Density-dependent diffusion in the SIS model

i-Notes.

S : susceptible individual

I : infected individual.

Local reactions

$S + I \xrightarrow{\alpha} 2I$ (infection)

$I \xrightarrow{\beta} S$ (recovery)

Random movement

D_1 diffusion coeff. for S

D_2 " " " I

Reaction-diffusion equations.

$$(1)\quad \begin{cases} \partial_t S = -\alpha S I + \beta I + D_1 \partial_x^2 S \\ \partial_t I = \alpha S I - \beta I + D_2 \partial_x^2 I \end{cases}$$

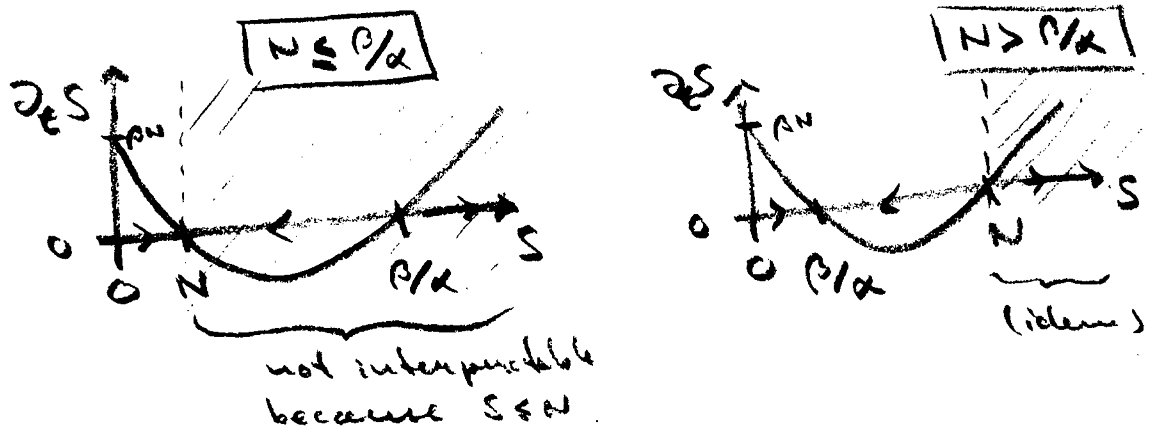
Assume that random movement is slow process compared with the local reactions, and define $\boxed{N := S + I}$ (loc. pop. dens.)

Then ① can be rewritten as

$$② \quad \begin{cases} \partial_t N = D_1 \partial_x^2 S + D_2 \partial_x^2 (N-S) & \text{slow} \\ \partial_t S = -\alpha S(N-S) + \beta(N-S) + D_1 \partial_x^2 S & \text{fast} \end{cases}$$

Fast dynamics:

$$\begin{cases} \partial_t N = 0 \\ \partial_t S = -\alpha S(N-S) + \beta(N-S) \end{cases}$$



Quasi-equilibrium (stable).

$$③ \quad S = \begin{cases} N & \text{if } N \le \beta/\alpha \\ \beta/\alpha & \text{if } N > \beta/\alpha \end{cases}$$

Slow dynamics.

$$\textcircled{4} \begin{cases} \partial_t N = D_1 \partial_x^2 S + D_2 \partial_x^2 (N-S) \\ 0 = -\alpha S(N-S) + \beta(N-S). \end{cases}$$

Subst ③ into ④ gives

$$\partial_t N = \begin{cases} D_1 \partial_x^2 N & \text{if } N \leq \beta/\alpha \\ D_2 \partial_x^2 (N - \beta/\alpha) & \text{if } N > \beta/\alpha \end{cases}$$

Since β/α is constant, we have $\partial_x^2 (N - \beta/\alpha) = \partial_x^2 N$ and so,

$$\partial_t N = \begin{cases} D_1 \partial_x^2 N & \text{if } N \leq \beta/\alpha \\ D_2 \partial_x^2 N & \text{if } N > \beta/\alpha. \end{cases}$$

Which we can also write as

$$\partial_t N = -\partial_x \left(-\mathcal{D}(N) \partial_x N \right)$$

where

$$\mathcal{D}(N) = \begin{cases} D_1 & \text{if } N \leq \beta/\alpha \\ D_2 & \text{if } N > \beta/\alpha \end{cases}$$

is a density-dependent diffusion coefficient.

Does spatial mobility make it easier for a disease to spread in an initially disease-free population?

Spatially well-mixed SIS model

$$(1)\begin{cases} \frac{dS}{dt} = -\alpha S I + \rho I \\ \frac{dI}{dt} = +\alpha S I - \rho I \end{cases}$$

$N := S + I$ is constant

linearization of (1) around $(S, I) = (N, 0)$ gives:

$$\dot{I} = (\alpha N - \rho) I$$

So, the disease can spread if $N > \cancel{\alpha/\beta}$ β/α but not if $N \le \beta/\alpha$

How is this if we allow individuals to move about?

Spatial SIS model

$$(2)\quad \begin{cases} \partial_t S = -\alpha S I + \beta I + D_1 \partial_x^2 S \\ \partial_t I = \alpha S I - \beta I + D_2 \partial_x^2 I \end{cases}$$

on an interval $[0, L]$ with reflecting boundaries.

Equilibrium equations

$$\begin{cases} 0 = -\alpha S I + \beta I + D_1 \partial_x^2 S \\ 0 = \alpha S I + \beta I + D_2 \partial_x^2 I \end{cases}$$

Disease-free equilibrium

$I = 0$ and $0 = D_1 \partial_x^2 S$

$\Leftrightarrow$ $I = 0$ and $S = ax + b$ (constant a, b)

Bnd. cnds: $\partial_x S = 0$ at $x = 0, L$ $\Rightarrow a = 0$

$\Rightarrow$ $\boxed{I = 0 \text{ and } S = b.}$

Note that

$\boxed{b = \frac{1}{L}\int_0^L S\,dx =: \bar{N}}$ is the (spatially) average pop density.

We investigate the stability of the disease-free equilibrium $(S, I) = (\bar{N}, 0)$.

Linearization of ② near $(\bar{N}, 0)$ gives:

③ $$\partial_t I = (\alpha \bar{N} - \beta) I + D_2 \partial_x^2 I$$

Substitute $I(t,x) = \underbrace{v(x)}_{\text{eigenfunction}} e^{\overbrace{\lambda}^{\text{eigenvalue.}} t}$;

$$\Rightarrow \quad \lambda v(x) = (\alpha \bar{N} - \beta) v(x) + D_2 v''(x)$$

which is a linear 2nd-order ordinary differential equation with general solution

④ $$\begin{cases} v(x) = A e^{a(\lambda) x} + B e^{-a(\lambda) x} \\ a(\lambda) := \sqrt{\dfrac{\lambda - \alpha \bar{N} + \beta}{D_2}} \end{cases}$$

where A and B are integration constants that can be determined from the bond conditions

From the boundary condition we have that $v'(x) = 0$ for $x = 0, L$.

This implies that

$$\begin{cases} a(\lambda) = 0 \text{ or} \\ A = B \text{ and } Ae^{a(\lambda)L} = Be^{-a(\lambda)L} \end{cases}$$

which is equivalent to

(5) $$\boxed{a(\lambda) = \frac{k\pi}{L}\, i}$$ (characteristic equation)

(for $k \in \mathbb{Z}$)

from which we solve the eigenvalues

(6) $$\boxed{\lambda_k := -D_2\left(\frac{k\pi}{L}\right)^2 + \alpha\bar{N} - \beta}$$ $(k \in \mathbb{Z}.)$

The corresponding eigenvectors are

(7) $$\boxed{v_k(x) = \cos\left(\frac{k\pi}{L}x\right)}$$ $(k \in \mathbb{Z})$

(just substitute (5) into (4))

The largest eigenvalue is found for $k=0$, which gives

$$\boxed{\lambda_0 = \alpha \bar{N} - \beta.}$$

The disease-free equilibrium is stable if $\lambda_0 < 0$ and unstable if $\lambda_0 > 0$.

The disease can spread in an initially disease-free pop. only if the disease-free equil is unstable.

<u>Hence</u>

Disease spreads if $\bar{N} > \beta/\alpha$.
Disease does not spread if $\bar{N} < \beta/\alpha$.

Which is the same result as in the spatially well-mixed situation on page <u>4</u>