Delay-differential equations; Local stability analysis.

Consider the equation

$$\textcircled{1} \quad \boxed{\dot{n} = f(n, n_T)}$$

where $n_T(t) := n(t-T)$, $T > 0$.

Suppose $n(t) = \bar{n}$ (constant) is an equilibrium of ①:

$$\boxed{0 = f(\bar{n}, \bar{n})}$$

Linearization (= 1st order Taylor approx) about the equilibrium gives

$$\textcircled{2} \quad \boxed{\dot{u} = au + bu_T}$$

where $u := n - \bar{n}$ and $u_T := n_T - \bar{n}$ and

$a := \partial_1 f(\bar{n}, \bar{n})$ and $b := \partial_2 f(\bar{n}, \bar{n})$

Substitute $u(t) = e^{\lambda t}$ to get the <u>characteristic equation</u>:

$$\boxed{\lambda = a + b e^{-\lambda T}}$$

where λ is an eigenvalue.

Write $\lambda = \mu + i\omega$ and separate the real and imaginary parts of the characteristic equation:

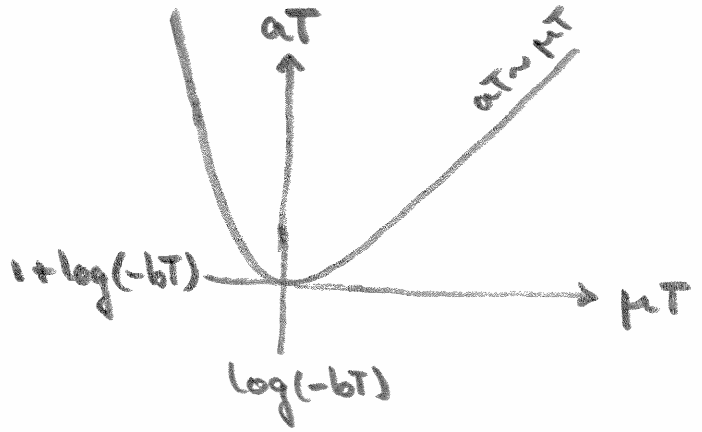
$$(3)\quad \boxed{\begin{aligned} \mu T &= aT + bT\, e^{-\mu T} \cos \omega T \\ \omega T &= - bT\, e^{-\mu T} \sin \omega T \end{aligned}}$$

Real eigenvalues

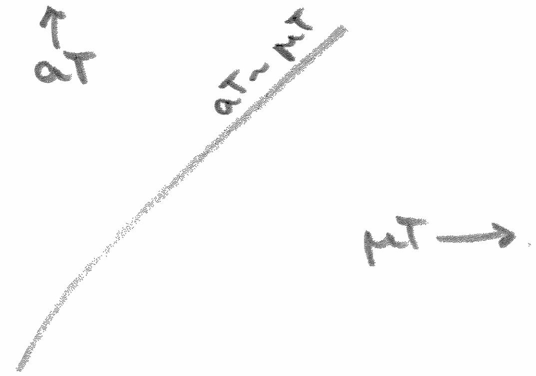
Substitute $\omega T = 0$ into (3) and rewrite as follows:

$$\boxed{aT = \mu T - bT\, e^{-\mu T}}$$

Case: $bT < 0$

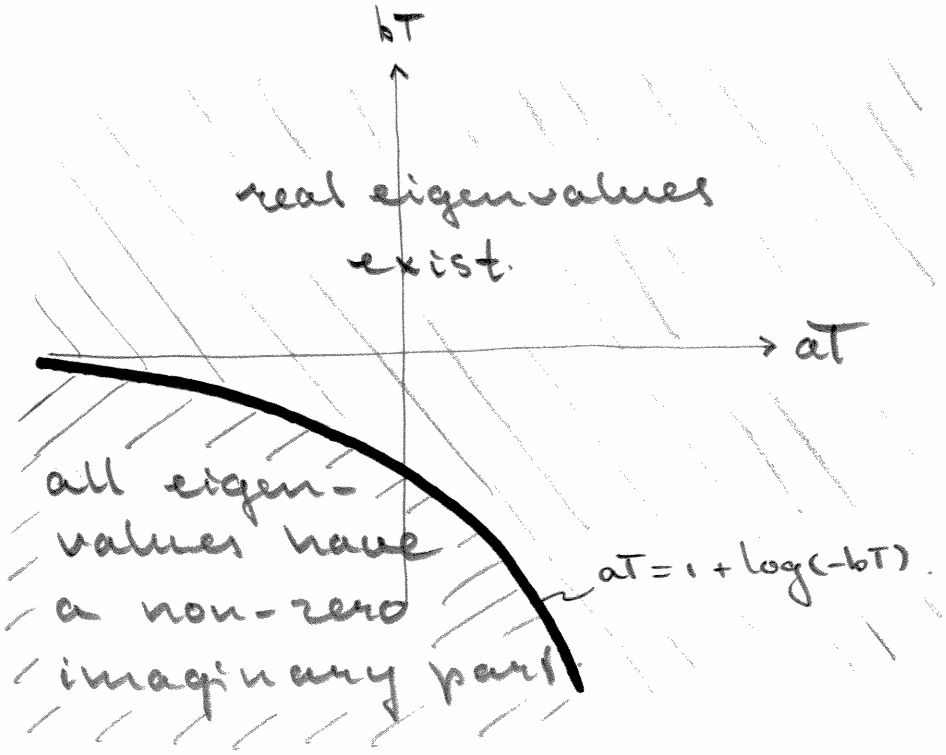


Case $bT \geq 0$



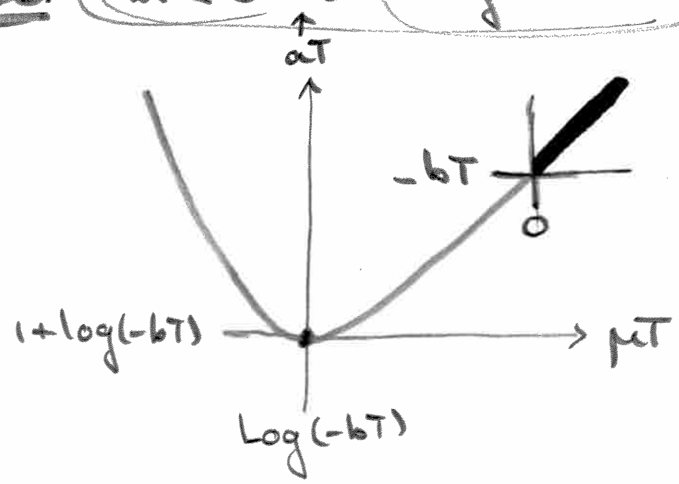
$\Rightarrow$ There exist no real eigenvalues if and only if $bT < 0$ & $aT < 1 + \log(-bT)$

Graphically, in the (aT, bT)-plane:



Positive (real) eigenvalues

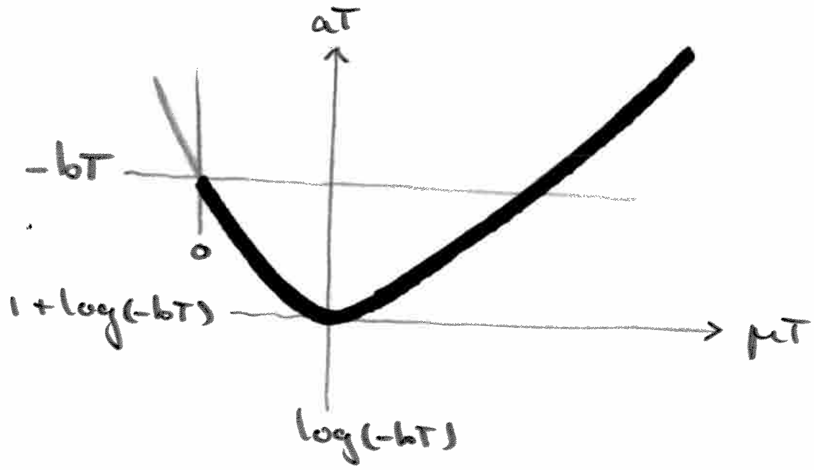
I case: $bT < 0$ & $\log(-bT) < 0$: $\Longleftrightarrow$ $\boxed{-1 < bT < 0}$



Pos. eig. val exists if and only if

$\boxed{aT > -bT}$

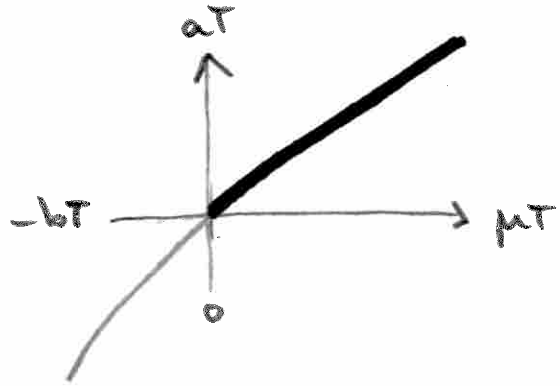
II case: $bT < 0$ & $\log(-bT) > 0$: $\iff$ $\boxed{bT < -1}$.



Pos. eig. val. exists if and only if

$$\boxed{aT > 1 + \log(-bT)}$$

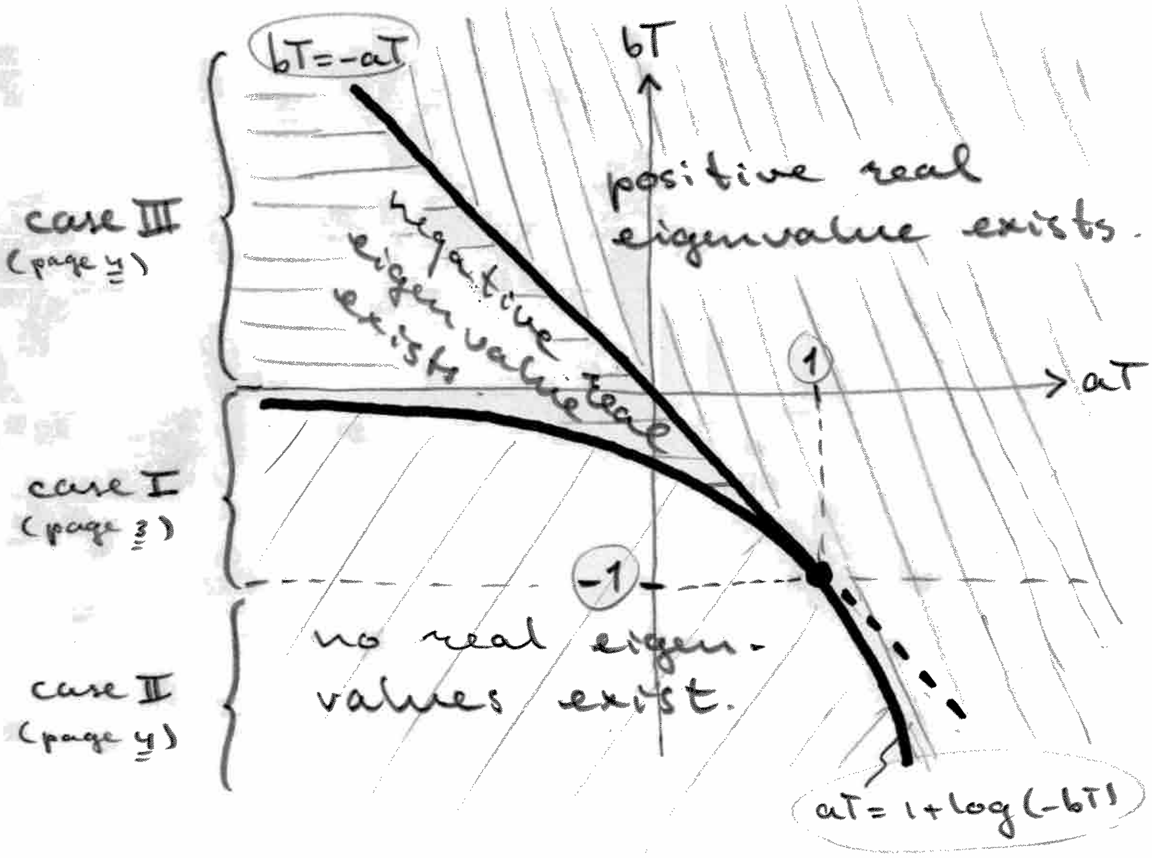
III case: $bT \geq 0$



Pos. eig. val exists if and only if

$$\boxed{aT > -bT}$$

Graphically, in the (aT, bT) plane,



<u>Note</u> that if positive (real) eigenvalues exist, then the equilibrium is unstable.

$$\boxed{\text{Complex eigenvalues}}$$

Crossing the imaginary axis at $\mu T = 0$:

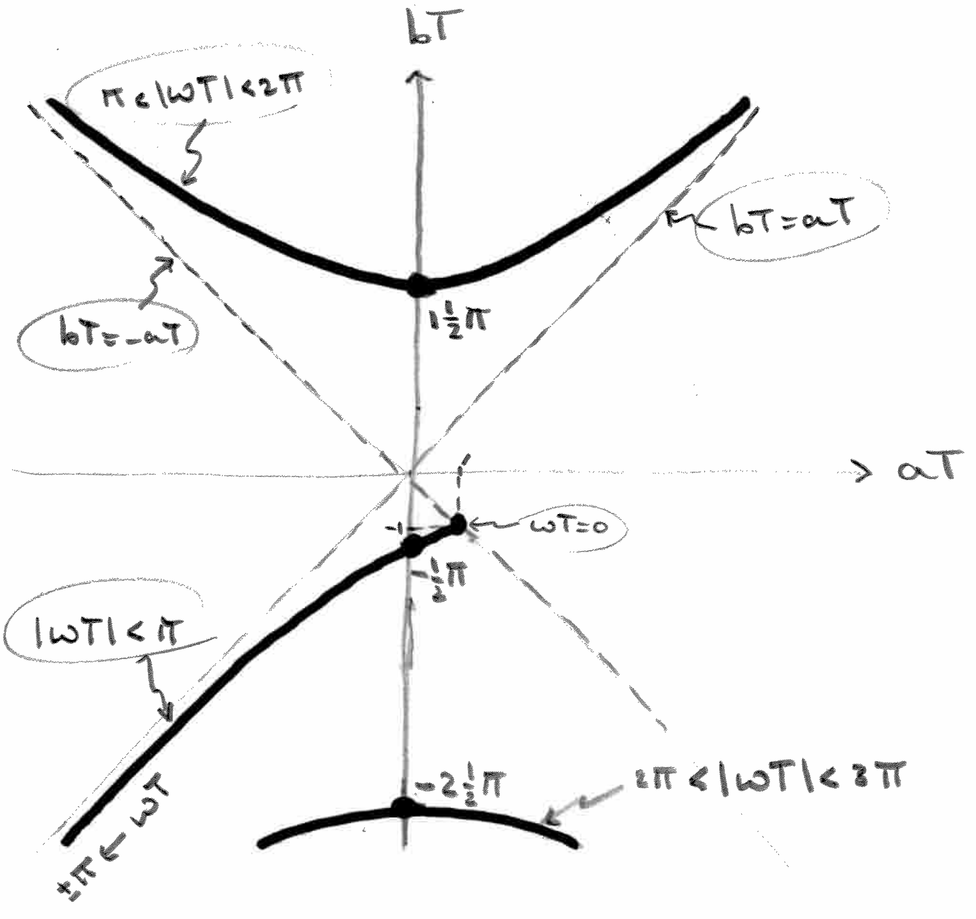
$$③ \Longrightarrow \boxed{\begin{aligned} 0 &= aT + bT\cos\omega T \\ \omega T &= \quad - bT \sin\omega T \end{aligned}}$$

(page 3)

Solve ④ for (aT, bT):

$$aT = \omega T \cot \omega T$$
$$bT = -\omega T / \sin \omega T.$$

Parameterized curves in the (aT, bT)-plane



For any (aT, bT) on one of the solid curves, at least one eig. val. has a zero real part.

Which of these curves correspond to the dominant eigenvalue?

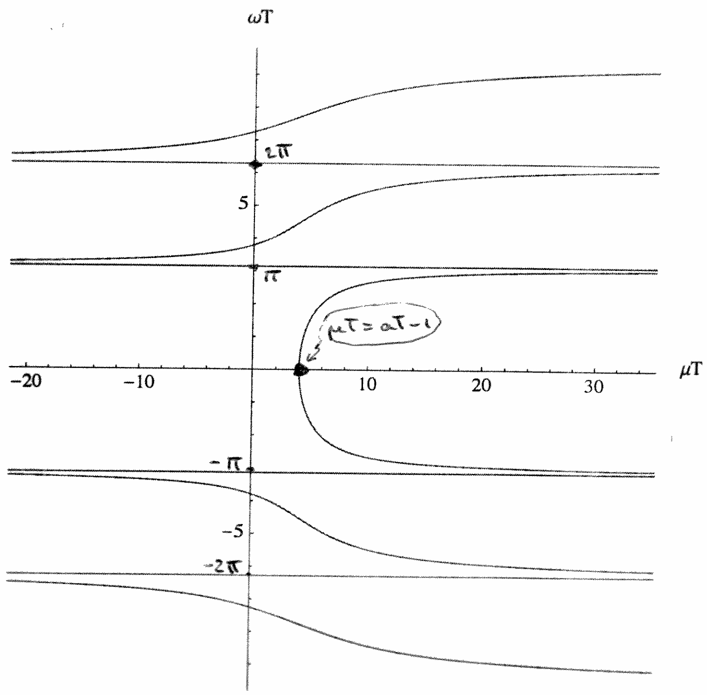
Write the char. equ. 3 (page 3) as

(4a) $\mu T = aT - \omega T \cot \omega T$ and

(4b) $$\mu T = \begin{cases} \log bT + \log\left(-\frac{\sin \omega T}{\omega T}\right) & \text{for } bT > 0 \text{ and } (2k-1)\pi < |\omega T| < 2k\pi \\ \log(-bT) + \log\left(\frac{\sin \omega T}{\omega T}\right) & \text{for } bT < 0 \text{ and } 2k\pi < |\omega T| < (2k+1)\pi \end{cases}$$

for $k = 0, 1, 2, \ldots$

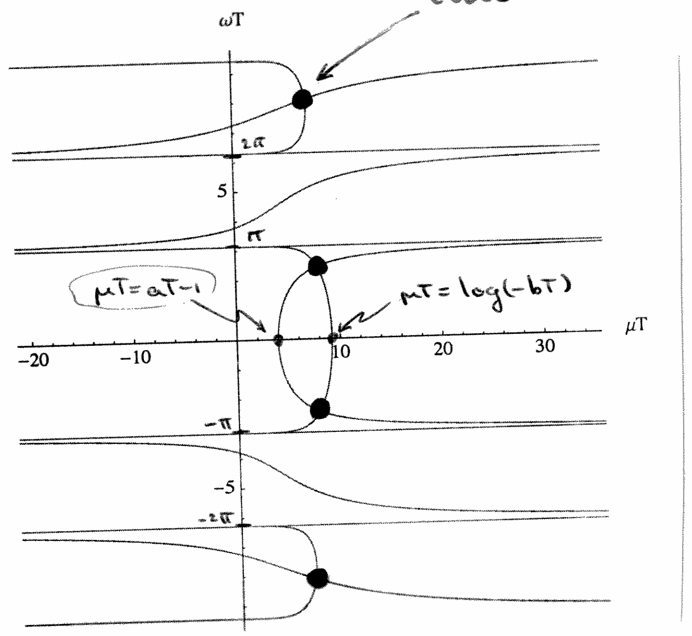
The graph of (4a):



To see where the eigenvalues are, this graph has to be intersected with the graph of (4b):

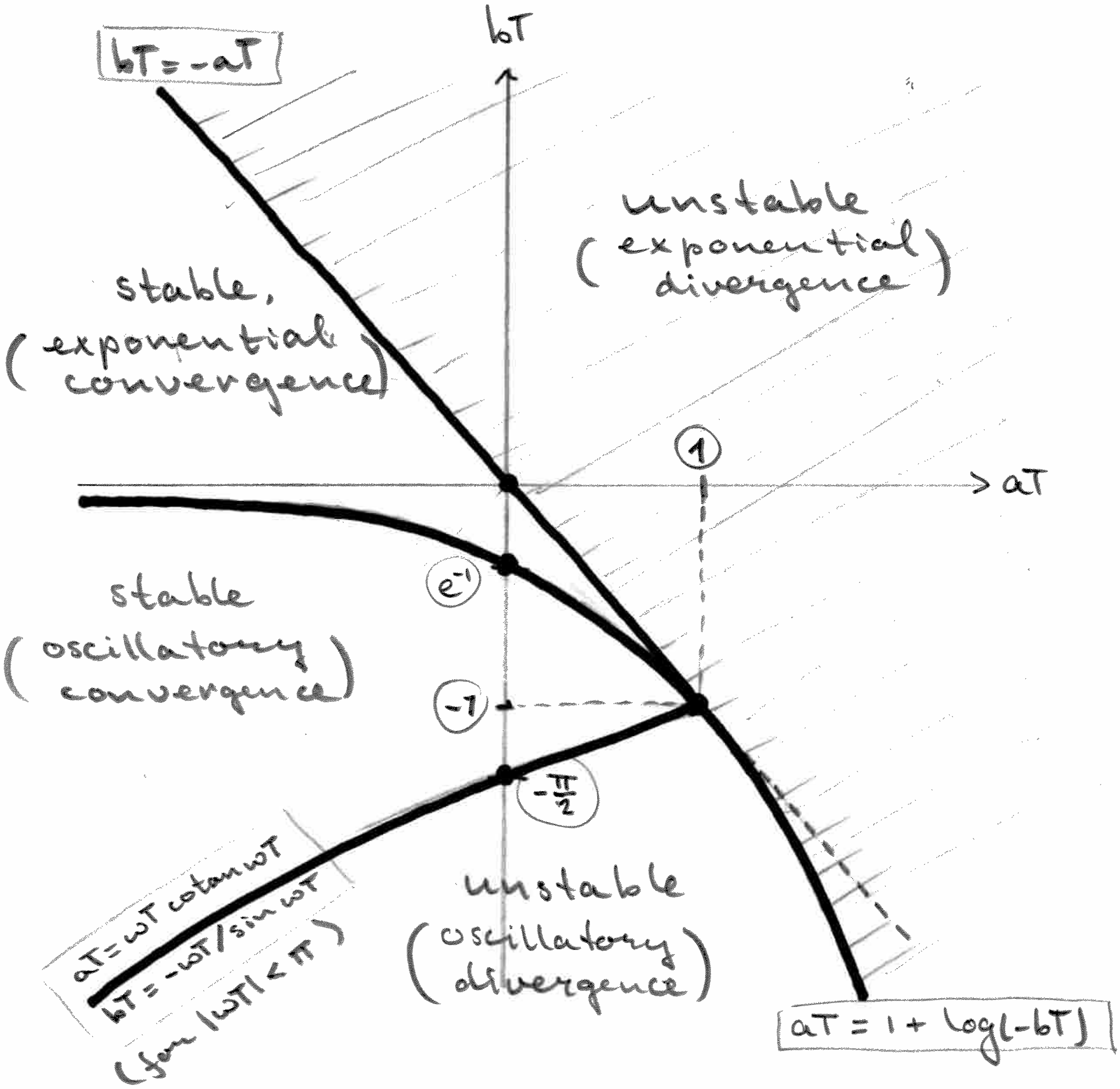
Graph of (4a) intersected with graph of (4b), case $bT<0$:

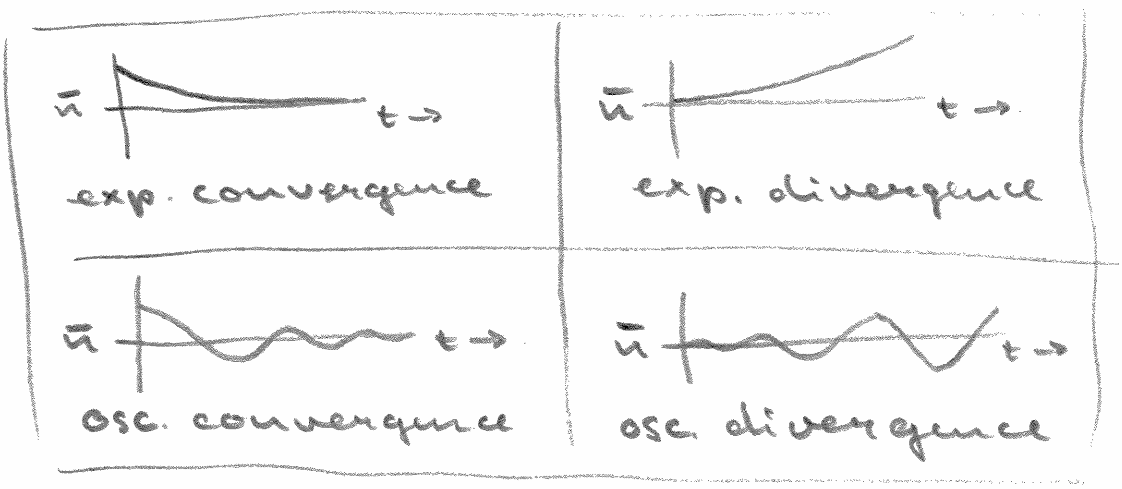
thick dots : eigenvalues.



The dominant eigenvalues (i.e., with the largest real part) are found in the strip $\boxed{|\omega T| < \pi}$

Complete linear stability bifurcation plot:





Example (prev. lecture, page 4)

$$\boxed{\dot{N} = \frac{\lambda\beta F(T)}{\gamma}(S_0 - N)N_T - \delta N} =: f(N, N_T)$$

λ : reproduction rate of territory owners.

β : territory colonization rate

δ : death rate territory owner

γ : " " free individual.

- $F(T)$: survival probability from birth to maturation.

S_0 : territory density

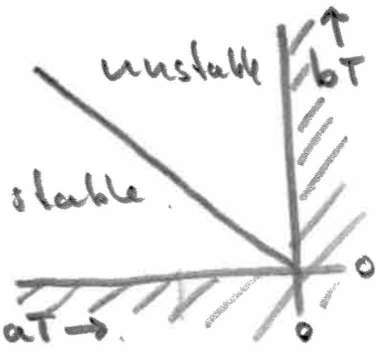
Equilibria:

$$N = 0 \quad \text{and} \quad \boxed{\bar{N} = S_0 - \frac{\delta\gamma}{\lambda\beta F(T)}} > 0.$$

$$a := \partial_1 f(\bar{N}, \bar{N}) = -\frac{\lambda\beta F(T)}{\gamma} S_0 < 0$$

$$b := \partial_2 f(\bar{N}, \bar{N}) = \delta > 0$$

Conclusion: $\bar{N}$ is stable whenever it's positive $\Leftarrow$



(Only left-upper quadrant of the bifurcation plot is feasible.)