Lineaariset mallit, spring 2012, Exercise 2, week 13

1. The trace of a square matrix is the sum of its diagonal elements. In other words, if $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix its trace is $\mathsf{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Show that (i) $\mathsf{tr}(\mathbf{A}+\mathbf{B}) = \mathsf{tr}(\mathbf{A}) + \mathsf{tr}(\mathbf{B})$ (ii) $\mathsf{tr}(\mathbf{AB}) = \mathsf{tr}(\mathbf{BA})$ and (iii) $\mathsf{tr}(\mathbf{A}) =$ the sum of the eigenvalues of $\mathbf{A}$, when $\mathbf{A}$ is symmetric. (*Hint*: In (iii) you can use the spectral decomposition of a symmetric matrix.)

2. Suppose the square matrix $\mathbf{A}$ $(n \times n)$ is an orthogonal projector or symmetric $(\mathbf{A} = \mathbf{A}')$ and idempotent $(\mathbf{A} = \mathbf{A}^2)$. Show that $\mathbf{A}$ is positive semidefinite, that is, $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0 \; \forall \; \mathbf{x} \in \mathbb{R}^n$ and that the rank of $\mathbf{A} =$ the trace of $\mathbf{A}$ or $\mathsf{r}(\mathbf{A}) = \mathsf{tr}(\mathbf{A})$. (*Hint*: the spectral decomposition of a symmetric matrix and Exercise 1.3)

3. Present the normal equations of the simple linear regression model $Y_1, ..., Y_n \perp\!\!\!\perp$, $Y_i \sim \mathsf{N}(\beta_1 + \beta_2 x_i, \sigma^2)$ in component form (without matrices) and show that the least squares estimates (obtained as solutions of the normal equations) can be expressed as

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad \text{and} \qquad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x},$$

where, for example, $\bar{y} = (y_1 + \cdots + y_n)/n$. Furthermore, present $\hat{\beta}_2$ by using the sample standard deviations and sample correlation coefficient computed from the data (y_i, x_i), $i = 1, ..., n$.

Note: Let $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} \sim \mathsf{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ be a general linear model and $\mathsf{S}(\boldsymbol{\beta}) = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \sum_{i=1}^{n} (y_i - \mathbf{x}_i'\boldsymbol{\beta})^2$ the related sum of squares function. Then $\partial \mathsf{S}(\boldsymbol{\beta}) / \partial \boldsymbol{\beta} = -2 \sum_{i=1}^{n} \mathbf{x}_i (y_i - \mathbf{x}_i'\boldsymbol{\beta}) = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}$ and setting the last expression equal to zero yields the normal equations $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{y}$.

4. (Continuation of Exercise 1.5) (i) Derive the expressions for the least squares estimates of the parameters $\mu_1, ..., \mu_p$ by using the formula of the normal equations. (ii) Derive the expectations, variances and covariances or the expectation vector and the covariance matrix of the least squares estimators.

5. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and let $f_1, ..., f_m$ be its component functions. Define

$$\frac{\partial}{\partial \mathbf{x}'} f(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n} f_1(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} f_m(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n} f_m(\mathbf{x}) \end{bmatrix}.$$

(i) Let $\mathbf{A}$ be a fixed $m \times n$ matrix and $\mathbf{x}$ $n \times 1$ a vector. Justify carefully the rule

$$\frac{\partial}{\partial \mathbf{x}'} \mathbf{A}\mathbf{x} = \mathbf{A}.$$

(ii) Let $\mathsf{S}(\boldsymbol{\beta})$ be as in the note of Exercise 2. Show (using part (i)) that

$$\frac{\partial^2}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} \mathsf{S}(\boldsymbol{\beta}) = 2\mathbf{X}'\mathbf{X},$$

which is positive definite when $\mathbf{X}$ is of full full column rank.