Model of Rosenzweig-MacArthur

$$(*)\begin{cases} \dfrac{dn}{dt} = rn\left(1-\dfrac{n}{K}\right) - \dfrac{\beta n}{1+\beta n T}\cdot p \\[2ex] \dfrac{dp}{dt} = \dfrac{\gamma \beta n}{1+\beta n T}\cdot p - \delta p \end{cases}$$

Special case of the model of Gause with

$$g(n) = rn\left(1-\frac{n}{K}\right) \qquad \text{(logistic)}$$

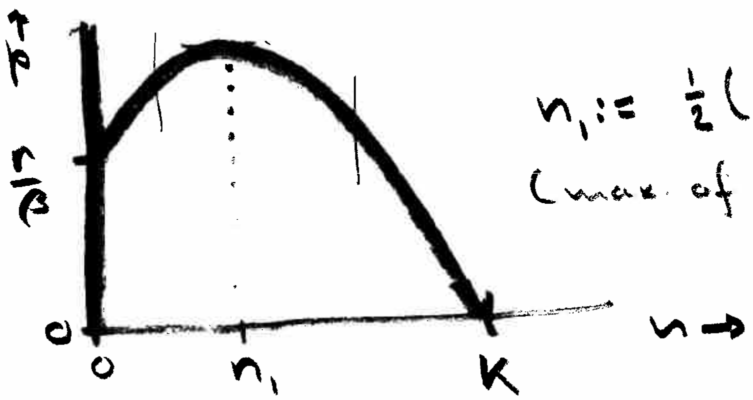
and

$$f(n) = \frac{\beta n}{1+\beta n T} \qquad \text{(Holling-2)}$$

Phase plane analysis

- zerocline of n:

$$\frac{dn}{dt} = 0 \iff n = 0 \text{ or } p = \frac{r}{\beta}(1+\beta T n)\left(1-\frac{n}{K}\right)$$

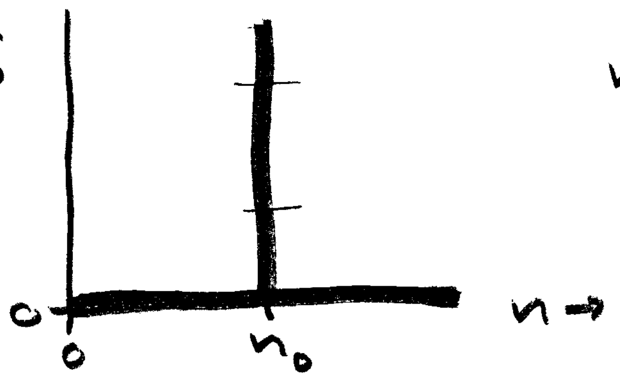


$$n_1 := \frac{1}{2}\left(K - \frac{1}{\beta T}\right)$$

(max. of the parabola)

- zerocline of p:

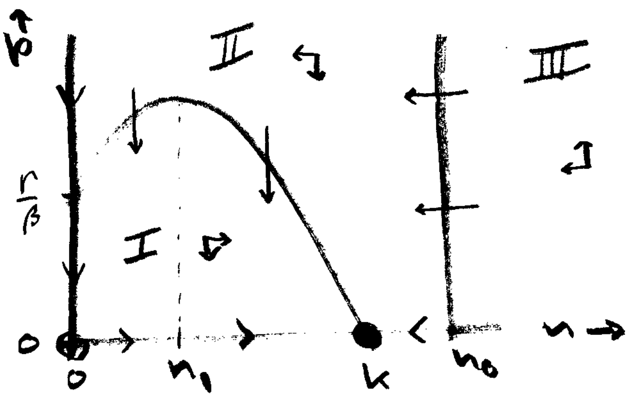
$$\frac{dp}{dt}=0 \iff p=0 \quad \text{or} \quad n=n_0$$



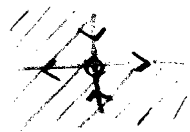
$$n_0 := \frac{\delta}{\beta(\gamma-\beta T)}$$

We consider the following three cases:

① $0<n_1<k<n_0$



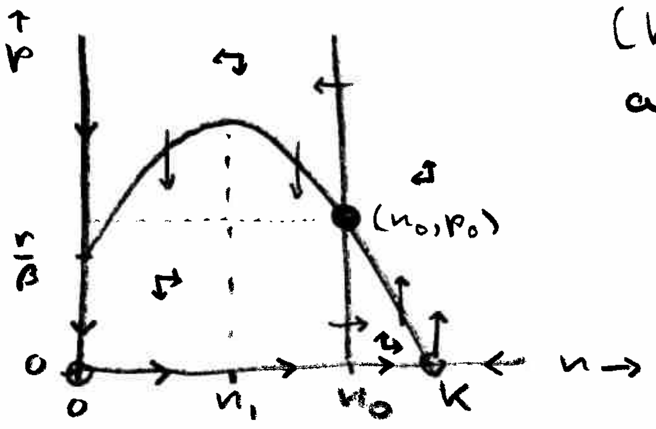
The origin $(0,0)$ is a saddle



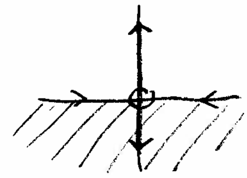
All orbits starting in region III must enter region II in finite time.
Once in region II, orbits converge to $(K,0)$, either directly or by first entering region I.

$\Rightarrow$ $(0,K)$ is globally stable

② $\boxed{0 < n_1 < n_0 < K}$



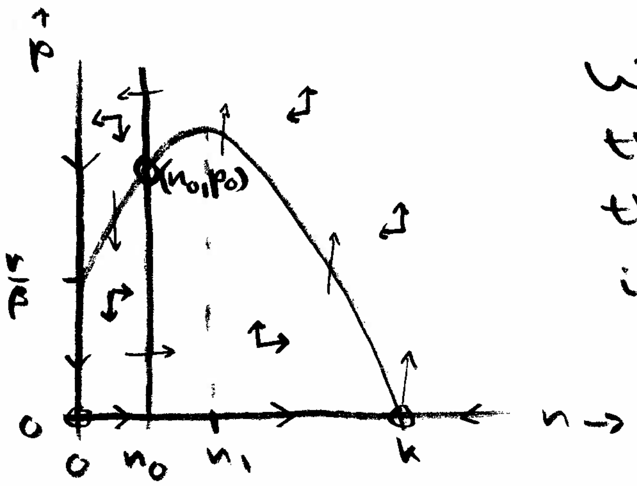
(K,0) has become a saddle



From our analysis of the Gause model (page 44-45, prev. lecture) we know immediately that (n_0, p_0) is <u>locally</u> stable.

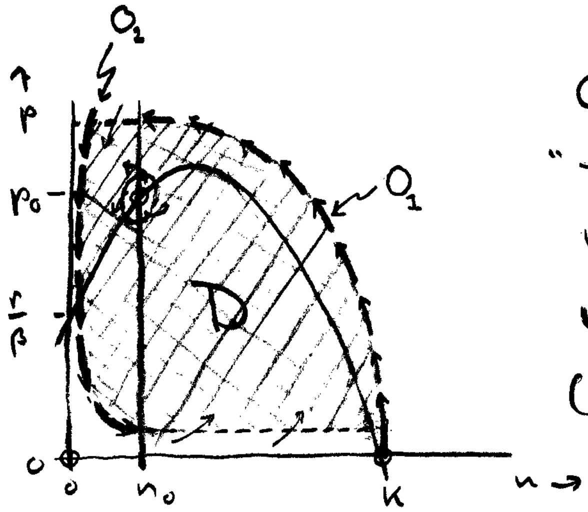
(Later we prove that (n_0, p_0) is in fact <u>globally</u> stable as well)

③ $\boxed{0 < n_0 < n_1 < K}$



We know from the Gause model that now (n_0, p_0) is <u>unstable</u>

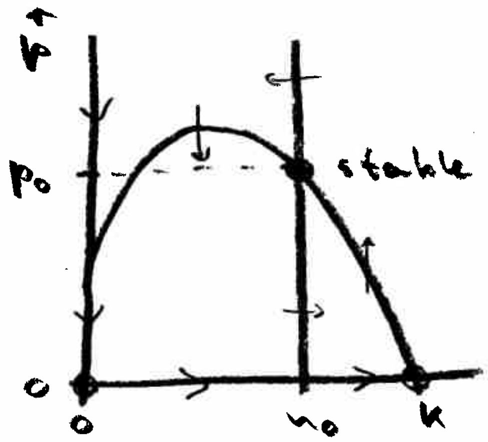
So, there are no stable equilibria at all! How does the system behave then? (<u>First study Appendix B</u>)



Construction of a "trapping region" (D) without equilibria, using two orbits (O_1 and O_2). The unstable focus has been excluded by cutting a hole around it. All orbits starting in the interior of the phase plane will eventually be trapped. Since the trapping region is bounded, and by construction does not contain an equilibrium, it contains a periodic orbit (→ Poincaré-Bendixon) all orbits starting in the interior of the phase-plane, therefore, converge to a periodic orbit in D.

(NB Poincaré-Bendixon does not say anything about the number of periodic orbits)

Back to case ②: $\boxed{0 < n_1 < n_0 < k}$



Implies
$\gamma - dT > 0$

We show that (n_0, p_0) is globally stable:

By Poincaré-Bendixon it is sufficient to exclude the existence of a periodic orbit.

That we do using Dulac's lemma with Dulac function

$$u(n,p) := \frac{p^{\alpha-1}}{f(n)}$$

where $\alpha > 0$ is to be chosen later, and where $f(n)$ is given on page 46 of this lecture's notes.

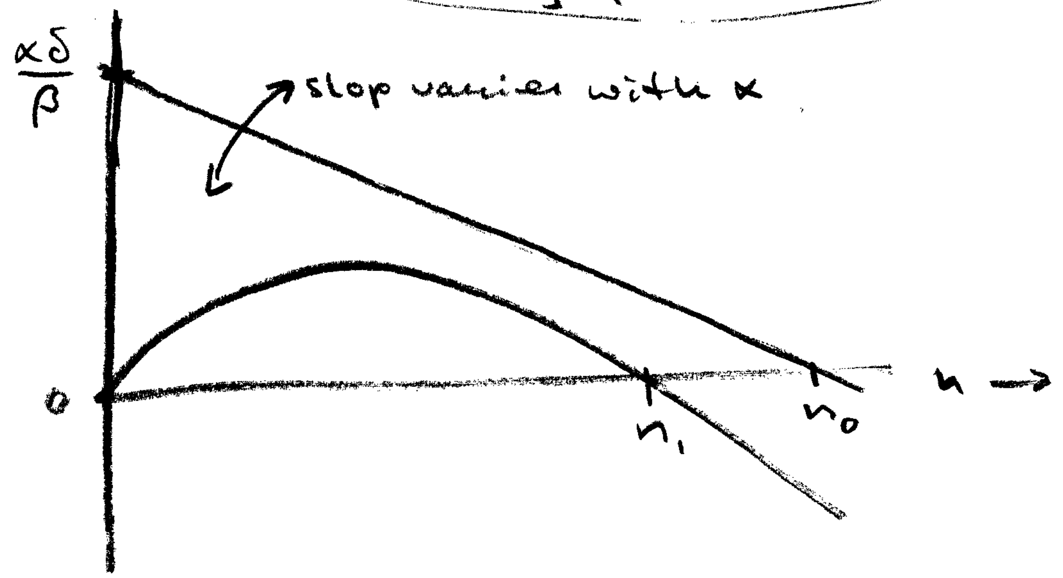
From equation ⊛ on page 46 we have

$$\operatorname{div}\left[u\begin{pmatrix} dn/dt \\ dp/dt \end{pmatrix}\right] =$$

$$= p^{\alpha-1}\Big(\Big(\frac{g(n)}{f(n)}\Big)' + \underbrace{\alpha\Big(\gamma - \frac{\delta}{f(n)}\Big)}_{\text{straight line through } (n_0,0) \text{ and neg. slope (see figure)}}\Big)$$

$$= \frac{p^{\alpha-1}}{n}\Big(\underbrace{r\beta n\Big(\beta T - \frac{1}{K} - \frac{2\beta T}{K}n\Big)}_{\text{parabola as a function of } n \text{ with zero in } n_1 \text{ (see figure)}} - \alpha\Big(\frac{\delta}{\beta} - \underbrace{(\gamma-\delta T)}_{>0}n\Big)\Big)$$

>0 follows from $0 < n_1 < n_0 < K$.



Obviously, $\alpha > 0$ can be chosen such that the straight line <u>always lies above</u> the parabolo, and hence $\operatorname{div}\left[u\begin{pmatrix} dn/dt \\ dp/dt \end{pmatrix}\right] < 0$ everywhere. ⊠