Lineaariset mallit, spring 2012, Exercise 1, week 12

1. The symmetric matrix $\mathbf{A}$ $(n \times n)$ is called positive definite (denoted $\mathbf{A} > \mathbf{0}$), if $\mathbf{x}'\mathbf{A}\mathbf{x} > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$. Show that the matrix $\mathbf{A}$ $(n \times n)$ is positive definite if and only if its eigenvalues are positive. Show further that a positive definite matrix is nonsingular (or invertible). (*Hint*: You can use the spectral decomposition of a symmetric matrix.)

2. Let $\mathbf{A}$ $(n \times n)$ be a positive definite matrix and $\mathbf{B}$ $(n \times k)$ a matrix of rank k (in other words, the columns of $\mathbf{B}$ are linearly independent). Show that $\mathbf{B}'\mathbf{A}\mathbf{B}$ $(k \times k)$ is positive definite and, hence, nonsingular.

3. Let $\mathbf{A}$ $(n \times n)$ be an idempotent matrix, that is, it satisfies $\mathbf{A} = \mathbf{A}\mathbf{A}$ (the notation $\mathbf{A}\mathbf{A} = \mathbf{A}^2$ will be used). Show that $\mathbf{I}_n - \mathbf{A}$ is also idempotent and that all eigenvalues of $\mathbf{A}$ are either 0 or 1. (*Hint*: Equations defining eigenvectors.)

4. Let $\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$ and $s^2 = (n-1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$ be the sample mean and sample variance of the data set $y_1, ..., y_n$. Show that

$$(n-1)\, s^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 = \mathbf{y}' (\mathbf{I}_n - \mathbf{J})\, \mathbf{y},$$

where $\mathbf{y} = [y_1 \ \cdots \ y_n]'$ and $\mathbf{J} = \mathbf{1}_n (\mathbf{1}_n' \mathbf{1}_n)^{-1} \mathbf{1}_n'$ ($\mathbf{1}_n = [1 \cdots 1]'$, $n \times 1$). Show further that $\mathbf{J}$ (and hence $\mathbf{I}_n - \mathbf{J}$) is symmetric and idempotent (or an orthogonal projector) by showing generally that the matrix $\mathbf{P} = \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'$ is symmetric and idempotent when $\mathbf{X}$ is an $n \times p$ matrix of rank p.

5. (One-way analysis of variance) Let $Y_{11}, ..., Y_{1n_1}, Y_{21}, ..., Y_{2n_2}, ..., Y_{p1}, ..., Y_{pn_p}$ be independent with $Y_{ji} \sim \mathsf{N}(\mu_j, \sigma^2)$ $(\mu_j \in \mathbb{R},\ \sigma^2 > 0)$. Present the set up as a special case of the general linear model by using the matrix form of the linear model. What is the rank of the matrix $\mathbf{X}$?