(03-04-2012)

Non-random motion: taxis

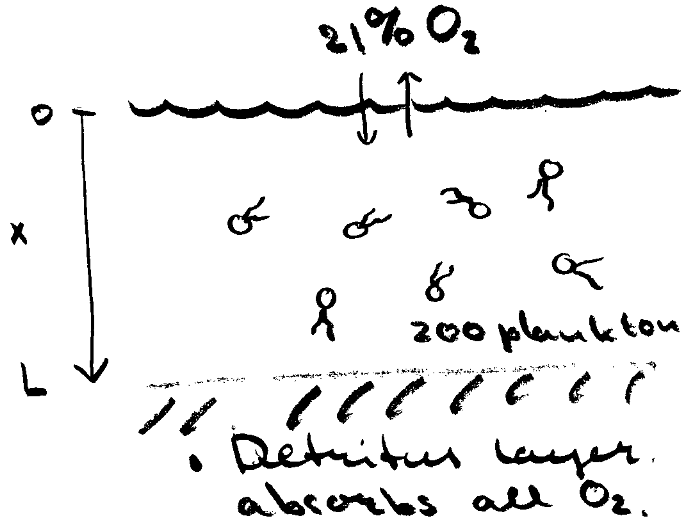
Taxis is the movement of particles or individuals up or down some density gradient.

$$\boxed{J = a \cdot n\, \partial_x c}$$

where $a \in \mathbb{R}$ is a constant, $n(t,x)$ the population density and $c(t,x)$ the density of some cond the individuals react to

- Positive taxis: movement up the gradient, i.e, $\boxed{a > 0}$
- Negative taxis: movement down the gradient, i.e, $\boxed{a < 0}$
- Auto-taxis: movement in response to the density gradient of its own kind, i.e, $\boxed{c = n}$

Example



- O_2-conc. in surface layer in equil with atmospheric O_2
- Random movement of zooplankton plus taxis toward higher O_2-concentrations

Balance equations

$$(1) \quad \boxed{\partial_t c = D_1 \partial_x^2 c - \alpha c n} \quad \text{(oxygen)}$$

diffusion — consumption by zooplankton

$$(2) \quad \boxed{\partial_t n = -\partial_x\left[-D_2 \partial_x n + \beta n \partial_x c\right]}$$

flux = diffusion + taxis

Boundary conditions

$$(3) \quad \begin{aligned} & c(t,0) = c_0 \quad \text{(const. conc. bnd)} \\ & c(t,L) = 0 \quad \text{(absorb. bnd)} \qquad \text{(refl. bnds)} \\ & -D_2 \partial_x n(t,x) + \beta n(t,x) \partial_x c(t,x) = 0 \quad \text{for } x = 0, L \end{aligned}$$

Density-dependent diffusion

Recall Fick's Law (p. 5 of 27-03-2012) for randomly moving particles:

$$J(t,x) = -\alpha(t,x)\,\partial_x n(t,x)$$

Notice that the diffusion coefficient $\alpha(t,x)$ may change with time or space (e.g., because individuals may be more active during some times of the day or year than at other times, or some places are more difficult to travel through than other places).

More generally, the diffusion coefficient may also depend on the pop. density. This may work in two directions, depending on the details of the model: crowding may increase or decrease the speed of movement, and so the diffusion coefficient may change accordingly.

Example

- i-states : N_1 single individual
  N_2 pair of individuals.

- i-processes: $2N_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} N_2$

This may, e.g, describe two individuals that meet and want to find out whether the other is friend or foe. Once that issue has been settled, they go their own separate ways again

Let's assume that the single individuals move randomly with a constant diffusion coefficient $D>0$ and that pairs do not move at all

$\Rightarrow$ • p-equations

$$\boxed{\begin{aligned} \partial_t n_1 &= -\alpha n_1^2 + 2\beta n_2 + D\partial_x^2 n_1 \\ \partial_t n_2 &= +\tfrac{1}{2}\alpha n_1^2 - \beta n_2 \end{aligned}}$$

- Time-scale separation:

Suppose that diffusion is a slow process compared to the formation of pairs and the breaking up of pairs (i.e, $D \ll \alpha, \beta$).

Then we can first consider the ~~fast~~ fast processes alone:

$$\boxed{\begin{aligned} \partial_t n_1 &= -\alpha n_1^2 + 2\beta n_2 \\ \partial_t n_2 &= +\tfrac{1}{2}\alpha n_1^2 - \beta n_2 \end{aligned}} \quad (\underline{\text{fast dynamics}})$$

Write $n := n_1 + 2n_2$ (tot. pop. dens.).
Then we get

$$\boxed{\partial_t n_1 = -\alpha n_1^2 + \beta(n - n_1)}$$

which equilibrates at

$$(*) \quad \boxed{n_1 = \frac{-\beta + \sqrt{\beta^2 + 4\alpha\beta n}}{2\alpha}} \quad (\text{quasi-equil})$$

Now we look at the dynamics of the slow variable $n := n_1 + 2n_2$:

$$\boxed{\partial_t n = D \partial_x^2 n_1}$$ (follows from eqs. on p. 4).

Subst. of n_1 by its quasi-equil value (⊛ on page 5) gives

(⊛⊛) $$\boxed{\partial_t n = D \partial_x^2 \left(\frac{-\beta + \sqrt{\beta^2 + 4\alpha\rho n}}{2\alpha} \right)}$$

which describes the dynamics in space of the total population. (Very handy if it is difficult to see whether individuals are actually interacting or not, like in small creatures like ants.)

I want to re-write (⊛⊛) into the form

(⊛⊛⊛) $$\boxed{\partial_t n = -\partial_x \underbrace{\left(-\mathcal{D}(n) \partial_x n \right)}_{\text{flux}}}$$

with a <u>density-dependent</u> diffusion coefficient $\mathcal{D}(n)$.

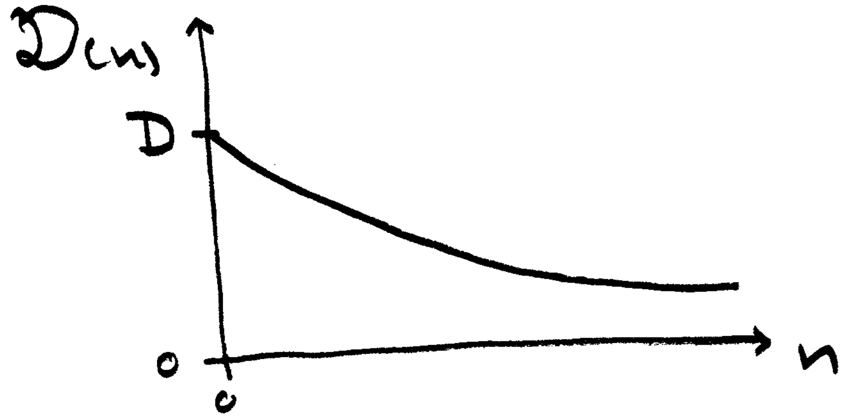
Comparing (**) with (***) we immediately see that

$$\mathcal{D}(n)\partial_x n = D\,\partial_x\left(\frac{-\beta+\sqrt{\beta^2+4\alpha\beta n}}{2\alpha}\right)$$

$$= D\sqrt{\frac{\beta}{4\alpha n+\beta}}\,\partial_x n$$

Hence, we find that the dens.-dep. diff coeff. $\mathcal{D}(n)$ is

$$\boxed{\mathcal{D}(n) = D\sqrt{\frac{\beta}{4\alpha n+\beta}}}$$



<u>Concl.</u>: At low pop. dens., the diff. coeff. is maximal and about equal to D; at high pop. dens., the diff. coeff. decreases about inversely with pop density.

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