STATISTICAL MECHANICS - EXERCISE 9

1. Let ϕ be a centered Gaussian random varible with respect to both the measures μ and ν. Let $E_\mu(\phi^2) = \sigma_\mu^2$ and $E_\nu(\phi^2) = \sigma_\nu^2$. Denote by $:\,:_\mu$ normal ordering with respect to the measure μ. Show that

$$:\phi^n:_\mu = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^m m!(n-2m)!} :\phi^{n-2m}:_\nu (\sigma_\nu^2 - \sigma_\mu^2)^m. \tag{1}$$

Solution: The arguments are essentially the same as in the problems last week. Using the definition of normal ordering, we can write

$$:e^{\lambda\phi}:_\mu = e^{\lambda\phi - \frac{1}{2}\sigma_\mu^2\lambda^2} =: e^{\lambda\phi}:_\nu e^{\frac{\lambda^2}{2}(\sigma_\nu^2 - \sigma_\mu^2)} = \sum_{n=0}^\infty \frac{\lambda^n}{n!} :\phi^n:_\nu \sum_{m=0}^\infty \frac{\lambda^{2m}}{2^m m!}(\sigma_\nu^2 - \sigma_\mu^2)^m. \tag{2}$$

Writing out the product of the two series and equating terms of same order on each sides gives the desired result.

2. Let ϕ and ψ be centered Gaussian random variables. Show that

$$E(:\phi^n::\psi^m:) = \delta_{nm} n! E(\phi\psi)^n. \tag{3}$$

Solution: Using the definition of normal ordering, we have

$$\begin{aligned} :e^{\lambda\phi}::e^{\mu\psi}: &= e^{\lambda\phi+\mu\psi} e^{-\frac{\lambda^2}{2}E(\phi^2) - \frac{\mu^2}{2}E(\psi^2)} \\ &=: e^{\lambda\phi+\mu\psi}: e^{\frac{1}{2}E((\lambda\phi+\mu\psi)^2) - \frac{\lambda^2}{2}E(\phi^2) - \frac{\mu^2}{2}E(\psi^2)} \\ &=: e^{\lambda\phi+\mu\psi}: e^{\lambda\mu E(\phi\psi)}. \end{aligned}$$

Recall that normal ordering was defined so that for any random variable V, $E(:e^V:) = 1$. Thus

$$E(:e^{\lambda\phi}::e^{\mu\psi}:) = e^{\lambda\mu E(\phi\psi)}. \tag{4}$$

The result now follows from expanding the exponentials and equating terms of the same order.

3. Consider the model where the Fourier transform of the covariance is $\frac{\chi(\frac{p}{\Lambda})}{p^2}$ and $\chi(p) = e^{-p^2}$. Consider the Feynman graph with four external legs, two vertices and one internal loop (see page 63 in the lecture notes - the graph labeled by $2m = 4$, $N = 2$). Show that for $d < 4$, the value of this graph is bounded as $\Lambda \to \infty$, for $d = 4$, it diverges logarithmically - its value goes like $\log \Lambda$ and for $d > 4$ it behaves like Λ^{d-4}.

Remark: In the original formulation, there was a $p^2 + r$ instead of p^2. This is not the model we have been considering in the lectures so the current version is more relevant to our course.

Solution: This is very similar to exercise 6.5. Finding the asymptotic behavior of this is equivalent to estimating the integral

$$
\begin{aligned}
I(q) &= \int_{\mathbb{R}^d} dp \frac{1}{p^2} \frac{1}{(q-p)^2} e^{-\frac{p^2}{\Lambda^2}} e^{-\frac{(p-q)^2}{\Lambda^2}} \\
&= \Lambda^d \int_{\mathbb{R}^d} dp \frac{1}{\Lambda^2 p^2} \frac{1}{(q-\Lambda p)^2} e^{-p^2} e^{-\left(p-\frac{q}{\Lambda}\right)^2} \\
&= \Lambda^{d-4} \int_{\mathbb{R}^d} \frac{1}{p^2} \frac{1}{(p-q\Lambda^{-1})^2} e^{-p^2} e^{-\left(p-\frac{q}{\Lambda}\right)^2}.
\end{aligned}
$$

Since the integral is cut off at large p, we see that the only singular behavior comes from around $p = 0$. We see that we are then dealing with precisely the same integral as in exercise 6.5. We see that for $d \geq 5$, $I(q) = \Lambda^{d-4}\mathcal{O}(1)$. For $d = 4$, $I(q) \sim -\Lambda^0 \log|q\Lambda^{-1}| \sim \log|\Lambda|$. For $d < 4$, we have $I(q) \sim \Lambda^{d-4}|q\Lambda^{-1}|^{d-4} = \mathcal{O}(1)$.

4. Consider a translation invariant kernel $K(x_1, x_2, x_3, x_4)$ and the potential

$$
V = \int_{(\mathbb{R}^d)^4} K(x_1, x_2, x_3, x_4) : \prod_{i=1}^{4} \phi(x_i) : dx_1 dx_2 dx_3 dx_4, \tag{5}
$$

where K is such that $V \in \mathcal{K}_\lambda$. Show that we can write this as $a \int : \phi(x)^4 : dx + \tilde{V}$, where $\tilde{V} \in \mathcal{K}_{\frac{\lambda}{2}}$ is less relevant than the ϕ^4 term.

Remark: In the original statement of the problem, the claim was that the perturbation was irrelevant, but the more accurate statement is that it is less relevant than the ϕ^4 term.

Solution: Abusing notation slightly, we use translation invariance to write $K(x_1, x_2, x_3, x_4) = K(x_2 - x_1, x_3 - x_1, x_4 - x_1)$. Writing K in terms of its Fourier transfom, we have

$$
K(x, y, z) = \int_{(\mathbb{R}^d)^3} \hat{K}(p_1, p_2, p_3) e^{ip_1 \cdot x + ip_2 \cdot y + ip_3 \cdot z} dp_1 dp_2 dp_3. \tag{6}
$$

We then write $\hat{K}$ in the following way:

$$
\hat{K}(p_1, p_2, p_3) = \hat{K}(0, 0, 0) + \int_0^1 \frac{d}{dt} \hat{K}(tp_1, tp_2, tp_3) dt. \tag{7}
$$

Thus

$$
\begin{aligned}
K(x, y, z) &= \int_{(\mathbb{R}^d)^3} \hat{K}(0, 0, 0) e^{ip_1 \cdot x + ip_2 \cdot y + ip_3 \cdot z} dp_1 dp_2 dp_3 \\
&+ \int_{(\mathbb{R}^d)^3} \int_0^1 \frac{d}{dt} \hat{K}(tp_1, tp_2, tp_3) dt e^{ip_1 \cdot x + ip_2 \cdot y + ip_3 \cdot z} dp_1 dp_2 dp_3 \\
&= \hat{K}(0, 0, 0) \delta(x) \delta(y) \delta(z) + \int_{(\mathbb{R}^d)^3} \int_0^1 \left(\sum_i p_i \partial_i \hat{K} \right) (tp_1, tp_2, tp_3) dt e^{ip_1 \cdot x + ip_2 \cdot y + ip_3 \cdot z} dp_1 dp_2 dp_3.
\end{aligned}
$$

The first term gives (recall that $\hat{f}(0) = \int f$)

$$
\hat{K}(0, 0, 0) \int_{(\mathbb{R}^d)^4} \delta(x_2 - x_1, x_3 - x_1, x_4 - x_1) : \prod_{i=1}^{4} \phi(x_i) : dx_1 dx_2 dx_3 dx_4 = \left(\int_{(\mathbb{R}^d)^3} K \right) \int_{\mathbb{R}^d} : \phi(x)^4 : dx, \tag{8}
$$

which is of the form we want. It remains to show that the second term is less relevant and is finite in the norm we claimed.

Consider now the second term

$$\sum_i \int_{(\mathbb{R}^d)^3} p_i \int_0^1 (\partial_i \hat{K})(tp_1, tp_2, tp_3) dt e^{ip_1 \cdot x + ip_2 \cdot y + ip_3 \cdot z} dp_1 dp_2 dp_3. \tag{9}$$

Let us define the functions H_i so that

$$\hat{H}_i(p_1, p_2, p_3) = \int_0^1 (\partial_i \hat{K})(tp_1, tp_2, tp_3) dt \tag{10}$$

and we have

$$\tilde{V} = \sum_i \int_{(\mathbb{R}^d)^4} \prod_j dx_j : \prod_j \phi(x_j) : \int_{(\mathbb{R}^d)^3} dp_1 dp_2 dp_3 e^{ip_1 \cdot (x_2 - x_1) + ip_2 \cdot (x_3 - x_1) + ip_3 \cdot (x_4 - x_1)} p_i \hat{H}_i(p_1, p_2, p_3). \tag{11}$$

Note that a p in Fourier space corresponds to $-i\frac{\partial}{\partial x}$ in real space so

$$\tilde{V} = \sum_i \int_{(\mathbb{R}^d)^4} \prod_j dx_j : \prod_j \phi(x_j) : (-i)(\partial_i H_i)(x_2 - x_1, x_3 - x_1, x_4 - x_1). \tag{12}$$

Moreover, $(\partial_i H_i)(x_2 - x_1, x_3 - x_1, x_4 - x_1) = \partial_{i+1} H_i(x_2 - x_1, x_3 - x_1, x_4 - x_1)$. Thus integrating by parts (assuming the H_i decay fast enough that no boundary terms are produced - this might follow from our definitions, but still should be checked if one wants to be rigorous) we find

$$\tilde{V} = i \sum_{i=1}^{3} \int_{(\mathbb{R}^d)^4} \prod_j dx_j : \frac{\partial}{\partial x_{i+1}} \prod_j \phi(x_j) : H_i(x_2 - x_1, x_3 - x_1, x_4 - x_1). \tag{13}$$

So we have succeeded in at least formally writing $\tilde{V}$ in a form we wished and since a derivative acting on the field ϕ always comes with a factor of L^{-1} in the renormalization map, we see that this indeed looks to be less relevant than the ϕ^4 term. What we still need to check is that this indeed is of the form we wish, i.e. that the kernel H is in some space $\mathcal{K}_{\lambda'}$.

The norm estimate is essentially the same as the one in the lecture notes on page 99. We note that ∂_k in Fourier space corresponds to multiplying by ix_k in real space. One then rescales the integration variable in the Fourier transform by t and then does the inverse scaling in the x integral. Then with some elementary estimates, one gets $||\tilde{V}||_{\frac{\lambda}{2}} \leq C(\lambda)||V||_\lambda$.