Lineaariset mallit, Spring 2012, Exercise 5, week 17

1. Continuation of Exercise 4.3. Estimate the parameter $\boldsymbol{\beta} = \left[\boldsymbol{\beta}_1' \ \ \boldsymbol{\beta}_2'\right]'$ under the constraint $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$. What is the distribution of the constrained estimator of $\boldsymbol{\beta}$? (*Hint*: Similarly to Exercise 4.4(ii) take the constraint $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ into account in the model and estimate the parameters of the resulting model.)

2. Continuation of Exercise 4.4. Derive the F-test for the null hypothesis $\mu_1 = \mu_2$ and show that an equivalent test can be based on a test statistic with a t_{n-2}-distribution $(n = n_1+n_2)$. (*Hint*: The matrix form of the model, the general formula of the F-test statistic, and the relation between the $F_{1,n-2}$-distribution and the t_{n-2}-distribution.)

3. Consider the simple linear regression model $Y_1, ..., Y_n \perp\!\!\!\perp$, $Y_i \sim \mathsf{N}\left(\beta_1 + \beta_2 x_i, \sigma^2\right)$. Derive the F-test for the null hypothesis $\beta_2 = 0$ and show that the test statistic can be expressed as

$$F = \frac{\hat{\beta}_2^2 \sum_{i=1}^n \left(x_i - \bar{x}\right)^2}{S^2},$$

where $\bar{x} = \left(x_1 + \cdots + x_n\right)/n$ and $S^2 = (n-2)^{-1} \sum_{i=1}^n (Y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2$ with $\hat{\beta}_1$ and $\hat{\beta}_2$ the least squares estimators of β_1 and β_2, respectively. (*Hint*: The matrix form of the model and the general formula of the F-test statistic. You also need the formula for the inverse of a 2×2 matrix.)

4. Continuation of the preceding one and Exercise 2.3. (i) Show that for the model of the preceding exercise

$$\text{SSE} = \sum_{i=1}^n \left(y_i - \bar{y}\right)^2 - \hat{\beta}_2^2 \sum_{i=1}^n \left(x_i - \bar{x}\right)^2 = \left(1 - r_{xy}^2\right)(n-1)\, s_y^2,$$

where r_{xy} is the correlation coefficient computed from the observations $\left(y_i, x_i\right)$, $i = 1, ..., n$, $s_y^2 = (n-1)^{-1} \sum_{i=1}^n \left(y_i - \bar{y}\right)^2$ and $\text{SSE} = \sum_{i=1}^n (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2$. (ii) Using (i), show that the F-test of the preceding exercise is equivalent to the t-test

$$\sqrt{n-2}\, r_{xy} \Big/ \sqrt{1 - r_{xy}^2} \overset{H}{\sim} t_{n-2}.$$

For which hypothesis can you also interpret this test? (*Hint*: The identity SST = SSR + SSE (see Exercise 3.3) where SSR can be rewritten in a suitable way by using the result of Exercise 2.3. Note also that $S^2 = \text{SSE}/\left(n-2\right)$).