Appendix C

(Stability analysis in discrete dynamical systems)

Consider the <u>one-dimensional</u> discrete dynamical system

(1) $n_{t+1} = f(n_t) \quad , \; t = 0,1,2,\ldots$

for given cont. diff. $f : \mathbb{R}_+ \to \mathbb{R}_+$ and suppose that

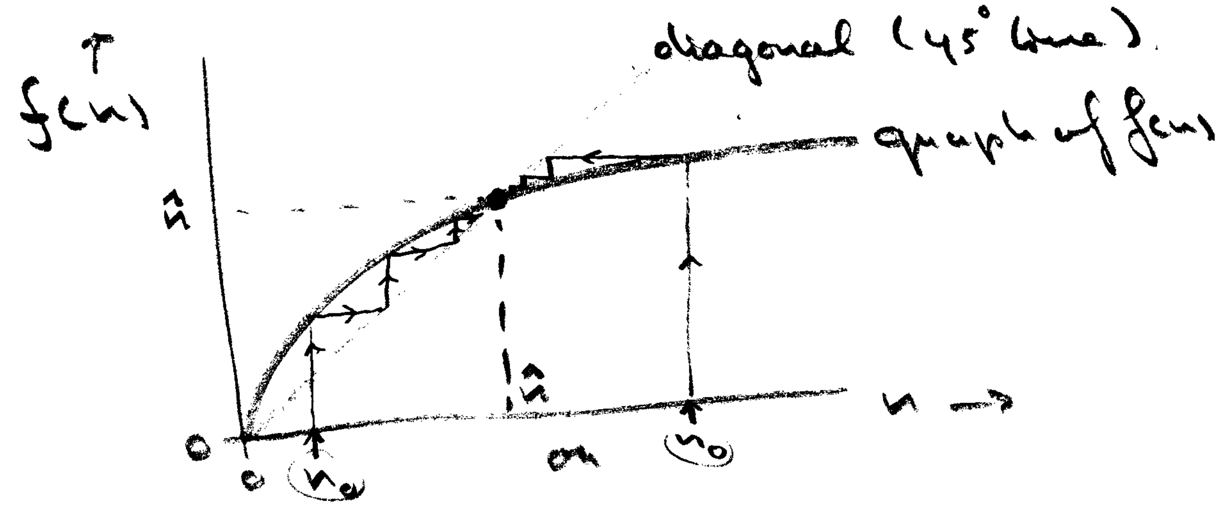
(2) $\hat{n} = f(\hat{n})$

(i.e., $\hat{n}$ is an equilibrium of (1)).

<u>Question</u>: Is $\hat{n}$ stable in the sense that $n_t \to \hat{n}$ as $t \to \infty$ for initial values n_0 in some open set?

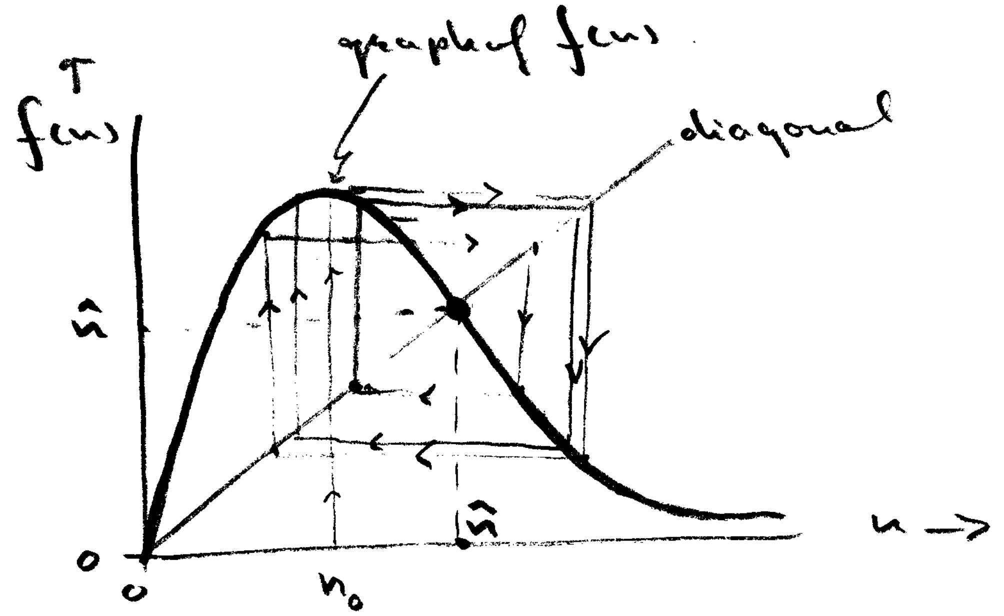
First answer: "Cobweb method"

Ex 1



<u>conclusion</u>: $\hat{n}$ is stable.

Ex 2



<u>conclusion</u>: $\hat{n}$ does <u>not</u> to be stable.

(The cobweb method gives global info about stability (or lack thereof), but is not always clear.

Local stability analysis

Linearize (1) about $\hat{n}$:

(3) $$n_{t+1} = f(\hat{n}) + f'(\hat{n})(n_t - \hat{n})$$

$$\overset{(2)}{=} \hat{n} + f'(\hat{n})(n_t - \hat{n})$$

and hence

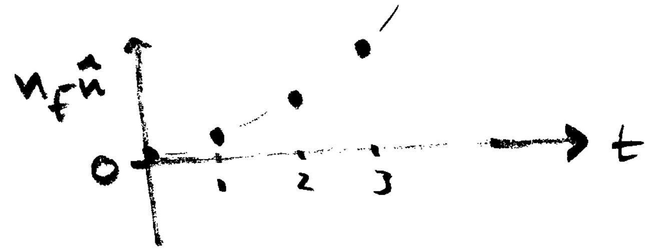
(4) $$n_{t+1} - \hat{n} = f'(\hat{n})(n_t - \hat{n})$$

This linearized system has solution

(5) $$n_t - \hat{n} = (n_0 - \hat{n})\left[f'(\hat{n})\right]^t$$

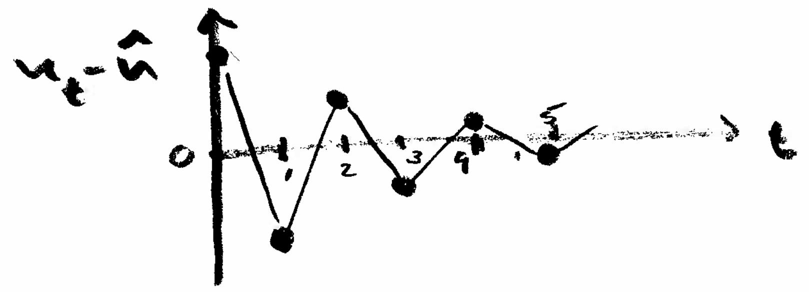
Hence

$\boxed{0 < f'(\hat{n})}$ $\Rightarrow$ monotone divergence

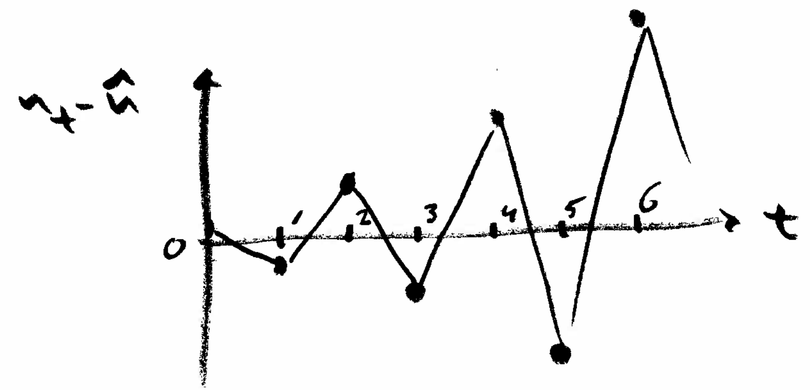


$0 < f'(\hat{n}) < 1$ $\Rightarrow$ monotone convergence

$\boxed{-1 < f'(\hat{u}) < 0} \Rightarrow$ non-monotonous convergence



$\boxed{f'(\hat{u}) < -1} \Rightarrow$ non-monotone divergence



<u>In short:</u> $\hat{u}$ in the linearized system ④ is <u>locally stable</u> if $|f'(\hat{u})| < 1$ and <u>unstable</u> if $|f'(\hat{u})| > 1$

The theorem of Hartman & Grobman (see Appendix A) says that if $|f'(\hat{u})| \neq 0$ and $|f'(\hat{u})| \neq 1$, then the non-linear system ① and the linearized system ④ are locally topologically equivalent

The local stability analysis directly generalizes to multi-dimensional systems.

(6) $n_{t+1} = f(n_t)$, $t = 0, 1, 2, \ldots$

with $f : \mathbb{R}^k_+ \longrightarrow \mathbb{R}^k_+$, cont. diff. and

(7) $\hat{n} = f(\hat{n})$

Linearization of (6) about the equilibrium $\hat{n}$ gives

(8) $n_{t+1} - \hat{n} = f'(\hat{n})(n_t - \hat{n})$

where $f'(\hat{n})$ is a $k \times k$ matrix of partial derivatives (Jacobi-matrix)

again, Hartman-Grobman says that if none of the eigenvalues of $f'(\hat{n})$ is zero or lies on the unit circle in the complex plane, then the systems (6) and (8) are topologically equivalent.

This motivates us to investigate the linear system

(9) $\boxed{x_{t+1} = A x_t \in \mathbb{R}^k}$ ($x=0$ is equil)

General solution:

(10) $x_t = A^t x_0$

but what does this say about the stability of $x=0$?

Let

$Ab_i = \lambda_i b_i$ $(i=1,\dots,k)$

where b_i is an eigenvector of A and λ_i the corresponding eigenvalue.

Let further

$B := (b_1, \dots, b_k)$ (matrix)

and

$\Lambda :=$ diagonal $(\lambda_1, \dots, \lambda_k)$.

(c.f. appendix A)

Then

$$AB = B\Lambda .$$

Suppose further that the eigenvectors $b_1, \ldots, b_k$ are linearly independent.

Then B^{-1} exists and

$$A = B \Lambda B^{-1}$$

Hence,

$$A^t = \overbrace{B\Lambda B^{-1} \cdot B\Lambda B^{-1} \cdot \ldots \cdot B\Lambda B^{-1} \cdot B\Lambda B^{-1}}^{t \text{ times}} =$$

$$= B \Lambda^t B^{-1} = B \begin{pmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_k^t \end{pmatrix} B^{-1}$$

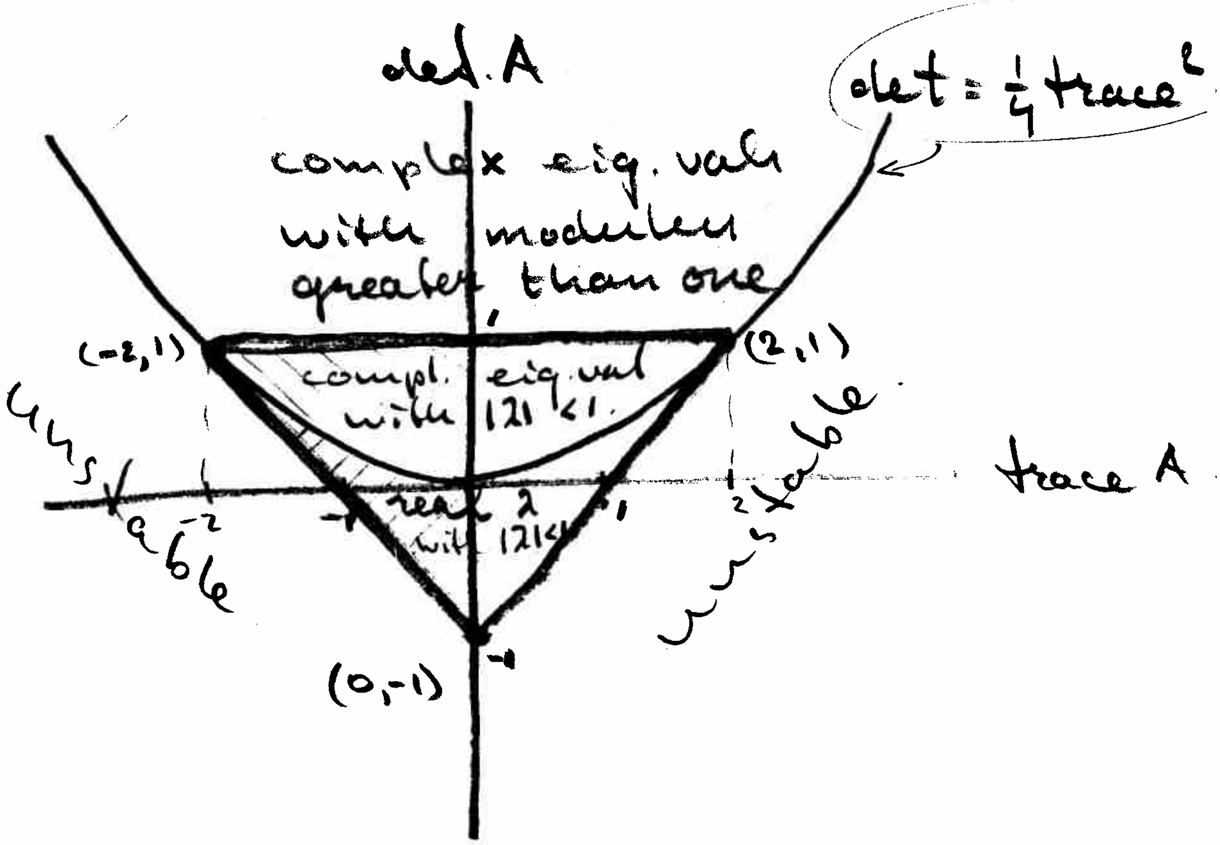
Subst into (10) gives

$$(11) \qquad \boxed{x_t = B \begin{pmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_k^t \end{pmatrix} B^{-1} x_0}$$

It immediately follows that if $|\lambda_i| < 1$ for all i, then $x = 0$ is stable. However, if there exists even a single i such that $|\lambda_i| > 1$, then $x = 0$ is unstable.

Planar systems.

Suppose $A \in \mathbb{R}^{2\times 2}$.
Then the eigenvalues λ_1 and λ_2 are fully determined by det A and trace A in the following way



Only for (trace A, det A) inside the triangle $(-2,1), (2,1), (0,-1)$ the $x = 0$ is stable.

Example

(model of Nicholson-Bailey).

$$\boxed{\begin{aligned} n_{t+1} &= \alpha n_t e^{-\beta p_t} \\ p_{t+1} &= n_t (1 - e^{-\beta p_t}) \end{aligned}}$$

(host)

(parasitoid).

Equilibrium

$n_{t+1} = n_t = \hat{n} > 0$ & $p_{t+1} = p_t = \hat{p} > 0$

$$\Longleftrightarrow \boxed{\hat{n} = \frac{\alpha \log \alpha}{(\alpha - 1)\beta}} \; \& \; \boxed{\hat{p} = \frac{\log \alpha}{\beta}}$$

provided $\alpha > 1$.

Jacobi-matrix

$$A := \begin{pmatrix} 1 & -\frac{\alpha \log \alpha}{\alpha - 1} \\ 1 - \frac{1}{\alpha} & \frac{\log \alpha}{\alpha - 1} \end{pmatrix}$$

obvious $(\alpha > 1)$

$$\text{trace } A = 1 + \frac{\log \alpha}{\alpha - 1} > 1$$

$$\det A = \frac{\alpha \log \alpha}{\alpha - 1} > 1$$

whenever $\alpha > 1$

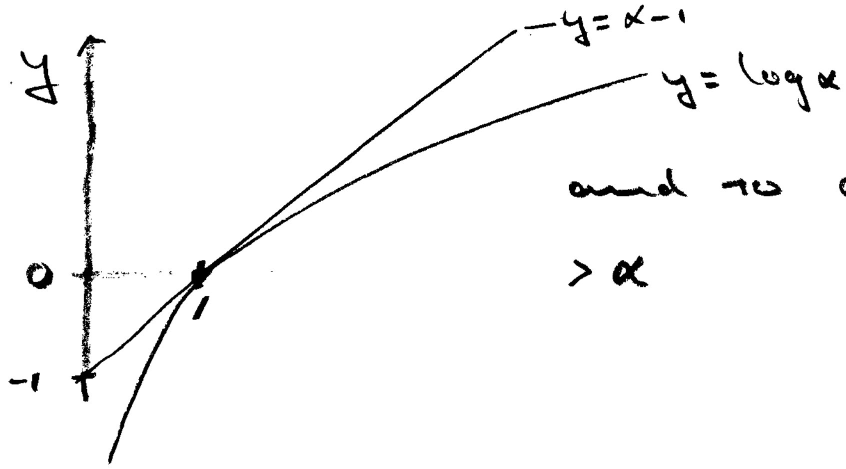
see next page

Since $\det A > 1$, $(\hat{u}, \hat{p})$ is unstable (see fig. on page 7)

To see that $\det A$ is greater than one, define

$$g(\alpha) := \det A = \frac{\alpha \log \alpha}{\alpha - 1}$$

$$\Rightarrow \quad g'(\alpha) = \frac{\alpha - 1 - \log \alpha}{(\alpha - 1)^2} > 0$$



and so $\alpha - 1 - \log \alpha > 0$.

$> \alpha$

$$\Rightarrow \quad g(\alpha) > \lim_{\alpha \downarrow 1} g(\alpha) = 1.$$