STATISTICAL MECHANICS - EXERCISE 6

1. Prove Wick's theorem.

Solution: Recall that for a Gaussian probability measure with covariance A^{-1}, the moments of the measure are given by

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \prod_{i=1}^{2n}\frac{\partial}{\partial f_{x_i}} e^{\frac{1}{2}(f,A^{-1}f)}\bigg|_{f=0}. \tag{1}$$

Expanding the exponential, we have

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \prod_{i=1}^{2n}\frac{\partial}{\partial f_{x_i}}\sum_{k=0}^{\infty}\frac{(f,A^{-1}f)^k}{2^k k!}\bigg|_{f=0}. \tag{2}$$

We see that only the $k = n$ term has $2n$ fs so it is the only one which survives the differentiation and setting $f = 0$. Thus writing out the inner product we have

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \prod_{i=1}^{2n}\frac{\partial}{\partial f_{x_i}}\frac{1}{2^n n!}\left(\sum_{\alpha,\beta} f_\alpha (A^{-1})_{\alpha,\beta} f_\beta\right)^n\bigg|_{f=0} \tag{3}$$

or written in another way,

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \prod_{i=1}^{2n}\frac{\partial}{\partial f_{x_i}}\frac{1}{2^n n!}\prod_{i=1}^{n}\sum_{\alpha_i,\beta_i} f_{\alpha_i}(A^{-1})_{\alpha_i,\beta_i} f_{\beta_i}\bigg|_{f=0} \tag{4}$$

So we see that only terms with $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\} = \{x_1, ..., x_{2n}\}$ contribute to the sum. Each such term contributes equally so in fact

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \frac{C_n}{2^n n!}\sum_P \prod_{i=1}^{n} A_{x_{P(2i-1)},x_{P(2i)}}, \tag{5}$$

where the sum is over all pairings of the points $\{x_1, ..., x_{2n}\}$ and C_n is the number of times a given pairing occurs in (4).

We can deduce C_n simply from the symmetries of (4). First of all, we note that since a covariance matrix is a symmetric matrix, we can always swap α_i and β_i so each pair gives a factor of 2 amounting in a factor of 2^n. On the other hand, we can permute the pairs in whatever way we wish so we get a factor of $n!$. There are no other symmetries so $C_n = n!2^n$. Thus

$$E\left(\prod_{i=1}^{2n}\phi_{x_i}\right) = \sum_P \prod_{i=1}^{n} A_{x_{P(2i-1)},x_{P(2i)}}. \tag{6}$$

2. Prove that (in some sense - such as for physicists or as a distribution or something)

$$(-\Delta)^{-1}_{xy} = \frac{c_d}{|x-y|^{d-2}}(1 + o(1)) \tag{7}$$

as $|x-y| \to \infty$ and calculate c_d.

Solution: Recall that,

$$G(x) = (-\Delta)^{-1}_{0,x} = \int_{[-\pi,\pi]^d} \frac{e^{ip\cdot x}}{\mu(p)} dp, \tag{8}$$

where $\mu(p) = \sum_{i=1}^d (2 - 2\cos p_i)$ and $dp = \prod_{i=1}^d \frac{dp_i}{2\pi}$ and $\Delta^{-1}_{x,y}$ is translation invariant. Making a change of variables $p = \frac{q}{|x|}$, we see that

$$G(x) = |x|^{-d+2} \int_{[-\pi|x|,\pi|x|]^d} \frac{e^{iq\cdot\hat{x}}}{|x|^2 \mu\left(\frac{q}{|x|}\right)} dq, \tag{9}$$

where $\hat{x} = \frac{x}{|x|}$. Noting that $|x|^2\mu(|x[^{-1}q) \to q^2$, one can show that (interpretation and amount of work this requires depends on how rigorous you want to be)

$$G(x) \sim |x|^{-d+2} \int_{\mathbb{R}^d} \frac{e^{iq\cdot\hat{x}}}{q^2} dq. \tag{10}$$

We are of course free to chose the q-coordinate system however we wish. Let us choose q_1 to be parallel to x. So we have

$$\int_{\mathbb{R}^d} \frac{e^{iq\cdot\hat{x}}}{q^2} dq = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \frac{e^{iq_1}}{q_1^2 + q^2} \frac{dq_1}{2\pi} dq. \tag{11}$$

Using the residue theorem one finds that

$$\int_{-\infty}^{\infty} \frac{e^{iq_1}}{q_1^2 + q^2} dq_1 = \pi \frac{e^{-|q|}}{|q|} \tag{12}$$

and

$$\begin{aligned} c_d &= \int_{\mathbb{R}^d} \frac{e^{iq\cdot\hat{x}}}{q^2} dq \\ &= \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \frac{e^{iq_1}}{q_1^2 + q^2} \frac{dq_1}{2\pi} dq \\ &= \frac{1}{2} \int_{\mathbb{R}^{d-1}} \frac{e^{-|q|}}{|q|} dq \\ &= \frac{1}{2} \frac{\alpha_d}{(2\pi)^{d-1}} \int_0^\infty r^{d-2} e^{-r} dr \\ &= \frac{1}{2} \frac{\alpha_d}{(2\pi)^{d-1}} \Gamma(d-1) \end{aligned}$$

Here α_d is the volume of the $d-2$ dimensional unit sphere (coming from integrating out the angular variables in our $d-1$ dimensional integral). If you are unfamiliar with the constant α_d, calculate the d-dimensional integral of $e^{-|x|^2}$ in two ways - as a product of 1-dimensional integrals and in spherical coordinates. You will find that

$$\alpha_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \tag{13}$$

so

(14) $$c_d = \frac{\Gamma(d-1)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{2^{d-1}\pi^{\frac{d-1}{2}}}.$$

We can simplify this further by using the following formula:

(15) $$\frac{\Gamma(2z)}{\Gamma(z)} = \frac{1}{\sqrt{2\pi}} 2^{2z-\frac{1}{2}} \Gamma\left(z + \frac{1}{2}\right).$$

Thus

(16) $$c_d = \frac{1}{\sqrt{2\pi}} 2^{d-1-\frac{1}{2}} \Gamma\left(\frac{d}{2}\right) \frac{1}{2^{d-1}\pi^{\frac{d-1}{2}}} = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}}.$$

Note that this is just the inverse of the volume of the d-dimensional unit sphere.

3. Consider a graph with vertices $x, y, x_1, ..., x_n$, simple edges $\{x, x_1\}$, $\{x_1, x_n\}$ and $\{x_n, y\}$ and double edges from x_i to x_{i+1} (so the graph looks like there is a straight line (going through x, x_1, x_n, y) and a chain of loops attached to it. Calculate the value of this graph and then sum the values over n.

Solution: Let us call our graph G_n. Let us begin by calculating the symmetry factor. There are n vertices we can connect x to. After this there are $n-1$ vertices we can connect y to. For each vertex we are connecting to x, there are 4 edges we can perform the connection with. Similarly for y. After this is done, there are 9 ways left we can connect x_1 to x_n, x_1 to x_2 and x_n to x_{n-1}. We then just continue counting combinations step by step and we find that the symmetry factor is

(17) $$n(G_n) = (n \cdot (n-1)4 \cdot 4 \cdot 3 \cdot 3) \cdot ((n-2)(n-3)4^2 \cdot 3^2) \cdots 2 = 2 \cdot 12^n n!.$$

The last factor of 2 comes from the fact that at the last step, there are only two ways to connect the points.

Using the standard rules for calculating the value of a diagram, we have

(18) $$val(G_n) = \sum_{x_1,\ldots,x_n} C(x_1 - x)C(y - x_n)C(x_n - x_1) \prod_{i=1}^{n-1} C(x_{i+1} - x_i)^2.$$

Writing C in terms of its Fourier transform: $C(x) = \int dp e^{ip\cdot x}\hat{C}(p)$, we have

(19) $$val(G_n) = \sum_{x_1,\ldots,x_n} \int \prod_{i=1}^{2n+1} (dp_i \hat{C}(p_i)) e^{ip_1\cdot(x_1-x)} e^{ip_2\cdot(y-x_n)} e^{ip_3\cdot(x_n-x_1)} \prod_{j=1}^{n-1} e^{ip_{2j+2}\cdot(x_{j+1}-x_j)} e^{ip_{2i+3}\cdot(x_{j+1}-x_j)}.$$

Noting that the sums amount to δ-functions, we see that

$$val(G_n) = \int \prod_{i=1}^{2n+1} (dp_i \hat{C}(p_i)) e^{-ip_1\cdot x} e^{ip_2 y} \delta(p_1 - p_3 - p_4 - p_5) \delta(-p_2 + p_3 + p_{2n} + p_{2n+1}) \times \prod_{j=1}^{n-2} \delta(-p_{2j+2} - p_{2j+3} + p_{2j+4} + p_{2j+5}).$$

Let us define $q_i = p_{2i+2}$ for $i \geq 1$, $p_1 = p$, $q = p_3$. The δ-functions imply that $p_{2i+1} = p - q - q_i$ so

$$\begin{aligned} val(G_n) &= \int dpdq \int \prod_{i=1}^{n-1} dq_i(\hat{C}(q_j)\hat{C}(p-q-q_j))\hat{C}(p)\hat{C}(q) \\ &\times \int dp_2 \delta(-p_2+q+(p-q))e^{-ip\cdot x}e^{ip_2\cdot y}\hat{C}(p_2) \\ &= \int dp e^{-ip\cdot(x-y)}\hat{C}(p)\hat{C}(-p)\int dq \hat{C}(q)I(p-q)^n, \end{aligned}$$

where $I(p) = \int dq\hat{C}(q)\hat{C}(p-q)$. Thus for the sum over all such graphs we find

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(-\lambda)^n}{n!} 2\cdot 12^n n! \int dp e^{-ip\cdot(x+y)}\hat{C}(p)\hat{C}(-p)\int dq\hat{C}(q)I(p-q)^n \\ &= 2\int dp e^{-ip\cdot(x-y)}\hat{C}(p)\hat{C}(-p)\int dq\hat{C}(q)\frac{I(p-q)^2}{1+12\lambda I(p-q)}. \end{aligned}$$

Note that since $\hat{C}(p) \leq \frac{1}{r}$, $I(p) \leq \frac{A}{r^2}$ for some constant A. So for small enough λ, $12\lambda I(p-q)$ is less than one and the geometric series converges as we claim above.

4. Show that for a general connected graph, the number of integrals in the Fourier representation we are left with after getting rid of the δ-functions is the number of independent cycles in the graph.

Solution: By Euler's theorem, for a connected graph one has n(vertices)-n(edges)+n(loops)=1. Let us write V for the number of internal vertices, I for the number of internal edges and E for the number of external edges. Of course n(edges)$= I+E$ and n(vertices)$= V+E$. Thus $L = n$(loops)$= I-V+1$.

We note that while there are V δ-functions coming from the 'local conservation of momentum', there is in fact a certain redundancy in them - there is also a 'global conservation of momentum': the total sum of Fourier modes incoming and outgoing to a graph is zero. Thus the number of relevant δ-functions in the integral is in fact $V-1$. After integrating them out, we are left with $I-(V-1) = L$ integrals.

5. Show that

(20)
$$\int \frac{1}{(q-p)^2q^2}dq \sim \begin{cases} \mathcal{O}(1), d\geq 5 \\ \log|p|, d=4 \\ |p|^{d-4}, d\leq 3 \end{cases}.$$

Solution: We shall use a trick (attributed to Feynman) commonly used in physics when evaluating Feynman diagrams. From a simple calculation, it follows that one can write (for positive x and y)

(21)
$$\frac{1}{xy} = \int_0^1 d\alpha \frac{1}{(\alpha x + (1-\alpha)y)^2}.$$

Using this and changing the order of integration, we find that

$$\begin{aligned} \int \frac{1}{(q-p)^2q^2}dq &= \int_0^1 d\alpha \int dq \frac{1}{(\alpha q^2+(1-\alpha)(q-p)^2)^2} \\ &= \int_0^1 d\alpha \int dq \frac{1}{(\alpha q^2 + (1-\alpha)q^2 - 2(1-\alpha)q\cdot p + (1-\alpha)p^2)^2} \\ &= \int_0^1 d\alpha \int dq \frac{1}{(q^2-2(1-\alpha)q\cdot p+(1-\alpha)p^2)^2}. \end{aligned}$$

We then perform a change of variable to get rid of the inner product: let $x = q-(1-\alpha)p$. Then $x^2 = q^2 - 2(1-\alpha)p\cdot q + (1-\alpha)^2p^2$ and

(22) $$\int \frac{1}{(q-p)^2q^2}dq = \int_0^1 d\alpha \int dx \frac{1}{(x^2+\alpha(1-\alpha)p^2)^2}.$$

Noting that the relevant contribution will come when q (and p) are close to zero, we get the leading behavior by considering any fixed bounded domain that contains a suitably large neighborhood of zero. So let us fix the integration domain (for the x integral) to be spherically symmetric around zero: take a ball $B(0,r)$. We then go into polar coordinates and find (note the abuse of notation: first x is a vector and then a scalar)

(23) $$\int \frac{1}{(q-p)^2q^2}dq = c_d \int_0^1 d\alpha \int_0^r \frac{x^{d-1}}{(x^2+\alpha(1-\alpha)p^2)^2}dx.$$

We then make a change of variables in this: write $x = \sqrt{\alpha(1-\alpha)p^2}y$. We find

(24) $$\int \frac{1}{(q-p)^2q^2}dq = c_d \int_0^1 d\alpha(\alpha(1-\alpha)p^2)^{\frac{d}{2}-2} \int_0^{\frac{r}{\sqrt{\alpha(1-\alpha)p^2}}} \frac{y^{d-1}}{(y^2+1)^2}dy.$$

As $p\to 0$, the y integral behaves like

(25) $$\int_1^{\frac{r}{\sqrt{\alpha(1-\alpha)p^2}}} y^{d-5}dy.$$

Now if $d\geq 5$, the y-integral gives something proportional to $((\alpha(1-\alpha))p^2)^{-\frac{d-4}{2}}$ which cancels with the prefactor so we see that the whole integral is bounded. For $d=4$, the y integral gives something proportional to $-\log(\alpha(1-\alpha))+\log|p|$. The prefactor is equal to one and the α-dependent part is integrable so we find that indeed for $d=4$, we have logarithmic divergence. For $d\leq 3$, the y-integral is bounded in p. The α-integral converges when the dimension is strictly larger than 2 in which case we find the behavior to be $|p|^{d-4}$ as claimed. In the $d=1$ and $d=2$ cases, the integral seems to be divergent even with the p term.

6. Show that the $\lambda\to\infty$, $r\to-\infty$ limit with a suitable relation between λ and r of the Ginzburg-Landau model is the Ising model. More precisely, consider the generating function for the correlation functions of the GL-model in finite volume:

(26) $$Z(h) = \int \prod_{x\in\Lambda_L} d\phi_x e^{-H_{GL}(\phi)} e^{-\sum_x \phi_x h_x}$$

and show that after suitably rescaling Z, h and ϕ one gets the generating function for the correlation functions of the Ising model in the $\lambda\to\infty$ and $r\to-\infty$ limit.

Solution: We begin by writing the Hamiltonian in a form more suggestive of the Ising model (e_k is the unit vector in the k direction):

$$\begin{aligned}
-H(\phi) &= -\frac{1}{2}\sum_{x,k}(\phi(x+e_k)-\phi(x))^2 - \frac{r}{2}\sum_x \phi(x)^2 - \lambda\sum_x \phi(x)^4 \\
&= \sum_{x,k}\phi(x)\phi(x+e_k) - \left(\frac{r}{2}+d\right)\sum_x \phi(x)^2 - \lambda\sum_x\phi(x)^4 \\
&= \sum_{x,k}\phi(x)\phi(x+e_k) - \lambda\sum_x\left(\phi(x)^2+\frac{r+2d}{4\lambda}\right)^2 + |\Lambda_L|\frac{(r+2d)^2}{16\lambda}.
\end{aligned}$$

Define $\alpha = \sqrt{\frac{|r+2d|}{4\lambda}}$ and $K=\exp(\lambda\alpha^4)$. Then for small enough r (so that α is negative)

$$Z(h) = K^{|\Lambda_L|} \int \prod_{x\in\Lambda_L} \left(d\phi(x) e^{h(x)\phi(x)} e^{-\lambda(\phi(x)^2-\alpha^2)^2} \right) \prod_{x,k} e^{\phi(x)\phi(x+e_k)}. \tag{27}$$

Let us first rescale ϕ and h: define $\varphi = \frac{\phi}{\alpha}$ and $b = \alpha h$. We then have

$$Z(h) = K^{|\Lambda_L|}\alpha^{|\Lambda_L|} \int \prod_{x\in\Lambda_L} \left(d\varphi(x) e^{b(x)\varphi(x)} e^{-\lambda\alpha^4(\varphi(x)^2-1)^2} \right) \prod_{x,k} e^{\alpha^2\varphi(x)\varphi(x+e_k)}. \tag{28}$$

We then make use of the Gaussian approximation to the δ-function:

$$\sqrt{\frac{t}{\pi}} e^{-tx^2} \to \delta(x) \tag{29}$$

as $t \to \infty$. We define

$$Z^{\lambda,\alpha}_{Ising}(b) = \frac{1}{K^{|\Lambda_L|}\alpha^{|\Lambda_L|}} \left(\frac{\pi}{\lambda\alpha^4} \right)^{\frac{1}{2}|\Lambda_L|} Z(h) \tag{30}$$

and the Gaussian approximation gives

$$\begin{aligned} Z^{\alpha}_{Ising}(b) &= \lim_{\lambda\to\infty} Z^{\lambda,\alpha}_{Ising}(b) \\ &= \int \prod_{x\in\Lambda_L} \left(d\varphi(x) e^{b(x)\varphi(x)} \delta(\varphi(x)^2 - 1) \right) \prod_{x,k} e^{\alpha^2\varphi(x)\varphi(x+e_k)} \\ &= \prod_{x\in\Lambda_L} \sum_{\varphi_x\in\{-1,1\}} e^{\sum_x b(x)\varphi(x)} e^{\alpha^2 \sum_{x\sim y} \varphi(x)\varphi(y)}, \end{aligned}$$

where $x \sim y$ means that x and y are nearest neighbors. Note that this is precisely of the Ising form with inverse temperature $\beta = \alpha^2$.