$$

This is a parameterized curve in the (a,b)-plane, parameterized by $\omega \in \mathbb{R}$.

where

$$\begin{cases} \overline{\cos(\omega z)} := \int_0^\infty \varphi(z) \cos(\omega z)\, dz \\ \overline{\sin(\omega z)} := \int_0^\infty \varphi(z) \sin(\omega z)\, dz. \end{cases}$$

(✱✱) describes a curve in the (a,b)-plane, parameterized by $\omega \in \mathbb{R}$, and is easy to plot numerically whenever φ is given.

Examples.

(i) $\boxed{\varphi(\tau) = \delta(\tau - T)}$ (Dirac-delta)

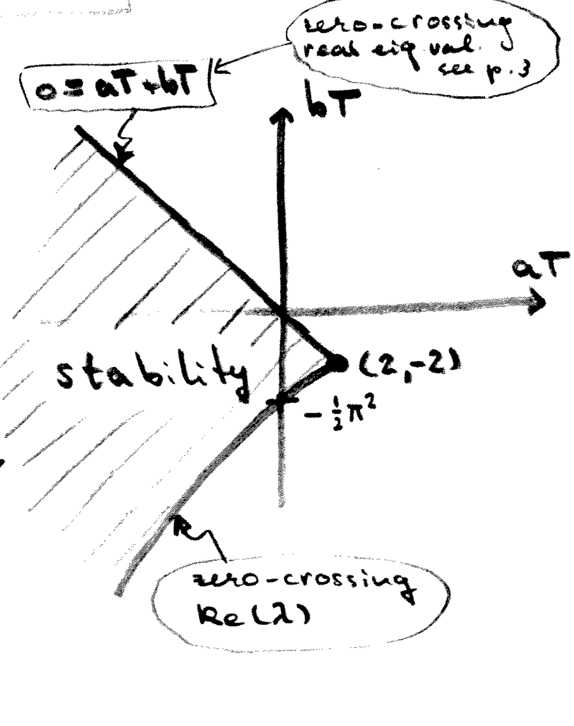
$$\Rightarrow \begin{cases} aT = \omega T \operatorname{cotan}(\omega T) \\ bT = \dfrac{-\omega T}{\sin(\omega T)} \end{cases}$$

cf. prev. lecture on the system $\frac{du}{dt} = f(u, u_T)$

which is what we already used in the previous lecture.

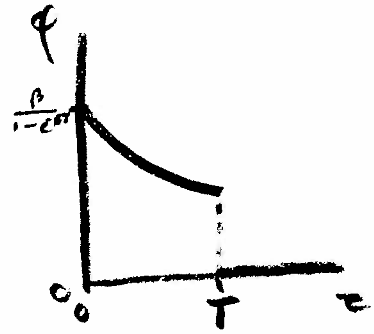
(ii) $\boxed{\varphi(\tau) = \begin{cases} \frac{1}{T} & \text{if } \tau \in [0,T] \\ 0 & \text{elsewhere} \end{cases}}$ (Uniform on [0,T])

$$\begin{cases} aT = \omega T \cdot \dfrac{\sin(\omega T)}{1 - \cos(\omega T)} \\ bT = \dfrac{-(\omega T)^2}{1 - \cos(\omega T)} \end{cases}$$



(iii) $$\boxed{\varphi(z) = \begin{cases} \dfrac{\beta e^{-\beta z}}{1 - e^{-\beta T}} & \text{for } z \in [0,T] \\ 0 & \text{elsewhere} \end{cases}}$$

This is the truncated exponential distribution



Where does that come from?

Suppose maturation age is uniformly distributed on the interval $[0,T]$ given that the juvenile does not die.

Suppose further that there is a constant death rate of juveniles.

Then the number of juveniles per unit of time maturing at time t is

$$\int_0^T \alpha\, n(t-z) \cdot \psi(z) \cdot e^{-\beta z}\, dz$$

→

where $\alpha n(t-\tau)$ is the birth rate at time $t-\tau$, and $\psi(\tau)$ the probability density of maturation time τ, and $e^{-\beta\tau}$ the probability of surviving till maturation.

If maturation time is uniformly distributed on $[0,T]$, then $\psi(\tau) = \begin{cases} \frac{1}{T} & \text{if } \tau \in [0,T] \\ 0 & \text{elsewhere.} \end{cases}$

Hence, the number of juveniles recruited into the population of adults at time t is

$$\int_0^T \alpha n(t-\tau)\psi(\tau)e^{-\beta\tau}d\tau =$$

$$= \frac{\alpha}{T}(1-e^{-\beta T})\int_0^T \varphi(\tau)n(t-\tau)d\tau$$

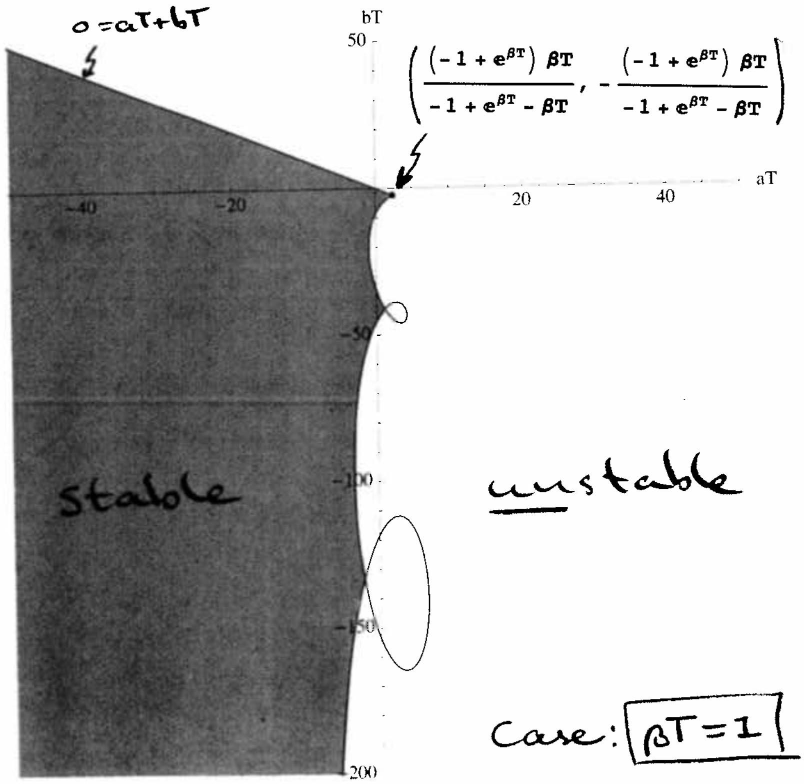
where φ is the truncated exponential distribution as given on the previous page.

Then from (⁂) on page 3 we get

$$\left\{\begin{aligned} aT &= \frac{\omega T\left(e^{\beta T}\beta T - \beta T \operatorname{Cos}[\omega T] + \omega T \operatorname{Sin}[\omega T]\right)}{-e^{\beta T}\omega T + \omega T \operatorname{Cos}[\omega T] + \beta T \operatorname{Sin}[\omega T]} \\ bT &= \frac{\left(-1 + e^{\beta T}\right)\omega T\left(\beta T^2 + \omega T^2\right)}{\beta T\left(-e^{\beta T}\omega T + \omega T \operatorname{Cos}[\omega T] + \beta T \operatorname{Sin}[\omega T]\right)} \end{aligned}\right.$$

parameterized curve in the (aT, bT)-plane.

for the zero transition of Re(λ)



Different kinds of delays in one model.

i-states: N : adult

E_a : egg of age $a \in [0, T]$

i-processes:

$N \xrightarrow{\alpha} N + E_0$ (reproduction)

$E_a \xrightarrow{\beta} \dagger$ (death of egg)

$N + E_a \xrightarrow{\gamma} N$ (egg eaten by adult)

$N \xrightarrow{\delta} \dagger$ (adult mortality)

$E_a \xmapsto{T} N$ (maturation after fixed time T)

p-level:

- The rate at which eggs were being produced at time $t-T$ is

$$\boxed{\alpha\, n(t-T)}$$

- The probability that an egg that was produced at time $t-T$ is still alife at time t is

$$\boxed{\underbrace{e^{-\beta T}}_{\text{survival dens-indep. hazards}} \cdot \underbrace{e^{-\gamma \int_0^T n(t-a)\,da}}_{\text{escape from being eaten by adults}}}$$

⇒ Population equation:

$$\frac{dn(t)}{dt} = \alpha n(t-T) \cdot e^{-\beta T} \cdot e^{-\gamma \int_0^T n(t-a)\,da} - \delta n(t)$$

Define:

$$\varphi(a) := \begin{cases} \frac{1}{T} & \text{for } a \in [0,T] \\ 0 & \text{elsewhere} \end{cases} \quad \left| \left(\begin{array}{c}\text{uniform} \\ \text{distr.}\end{array}\right)\right.$$

Then the pop. equ. becomes

$$\frac{dn}{dt} = \alpha n_T e^{-\beta T - \gamma T n_\varphi} - \delta n$$

which is of the form

$$\frac{dn}{dt} = f(n, n_T, n_\varphi)$$

i.e., with two types of delay, one fixed (n_T) and one distributed (n_φ)

Non-trivial equilibrium:

$$\bar{u} = \frac{1}{\gamma T}\left(\log\frac{\alpha}{\delta} - \beta T\right) > 0$$

Linearization about the equilibrium $\bar{u}$ gives

$$\frac{du}{dt} = -\delta u + \delta u_T - \delta\left(\log\frac{\alpha}{\delta} - \beta T\right) u_\varphi$$

The corresponding characteristic equation is

$$\boxed{\lambda T = \left(aT \cdot \lambda T + bT\right) \cdot \frac{1 - e^{-\lambda T}}{\lambda T}}$$

where

$$\begin{cases} a := -\delta < 0 \\ b := -\delta\left(\log\frac{\alpha}{\delta} - \beta T\right) < 0 \end{cases}$$

- To find zero-crossings of real eigenvalues, let $|\lambda \longrightarrow 0|$:

$$\Rightarrow \boxed{0 = bT}$$

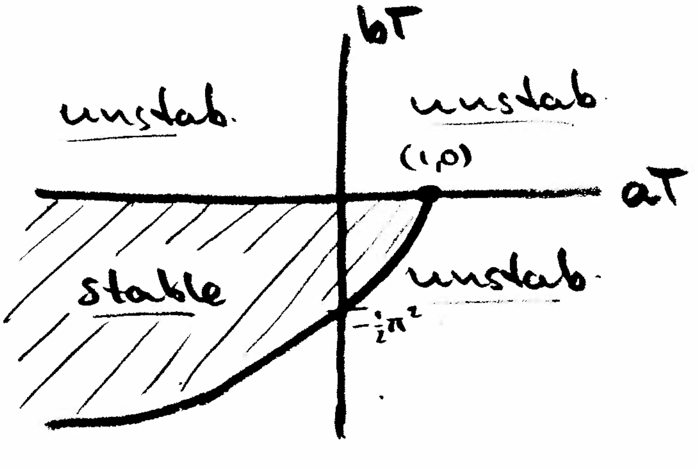
- To find zero-crossings of $\mathrm{Re}(\lambda)$ for complex λ, put $\boxed{\lambda = i\omega \neq 0}$, and separate the real and complex parts of the characteristic equation:

$$\begin{cases} 0 = aT\,(1-\cos\omega T) + bT \cdot \dfrac{\sin\omega T}{\omega T} \\[2ex] bT = aT \sin\omega T - bT\, \dfrac{1-\cos\omega T}{\omega T} \end{cases}$$

Solving for aT and bT gives

$$\begin{cases} aT = \frac{1}{2}\,\omega T \cdot \dfrac{\cos\omega T}{\sin\omega T} \\[2ex] bT = -\frac{1}{2}\,(\omega T)^2 \end{cases}$$

which describes a curve in the (aT, bT)-plane, parameterized by ωT.



Remember that only $aT<0$ and $bT<0$ are realizible in this model (see page 10).