STATISTICAL MECHANICS - EXERCISE 7

1. Consider a block spin argument for a system with Hamiltonian $H(\phi) = (\phi, (-\Delta + \Delta^2)\phi)$. Show that this converges to the Gaussian fixed point.

Solution: The Fourier transform of the correlation function is

$$\langle \phi(x)\phi(y)\rangle = \int_{[-\pi,\pi]^d} dp \frac{e^{ip\cdot(x-y)}}{\mu(p)(\mu(p)+1)}$$

Applying the coarse graining procedure as in the lecture notes, we have

$$\langle C_L^n \phi(x) C_L^n \phi(y)\rangle = L^{-2nd} \sum_{u,v} L^{-2n} \int_{[-L^n\pi, L^n\pi]^d} dp \frac{e^{ip\cdot(x-y+L^{-n}(u-v))}}{\mu(L^{-n}p)(1+\mu(L^{-n}p))}.$$

We note that $L^{2n}\mu(L^{-n}p) \to p^2$ as in the notes and $1 + \mu(L^{-n}p) \to 1$ so we see that indeed the limit of this is the Gaussian fixed point.

Remark: I'm not quite sure about the approach to the next problem or the formulation of the statement. Perhaps it is correct the way it is, but I am missing something in the calculation or misunderstood something. I think that at least morally, the idea should be to Fourier transform, show that the Fourier transform is a product of two things. The zeroes of one cancel the singularities of the other and the entire Fourier transform is analytic and we have exponential decay. To me it seems D_0 is not translation invariant so even Fourier transforming (a one-dimensional Fourier transform that is) isn't possible as it is. If you notice I'm missing something or if you come up with something else, feel free to contact me. Even though, let us look at some of the type of arguments and issues I think might be involved in a proof.

2. Let $G_0 = (-\Delta)^{-1}$ with periodic boundary conditions (period L^N with the $p = 0$ mode removed from the Fourier expansion), C the block spin map and $G_1 = CG_0C^T$. Show that for $D_0 = G_0 C^T G_1^{-1}$, $(D_0)_{x,y}$ decays exponentially in $|x - y|$.

Solution: One way to show that a function $f(x)$ has exponential decay (i.e. that there are some positive α and C so that for large enough x, $|f(x)| \leq Ce^{-\alpha|x|}$) is to show that its Fourier transform is analytic. This argument holds (at least a contour integral proof for it) only for functions with continuous variables so we must agree first what we mean by exponential decay - do we mean it in the $L \to \infty$ limit or do we mean that the correlation length is a function of L-finite for each finite L, but might not have a finite limit as $L \to \infty$. If we take the interpretation that we are interested in finte L, we must note that we have periodic boundary conditions and we should consider distances smaller than the period. Moreover, we have to note that singularities correspond to things exploding as $L \to \infty$.

Without worrying too much about the interpretatoin, let us write stuff in Fourier language.

$$(G_0)_{x,y} = \frac{1}{L^{Nd}} \sum_{p \in \left(\frac{2\pi}{L^N}\mathbb{Z}\right)^d : p\neq 0, |p_i|<\pi} \frac{1}{\mu(p)} e^{ip\cdot(x-y)}. \tag{1}$$

Let us consider how the block spin map acts on an element in the Fourier basis: (write $B_L(0) = \{y : |y_i| < \frac{L}{2}\}$)

$$Ce^{ip\cdot x} = L^{\frac{a}{2}-d} \sum_{u\in B_L(0)} e^{ip\cdot(Lx+u)} = L^{\frac{a}{2}-d} e^{ip\cdot(Lx)} \sum_{u\in B_L(0)} e^{ip\cdot u} = L^{\frac{a}{2}-d} e^{ip\cdot(Lx)} f(p), \tag{2}$$

where $f(p) = \prod_{i=1}^d \frac{\sin\frac{Lp_i}{2}}{\sin\frac{p_i}{2}}$. We then have (note that the C acts on the $e^{ip\cdot x}$ and C^T on the $e^{-ip\cdot y}$)

$$(G_1)_{x,y} = \frac{1}{L^{Nd}} L^{a-2d} \sum_{p\in\left(\frac{2\pi}{L^N}\mathbb{Z}\right)^d : p\neq 0, |p_i|<\pi} \frac{1}{\mu(p)} e^{iLp\cdot(x-y)} f(p)^2. \tag{3}$$

Let us rescale the summation variable: $p' = Lp$ -

$$(G_1)_{x,y} = \frac{L^{a-3d}}{L^{(N-1)d}} \sum_{p'\in\left(\frac{2\pi}{L^{N-1}}\mathbb{Z}\right)^d : p'\neq 0, |p_i'|<L\pi} \frac{1}{\mu\left(\frac{p'}{L}\right)} f\left(\frac{p'}{L}\right)^2 e^{ip'\cdot(x-y)}.$$

We want to write this as a Fourier transform, but we are now summing over p' on a scale $L\pi$ instead of π. To do this, we split the sum over $[-L\pi, L\pi]^d$ into a sum over boxes of size $[-\pi,\pi]^d$:

$$(G_1)_{x,y} = \frac{L^{a-3d}}{L^{(N-1)d}} \sum_{p'\in\left(\frac{2\pi}{L^{N-1}}\mathbb{Z}\right)^d : p'\neq 0, |p_i'|<\pi} \sum_{M\in\mathbb{Z}^d : |M_i|<\frac{L}{2}} \frac{1}{\mu\left(\frac{p'+2\pi M}{L}\right)} f\left(\frac{p'+2\pi M}{L}\right)^2 e^{i(p'+2\pi M)\cdot(x-y)} \tag{4}$$

As x, y and M are in $\mathbb{Z}^d$, $e^{2\pi i M\cdot(x-y)} = 1$ and we find

$$(G_1)_{x,y} = \frac{L^{a-3d}}{L^{(N-1)d}} \sum_{q\in\left(\frac{2\pi}{L^{N-1}}\mathbb{Z}\right)^d : q\neq 0, |q_i|<\pi} e^{iq\cdot(x-y)} \sum_{M\in\mathbb{Z}^d : |M_i|<\frac{L}{2}} \frac{1}{\mu\left(\frac{q+2\pi M}{L}\right)} f\left(\frac{q+2\pi M}{L}\right)^2 \tag{5}$$

and we have written $(G_1)_{x,y}$ in the form of a Fourier transform (note that while G_0 is L^N periodic, the coarse graining procedure essentially changes the scale on which we look at things by a factor of L. Thus G_1 is L^{N-1} periodic and the natural scale Fourier modes live on is $L^{-(N-1)}$ so this indeed is 'the' Fourier transform of G_1).

$$\hat{G}_1(p) = L^{a-3d} \sum_{M\in\mathbb{Z}^d : |M_i|<\frac{L}{2}} \frac{1}{\mu\left(\frac{p+2\pi M}{L}\right)} f\left(\frac{p+2\pi M}{L}\right)^2. \tag{6}$$

If we are considering singularities purely in p (namely we don't call things blowing up with L singularities), this can have singularities only in the $M = 0$ block in which case we have for $p \to 0$

$$L^{a-3d} \frac{1}{\mu\left(\frac{p}{L}\right)} f\left(\frac{p}{L}\right) \sim L^{a-3d} \frac{L^2}{p^2} L^{2d} = \frac{1}{p^2} L^{a-d+2}. \tag{7}$$

Note that if $a - d + 2 = 0$, we have a singularity of the form p^{-2} (which one would expect from a covariance in a Gaussian type model), but it is not clear that there are no other singularities in the $L \to \infty$ limit.

My interpretation is that one would then like to calculate the Fourier transform of G_0C^T and note that the Fourier transform of $G_0C^TG_1^{-1}$ is the product of the Fourier transform of G_0C^T and that of G_1^{-1}. The hope would then be to show that the Fourier transform of G_0C^T has a pole (of max order 2) at $p = 0$ which is exactly cancelled by the (second order) zero of $\hat{G}_1^{-1}$ at $p = 0$. The problem is of course that $(G_0C^T)_{x,y}$ is not a function of $x - y$ and the Fourier transform does not exist (in the way we wish to use it - one could of course transform in both x and y or something else).

Remark: There were some errors in the original formulation of the problem. They should be corrected here.

3. The aim of this problem is to consider the central limit theorem from the point of view of the renormalization group. We shall not consider the CLT in it's full generality to make our RG approach simpler

Consider 2^n independent identically distributed real random variables Q_i . Let us assume that the distribution of Q_i has a density $\rho(q)$ and we assume that $\rho(q)$ has 'enough' regularity - if you are interested in the regularity assumptions we need, think about what regularity we need at each step. Moreover, we assume that Q_i are centered ($\int q\rho(q)dq = 0$) and unit variance $\int q^2\rho(q)dq = 1$).

The central limit theorem states that $2^{-\frac{n}{2}}\sum_i Q_i$ converges to a standard Gaussian. Consider now the distribution of $\sum_i Q_i$. THis is given by the convolution of the distributions of Q_i. Following the RG-philosohy, one would like to calculate this convolution in steps - at the first step, pair up the distributions and calculate the convolutions of these pairs. At the next step, pair up again and convolve. Following this idea, introduce the following mapping acting on densities:

$$(\mathcal{T}_\lambda\rho)(q) = \lambda\int_{\mathbb{R}}\rho(q')\rho(\lambda q - q')dq' \tag{8}$$

.

What we would morally like to do is to show that $\mathcal{T}_\lambda^n\rho$ converges to a Gaussian distribution as $n \to \infty$ if we pick λ correctly.

a) Let $\hat{\rho}$ be the Fourier transform of ρ and $w = \log\hat{\rho}$. With some abuse of notation, lift $\mathcal{T}_\lambda$ to act on w, i.e. define $\mathcal{T}_\lambda w = \log\widehat{\mathcal{T}_\lambda\rho}$. Show that when acting on w, $\mathcal{T}_\lambda$ is linear.

b) Assuming we can expand w as a series around 0, describe the fixed points of $\mathcal{T}_\lambda$ (when acting on functions of the form w).

c) Study the eigenfunctions and eigenvalues of $\mathcal{T}_\lambda$ (again when acting on the functions w) and try to discuss the stability of different eigenfunctions and how they are related to the central limit theorem.

Solution: a) Plugging in$\rho(q) = \frac{1}{2\pi}\int dke^{ikq}\hat{\rho}(k)$, the definition of $\mathcal{T}_\lambda$ becomes

$$\begin{aligned}(\mathcal{T}_\lambda\rho)(q) &= \frac{\lambda}{(2\pi)^2}\int dq'dk_1dk_2e^{ik_1q'+ik_2(\lambda q-q')}\hat{\rho}(k_1)\hat{\rho}(k_2)\\ &= \frac{\lambda}{2\pi}\int dk_1dk_2e^{ik_2\lambda q}\delta(k_1-k_2)\hat{\rho}(k_1)\hat{\rho}(k_2)\\ &= \frac{1}{2\pi}\int dke^{ikq}\hat{\rho}\left(\frac{k}{\lambda}\right)^2.\end{aligned}$$

Thus $\widehat{\mathcal{T}_\lambda \rho}(k) = \hat{\rho}\left(\frac{k}{\lambda}\right)^2$ and $\mathcal{T}_\lambda w(k) = 2w\left(\frac{k}{\lambda}\right)$ which is indeed a linear map.

b) Let w^* be a fixed point of $\mathcal{T}_\lambda$. By a), this means that for each $k \in \mathbb{R}$,

$$w^*(k) = 2w^*\left(\frac{k}{\lambda}\right). \tag{9}$$

Assuming w^* can be expanded as a power series around $k = 0$, this becomes

$$\sum_{j=0}^{\infty} \frac{w_j}{j!} k^j = \sum_{j=0}^{\infty} 2\frac{w_j}{j!} \lambda^{-j} k^j \tag{10}$$

which means that for each j, we must have $w_j = 2\lambda^{-j} w_j$. This implies that either $w_j = 0$ for all j or there is a single j for which $2 = \lambda^j$ and $w_j = 0$ for other j.

Let us investigate how w_j are actually related to the density ρ. To do this, we consider the moments of ρ:

$$\begin{aligned}
\int q^l \rho(q) dq &= \int dq q^l \int \frac{dk}{2\pi} e^{ikq} \hat{\rho}(k) \\
&= \int dq \int \frac{dk}{2\pi} \left(\left(\frac{1}{i}\frac{d}{dk} \right)^l e^{ikq} \right) e^{w(k)} \\
&= \int dq \int \frac{dk}{2\pi} e^{ikq} \left(-\frac{1}{i}\frac{d}{dk} \right)^l e^{w(k)} \\
&= \int dk \delta(k) \left(-\frac{1}{i}\frac{d}{dk} \right)^l e^{w(k)} \\
&= \left(-\frac{1}{i}\frac{d}{dk} \right)^l e^{w(k)} \Bigg|_{k=0}.
\end{aligned}$$

Here we used some tricks such as integration by parts which requires some regularity from ρ, but we won't focus on that.

First of all ρ is the density of a probability measure, so $\int \rho = 1$. This implies that $e^{w(0)} = 1$ so $w(0) = w_0 = 0$. Secondly, we assumed ρ to be centered: $\int q\rho(q)dq = 0$. This implies that $w'(0)e^{w(0)} = 0$ so we want $w'(0) = w_1 = 0$. Finally we wanted unit variance. Thus $\int q^2 \rho(q) dq = 1$ and $-(w''(0)e^{w(0)} + (w'(0))^2 e^{w(0)}) = -w''(0) = -w_2 = 1$. So we see that for the fixed point, the only possibility is that $\lambda^2 = 2$, $w_2 = -1$ and $w_j = 0$ for all other j, i.e. $w(k) = -\frac{1}{2}k^2$ and $\hat{\rho}(k) = e^{-\frac{1}{2}k^2}$. Inverting the Fourier transform

$$\rho(q) = \int \frac{dk}{2\pi} e^{ikq} e^{-\frac{1}{2}k^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}q^2}, \tag{11}$$

so we see that the only possible fixed point satisfying our asssumptions (and some decent amount of regularity) is the standard normal distribution.

Note that if we didn't assume that the distribution is centered, we could have gotten something with $w_1 \neq 0$. This would be in the scope of the law of large numbers instead of the central limit theorem.

c) Consider now w^* - the Gaussian fixed point and consider a perturbation to this: δw. Let us assume that δw can be expanded as a power series around zero and that $w^* + \delta w$ corresponds to a ρ of the form we have been considering - i.e. if $\delta w(k) = \sum_{j=0}^{\infty} \frac{1}{j!}\delta w_j k^j$, then $\delta w_0 = \delta w_1 = \delta w_2 = 0$. We then have

$$(\mathcal{T}_\lambda(w^* + \delta w))(k) = w^*(k) + \sum_{j=3}^{\infty} \frac{1}{j!} 2^{1-\frac{j}{2}} \delta w_j k^j. \tag{12}$$

From the point of view of eigenvalues and eigenvectors, k^j is an eigenvector with eigenvalue $2^{1-\frac{j}{2}}$. For $j \geq 3$, these eigenvalues are all less than one (and positive) so we see that any such perturbation δw is irrelevant in the RG-point of view: $\mathcal{T}_\lambda^n \delta w \to 0$ as $n \to \infty$. So we see that with sufficient regularity assumptions, we have convergence to the normal distribution.

Remark: The original formulation of the problem was slightly incorrect again. The function is not exactly $(-\Delta + N^{-2})$ but it is something very similar - we could also think of F being the covariance of the discrete two dimensional Gaussian free field.

4. Let N be a positive integer and for $x, y \in \mathbb{Z}^2$ define $x \sim_N y$ if $x - y \in (N\mathbb{Z})^2$. Let τ be an exponentially distributed random variable with parameter N^{-2} and let $\{w_m\}_{m=0}^{\infty}$ be a simple random walk on $\mathbb{Z}^2$ independent of τ. Let

$$F(x, y) = E^x \left(\sum_{m=0}^{\tau} \mathbf{1}(w_m \sim_N y) \right). \tag{13}$$

Compare F to $(-\Delta + N^{-2})^{-1}$, where Δ is the lattice Laplacian with period N in both directions.

Solution: **4.** Just calculating, we find

$$\begin{aligned}
E^x \left(\sum_{m=0}^{\tau} \mathbf{1}(w_m \sim_N y) \right) &= \sum_{n=0}^{\infty} P(\tau \in [n, n+1)) \sum_{m=0}^{n} P^x(w_m \sim_N y) \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left(e^{-\frac{n}{N^2}} \left(1 - e^{-\frac{1}{N^2}}\right) \right) P^x(w_m \sim_N y) \\
&= \sum_{m=0}^{\infty} e^{-\frac{m}{N^2}} P^x(w_m \sim_N y) \\
&= \sum_{z \in \mathbb{Z}^2} \sum_{m=0}^{\infty} e^{-\frac{m}{N^2}} P^x(w_m = y + Nz).
\end{aligned}$$

Let $\psi(p)$ be the characteristic function of a single step in our simple random walk:

$$\psi(p) = E(e^{ip \cdot X}) = \frac{1}{4}(e^{ip_1} + e^{-ip_1} + e^{ip_2} + e^{-ip_2}) = \frac{1}{2}(\cos p_1 + \cos p_2). \tag{14}$$

As $P^x(w_m = y)$ is an m-fold convolution of the distributions of the elementary steps in the random walk, we see that the Fourier representation of this function is just the mth power of ψ. Thus

$$E^x\left(\sum_{m=0}^{\tau}\mathbf{1}(w_m \sim_N y)\right) = \sum_{z\in\mathbb{Z}^2}\sum_{m=0}^{\infty}\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} e^{-\frac{m}{N^2}}\psi(p)^m e^{ip\cdot(y-x+Nz)}dp$$
$$= \sum_{z\in\mathbb{Z}^2}\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\frac{e^{ip\cdot(y-x+Nz)}}{1-e^{-\frac{1}{N^2}}\psi(p)}dp.$$

Note that for $p \in [-\frac{\pi}{N}, \frac{\pi}{N}]^2$,

$$\frac{1}{(2\pi)^2}\sum_{z\in\mathbb{Z}^2} e^{iNp\cdot z} = \delta(Np) = \frac{1}{N^2}\delta(p). \tag{15}$$

On the other hand, the summand is $\frac{2\pi}{N}$ periodic (in both directions) as a function in p. Thus performing the integration of the delta functions the z-sum gives we have

$$E^x\left(\sum_{m=0}^{\tau}\mathbf{1}(w_m \sim_N y)\right) = \frac{1}{N^2}\sum_{p:p_i=\frac{2\pi n_i}{N},|n_i|\le\frac{N}{2}}\frac{e^{ip\cdot(y-x)}}{1-e^{-\frac{1}{N^2}}\psi(p)}. \tag{16}$$

To leading order in N^{-2}, this agrees with $(-\Delta + N^{-2})$.