Travelling wave in the SIS model

S : susceptible individual
I : infected individual.

$$S + I \xrightarrow{\alpha} 2I \quad \text{(infection)}$$

$$I \xrightarrow{\beta} S \quad \text{(recovery)}$$

Pop. equations

$$\begin{cases} \partial_t S = -\alpha S I + \beta I + D\,\partial_x^2 S \\ \partial_t I = +\alpha S I - \beta I + D\,\partial_x^2 I \end{cases}$$

$N := S + I$ (total pop. density)

$$\Rightarrow \begin{cases} \partial_t N = D\,\partial_x^2 N \\ \partial_t I = \alpha (N - I) I - \beta I + D\,\partial_x^2 I \end{cases}$$

Notice:

- Equation for N is indep. of I.
- Take $N(x,t) = K > 0 \;\; \forall x,t$ as solution.

Substitute $N = K$ into equation for I:

(*) $$\boxed{\partial_t I = \alpha (K - I) I - \beta I + D \partial_x^2 I}$$

with boundary conditions

$\boxed{I(-\infty, t) = I_0}$, $\boxed{I(+\infty, t) = 0}$ $\forall t$

for given $0 < I_0 \leq K$.

Try travelling wave solution

$I(x,t) = I(x - \underbrace{v}_{\text{wave speed}} t)$

Substitution into (*) gives

(**) $$\boxed{-v I' = \alpha (K - \tfrac{\beta}{\alpha}) I - \alpha I^2 + D I''}$$

with bnd. cond $\boxed{I(-\infty) = I_0}$, $\boxed{I(+\infty) = 0}$

Assume that

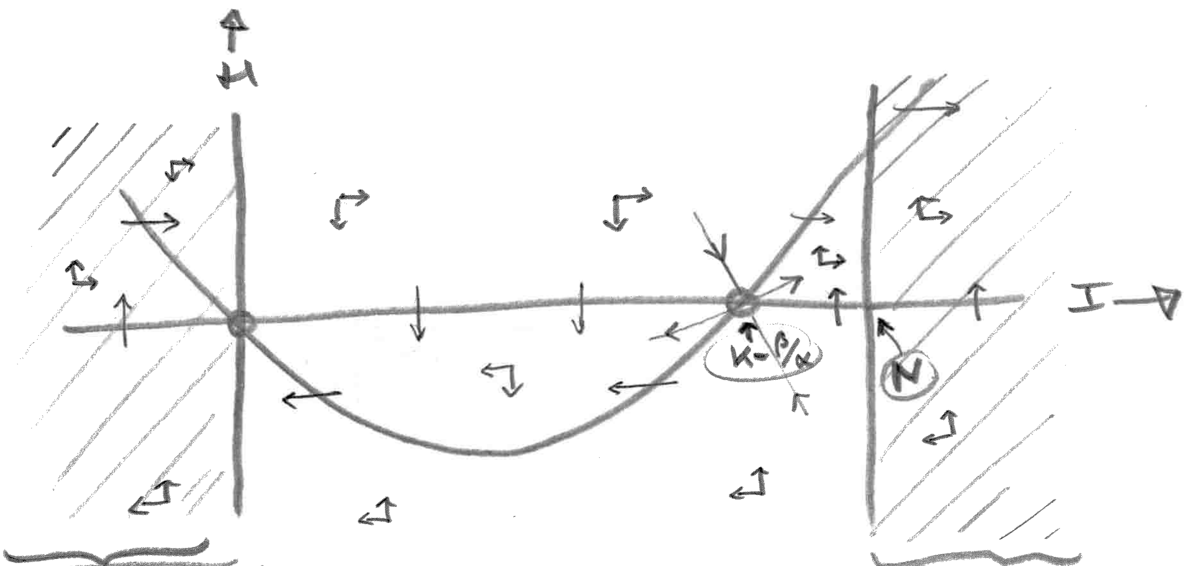
$\boxed{K > \frac{\beta}{\alpha}}$ (i.e., in a spatially homogeneous situation the epidemic would spread rather than die out)

Phase-plane analysis of (**)

$$(***)\begin{cases} I' = H \quad \leftarrow \text{help variable} \\ H' = \frac{\alpha}{D} I\left(I-(K-\frac{\beta}{\alpha})\right) - \frac{\nu}{D}\cdot H \end{cases}$$

I-isocline: $I' = 0 \iff H = 0$.

H-isocline: $H' = 0 \iff H = \frac{\alpha}{\nu} I\left(I-(K-\frac{\beta}{\alpha})\right)$.



no interpretation ($I<0$)

no interpretation ($I>K$)

<u>Equilibria:</u> $(I,H) = (K-\beta/\alpha, 0)$
is obviously a saddle.

$(I,H) = (0,0)$
not clear.
Local stab. anal. needed.

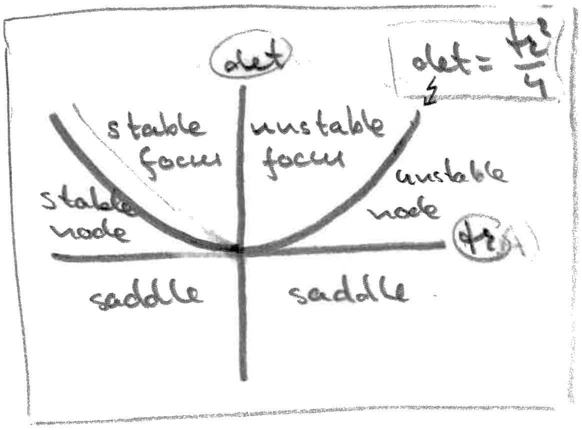
Linearization of (***) about the equilibrium (0,0):

$$\begin{pmatrix} I \\ H \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{\alpha}{D}\left(k-\frac{\beta}{\alpha}\right) & -\frac{v}{D} \end{pmatrix}}_{\text{matrix } A} \begin{pmatrix} I \\ H \end{pmatrix}.$$

<u>Stability</u>:

$$\begin{cases} \det(A) = \frac{\alpha}{D}\left(k-\frac{\beta}{\alpha}\right) > 0 \\ tr(A) = -\frac{v}{D} < 0 \end{cases}$$

$\Rightarrow$ (0,0) is <u>stable</u>.



<u>Node or focus</u>:

$$\text{node} \Leftrightarrow \det \leq \frac{tr^2}{4} \Leftrightarrow \boxed{|v| \geq 2\sqrt{\alpha D\left(k-\beta/\alpha\right)}} \quad (***)$$

<u>What is the significance of this?</u>

- Orbits in the (I,H) plane give info about the profile of the travelling wave

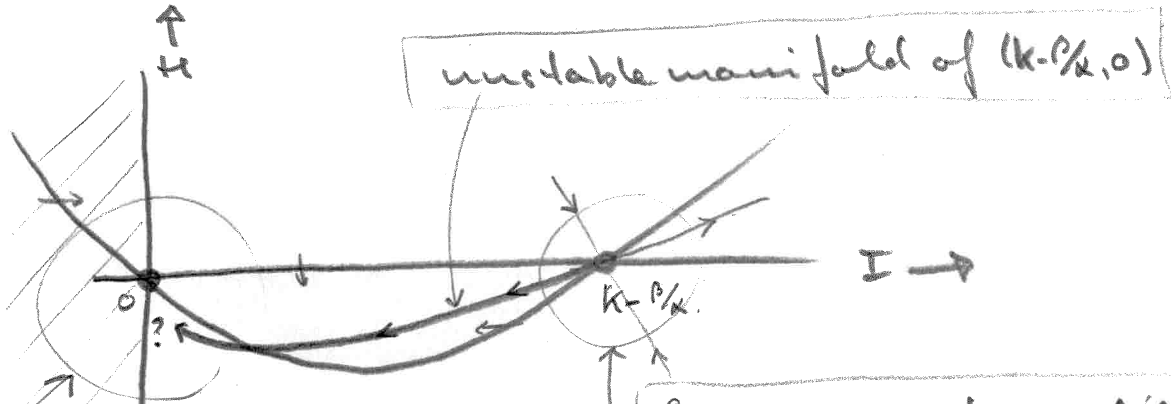
(fixed shape, translated in time).

- Not every orbit is interpretable as a travelling wave. To begin with, only orbits on which I stays in [0, k] are interpretable. →

Moreover, only orbits that satisfy the boundary conditions $I(-\infty) = I_0 \in (0, k]$ and $I(+\infty) = 0$ are valid solutions.

(~~E.g.~~ the equilibria $(0,0)$ and $(k - \beta/\alpha, 0)$ satisfy $I \in [0, k]$ for all x, but they do not satisfy both boundary conditions simultaneously.)

The only candidate for a travelling wave solution is one of the unstable manifolds of $(k - \beta/\alpha, 0)$:



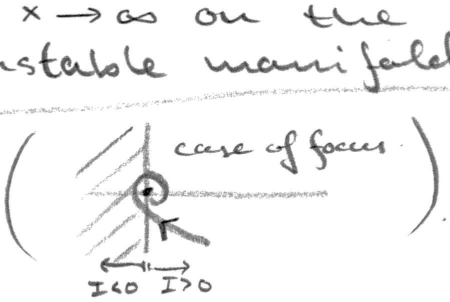
On the unstab. manifold $I \to k - \beta/\alpha$ as $x \to -\infty$.

⇓

If $(0,0)$ is a focus, then I becomes periodically negative as $x \to \infty$ on the unstable manifold.

⟹

Necessary condition for the unstab. manifold to represent a travelling wave solution is that $I_0 = k - \beta/\alpha$ and $(0,0)$ is <u>not</u> a focus.

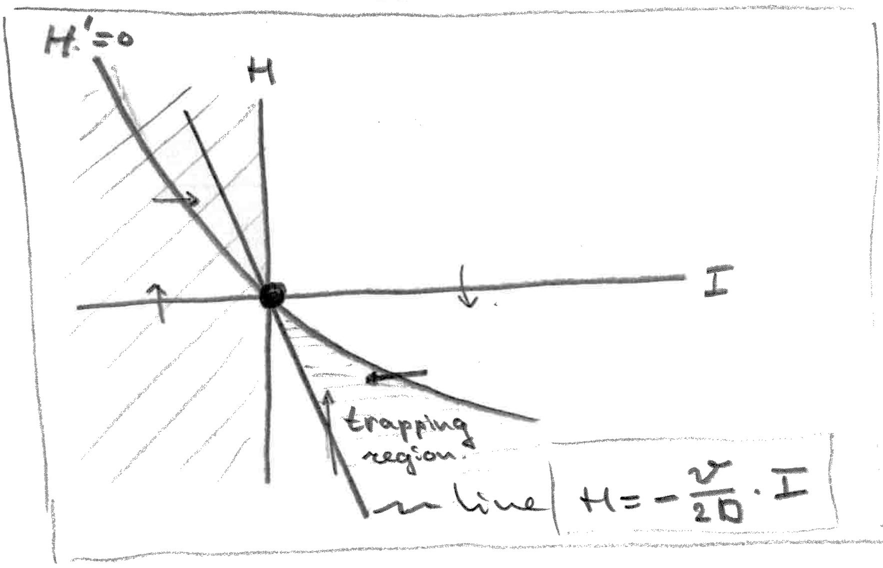


$\Rightarrow$ <u>Necessary</u> condition for the unstable manifold to represent a travelling wave solution:

$$\boxed{I_0 = k - \beta/\alpha} \quad \& \quad \boxed{v \geq 2\sqrt{\alpha D(k-\beta/\alpha)}}$$

(see (**) on page 4)

In fact, these conditions are also <u>sufficient</u>:



$$\left.\frac{dH}{dI}\right|_{H=-\frac{vI}{2D}} \leq -\frac{v}{2D} \iff \frac{\alpha I}{D} \geq \frac{\alpha}{D}\left(k-\frac{\beta}{\alpha}\right) - \left(\frac{v}{2D}\right)^2$$

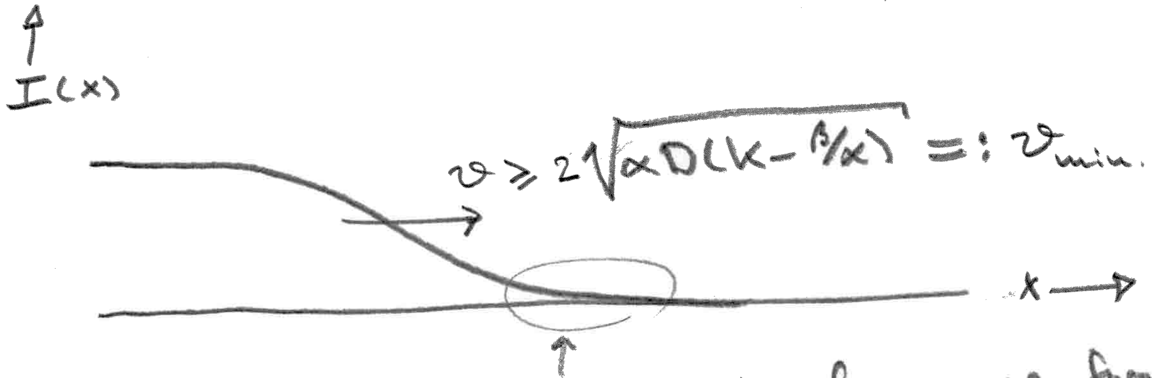
which is satisfied for all I whenever

$$\boxed{v \geq 2\sqrt{\alpha D\left(k-\frac{\beta}{\alpha}\right)}}$$

c.f.

<u>Conclusion</u> $I(x) \in (0, k-\beta/\alpha)$ on the unstable manifold $\forall x \in \mathbb{R}$, with $I(-\infty) = k-\beta/\alpha > 0$ and $I(+\infty) = 0$.

Corresponding travelling wave:



exponential wave front follows from the linearization near (0,0).

$$\begin{pmatrix} I \\ H \end{pmatrix}' = A \begin{pmatrix} I \\ H \end{pmatrix}, \quad \boxed{A = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha}{D}(K - \beta/\alpha) & -\frac{v}{D} \end{pmatrix}}.$$

(→p. 4).

$$\Rightarrow \begin{cases} I(x) \propto e^{+\lambda_{dom} x} \quad \text{where} \\ \lambda_{dom} = -\frac{v}{2D}. \end{cases}$$

What about the wave speed?

- v depends on initial condition.
- By choosing $I(x,0)$ large enough for large values of x, an arbitrary high wave speed can be realized.
- $\boxed{v = v_{min}}$ is the asymptotic wave speed for initial conditions $I(x,0)$ with compact support in space.