STATISTICAL MECHANICS - EXERCISE 8

1. Let ϕ be a centered Gaussian random variable of variance σ^2.

a) Write $:\phi^n:$ in terms of ϕ and σ.
b) Try to invert the formula you got, i.e. write ϕ^n in terms of $:\phi^m:$ and σ.

Solution: a) By the definition of normal ordering, $:e^{\lambda\phi}:= e^{\lambda\phi}e^{-\frac{1}{2}\lambda^2\sigma^2}$. Plugging in the power series representations of the functions we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}:\phi^n:\lambda^n=\sum_{n=0}^{\infty}\frac{1}{n!}\phi^n\lambda^n\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-\sigma^2}{2}\right)^m\lambda^{2m}.$$

Recall that the product of two series (say when both are absolutely convergent) $\sum_{n=0}^{\infty}a_n\sum_{m=0}^{\infty}b_m=\sum_{n=0}^{\infty}c_n$, where $c_n=\sum_{k=0}^{n}a_{n-k}b_k$. Let us write $a_n=\frac{1}{n!}\phi^n\lambda^n$ and $b_m=0$ when m is odd and $b_{2m}=\frac{1}{m!}\left(-\frac{\sigma^2}{2}\right)^m\lambda^{2m}$. We certainly satisfy the absolute convergence criterion so we can use the above formula to calculate the product of the series. For the convolution, we find

$$\begin{aligned}c_n&=\sum_{k=0}^{n}a_{n-k}b_k\\&=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}a_{n-2k}b_{2k}\\&=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{(n-2k)!}\phi^{n-2k}\frac{1}{k!}\left(-\frac{\sigma^2}{2}\right)^k\lambda^n.\end{aligned}$$

Equating the coefficients of the λ^n terms on both sides, we find that

$$:\phi^n:=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!}\phi^{n-2k}\left(-\frac{\sigma^2}{2}\right)^k. \tag{1}$$

b) This part is essentially identical: write $e^{\lambda\phi}=:\,:e^{\lambda\phi}:e^{\frac{1}{2}\lambda^2\sigma^2}$ and multiply out the series. The result is

$$\phi^n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{(n-2k)!k!}:\phi^{n-2k}:\left(\frac{\sigma^2}{2}\right)^k. \tag{2}$$

2. Continuing from the previous problem; if $\sigma=1$, show that $:\phi^n:=H_n(\phi)$, where H_n is the nth Hermite Polynomial.

Solution: By the definition of normal ordering, we have for general σ

$$:e^{\lambda\phi}:=e^{\lambda\phi-\frac{1}{2}\lambda^2\sigma^2}. \tag{3}$$

Writing this slightly differently, we have

(4) $$\sum_{n=0}^{\infty} \frac{(\lambda\sigma)^n}{n!} : \left(\frac{\phi}{\sigma}\right)^n := e^{\lambda\sigma\frac{\phi}{\sigma} - \frac{1}{2}(\lambda\sigma)^2}.$$

If we write $t = \lambda\sigma$ and $\psi = \frac{\phi}{\sigma}$, we have

(5) $$\sum_{n=0}^{\infty} \frac{t^n}{n!} : \psi^n := e^{t\psi - \frac{1}{2}t^2}.$$

We note that the (probabilist's) Hermite polynomials can be defined through the generating function:

(6) $$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{tx - \frac{t^2}{2}}.$$

So we conclude that

(7) $$: \psi^n := H_n(\psi),$$

and for $\sigma = 1$ this becomes $: \phi^n := H_n(\phi)$.

Remark: In the original formulation of the problem, there was a problem where one was asked to show that the partition function is a martingale. While this is correct at least morally, it is not exactly correct for our representation of the field. For the correct statement, one would have to introduce yet another representation of the hierarchical field - a branching random walk. The main difference is that we look at the field from a sort of inverse point of view: the term corresponding to the scale of the entire box - $a_{n,\{0,...,2^n-1\}^2}$ in our formulation below - is taken to be independent of n. When one looks at the problem from this point of view (more in the spirit of taking a continuum limit or taking a square of fixed size and partitioning it into finer and finer discretizations) there is a certain self similarity that allows powerful martingale arguments. This being said, we shall not delve more into the branching random walk since it would require a fair amount of notation and would not be so relevant to the rest of our discussion.

3. Consider the two-dimensional hierarchical field from a slightly different point of view: consider the square $\{0, ..., 2^n - 1\}^2$ and let $\mathcal{B}_k$ be the collection of squares in $\mathbb{Z}^2$ with side length 2^k and $\mathcal{BD}_k$ subset of $\mathcal{B}_k$ consisting of squares of the form $\{0, ..., 2^k - 1\}^2 + (i, j)2^k$ for some integers i, j. Also for $x \in \mathbb{Z}^2$, let $\mathcal{B}_k(x)$ consist of squares $B \in \mathcal{B}_k$ so that $x \in B$ and define $\mathcal{BD}_k(x)$ similarly.

Let $\{a_{k,B}\}_{k\geq 0, B\in\mathcal{BD}_k}$ be a family of i.i.d. standard Gaussians. We then define the field

(8) $$\phi_n(z) = \sum_{k=0}^{n} \sum_{B\in\mathcal{BD}_k(z)} a_{k,B}.$$

a) Calculate the covariance of the field ϕ_n

b) Consider then a modification of our field ϕ_n. Let $\mathcal{B}_k^n$ be the subset of $\mathcal{B}_k$ where the squares have the property that their lower left corner is in $\{0, ..., 2^n - 1\}^2$. Let $\{b_{k,B}\}_{k\geq 0, B\in\mathcal{B}_k^n}$ be a family of independent centered Gaussian random variables and $b_{k,B}$ having variance 2^{-2k}. Then define $b_{k,B}^n = b_{k,B}$ if $B \in \mathcal{B}_k^n$ and $b_{k,B}^n = b_{k,B'}$ if B' is a translate of B by $(i, j)2^n$ for some integers i, j. Then define

(9) $$\psi_n(z) = \sum_{k=0}^{n} \sum_{B\in\mathcal{B}_k(z)} b_{k,B}^n.$$

Calculate the covariance $\langle \psi_n(z)\psi_n(w)\rangle$ and estimate it in the limit of large $d_n(z,w)$ (here d_n is the distance on the torus). Compare ψ and ϕ. Note that the complications with the translations are there only to make the field periodic so that the covariance is simpler.

Solution: a) We have

$$\begin{aligned}
\langle \phi_n(z)\phi_n(w)\rangle &= \sum_{k=0}^{n}\sum_{k'=0}^{n}\sum_{B\in\mathcal{BD}_k(z)}\sum_{B'\in\mathcal{BD}_{k'}(w)} E(a_{k,B}a_{k',B'})\\
&= \sum_{k=0}^{n}\sum_{k'=0}^{n}\sum_{B\in\mathcal{BD}_k(z)}\sum_{B'\in\mathcal{BD}_{k'}(w)} \delta_{k,k'}\delta_{B,B'}\\
&= \sum_{k=0}^{n}\sum_{B\in\mathcal{BD}_k(z)}\sum_{B'\in\mathcal{BD}_k(w)} \delta_{B,B'}.
\end{aligned}$$

Note that $\mathcal{BD}_k(z)$ always contains only a single element (the unique square of sidelength 2^k of the form $\{0,...,2^k-1\}^2+2^k(i,j)$ containing the point z). So the sum over B and B' is equal to one, if there is such a box containing both of the points and it is equal to zero if there is not.

This can also be interpreted in terms of tree structure: we have a tree of height n and at each level k there is branching into 4 directions. We can identify each vertex at level k with a square of size $2^k\times 2^k$ (the tree picture is probably easiest to grasp if you consider the $d=1$ case and you're dealing with a binary tree). The covariance in this case becomes simply a function of the distance between points in the ultrametric distance - the number of generations the points differ from each other in (again drawing a picture in the $d=1$ case may clarify this).

b) For $x,y\in\mathbb{Z}^2$, $x=(x_1,x_2)$ and $y=(y_1,y_2)$ let $t_i(x,y)=\min(|x_i-y_i|,|x_i-y_i-2^n|,|x_i-y_i+N|)$ (i.e. the minimum amount the i coordinates differ in the periodic distance). We then have

$$\begin{aligned}
\langle \psi_n(x)\psi_n(y)\rangle &= \sum_{k,k'=0}^{n}\sum_{B\in\mathcal{B}_k(x)}\sum_{B'\in\mathcal{B}_{k'}(y)} E(b^n_{k,B}b^n_{k',B'})\\
&= \sum_{k,k'=0}^{n}\sum_{B\in\mathcal{B}_k(x)}\sum_{B'\in\mathcal{B}_{k'}(y)} 2^{-2k}\delta_{k,k'}\mathbf{1}(B=B'+(i,j)2^k).
\end{aligned}$$

Now the number of squares of side length 2^k containing both x and y (and taking into account the periodicicty) is simply $(2^k-t_1(x,y))(2^k-t_2(x,y))$. Noting that $k=\lceil\log_2(d^n_\infty(x,y)+1)\rceil$ (where d^n_∞ is the sup distance with periodicity taken into account) is the minimum scale where we can even have a box containing both points, we find that

$$\begin{aligned}
\langle \psi_n(x)\psi_n(y)\rangle &= \sum_{k=\lceil\log_2(d^n_\infty(x,y)+1)\rceil}^{n} 2^{-2k}(2^k-t_1(x,y))(2^k-t_2(x,y))\\
&= \sum_{k=\lceil\log_2(d^n_\infty(x,y)+1)\rceil}^{n}\left(1-\frac{t_1(x,y)}{2^k}-\frac{t_2(x,y)}{2^k}+\frac{t_1(x,y)t_2(x,y)}{4^k}\right).
\end{aligned}$$

This is as closed a form of the covariance as we can get. Comparing with the purely hierarchical model, we note that this is something between the hierarchical model and the actual free field. This has a similar rather explicit construction as the hierarchical model, but it has the benefit of having a translation invariant covariance. On the other hand, the covariance and the field it self is still rather complicated and one can expect that proving non-trivial things for it will be complicated.

For asymptotic estimates, one can use inequalities such as $a+b-ab \geq 0$ for $0 \leq a, b \leq 1$ and $a+b-ab \leq a+b$ to estimate the covariance and find that there is a constant C (independent of n) so that

$$|\langle \psi_n(x)\psi_n(y)\rangle - (n - \log_2 d^n(x,y))| \leq C, \tag{10}$$

where d^n is the periodic Euclidean distance. This in fact means that in some sense ψ_n is just as good a 'discrete 2-dimensional Gaussian free field' as any other definition. This is because taking a continuum limit of this correlation function will lead to the correlation function of the 2-dimensional continuum Gaussian free field.

As a recap, the field ψ_n is similar to ϕ_n as it's constructed in a rather similar manner, but it is different in that it has a translation invariant covariance and its continuum limit should be the actual 2-dimensional Gaussian free field.