Statistical mechanics - Exercise 5.

1. Calculate $\langle \varphi_x \varphi_y \rangle_\lambda$ to order λ^2 in the Ginzburg-Landau model.

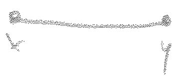
Solution: Write $\mu_c(d\varphi)$ for our Gaussian measure we wish to calculate

$$\langle \varphi_x \varphi_y \rangle_\lambda = \frac{\int \varphi_x \varphi_y \, e^{-\lambda \sum_z \varphi_z^4} \mu_c(d\varphi)}{\int e^{-\lambda \sum_z \varphi_z^4} \mu_c(d\varphi)}$$

to order λ^2.

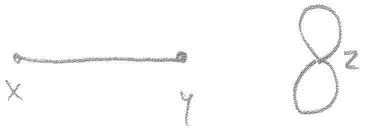
We consider expansions for the numerator and denominator separately.

At order λ^0, we only have the following diagram:

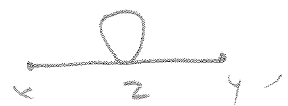


(no powers of λ so no vertices)

At order λ^1, we can have two types of diagrams

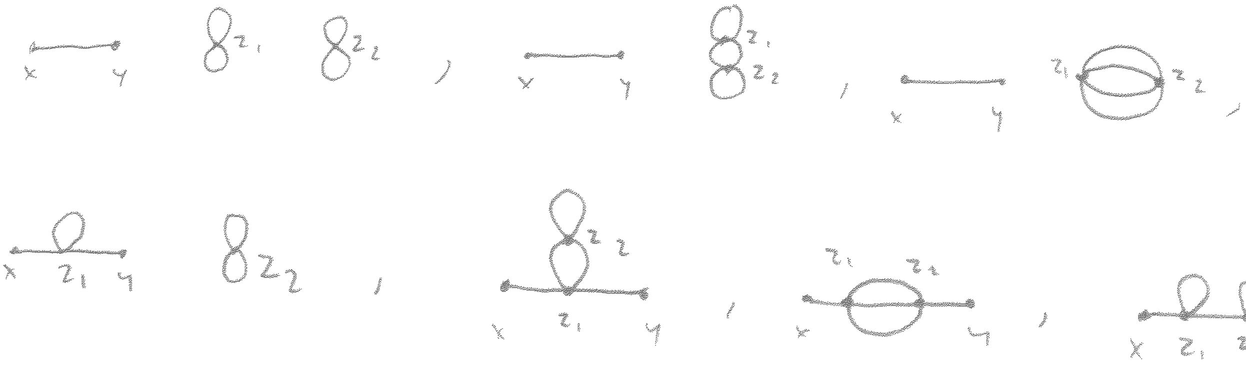


and



where we sum over z.

At order λ^2, we have the following ones

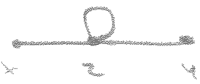


We still have to figure out the combinatorial weights of each such diagram.

At order λ^0, corresponds to $\langle \varphi_x \varphi_y \rangle_{\lambda=0}$ and it has no special weight.

The sum of the graphs at order λ^1 corresponds to $\langle \varphi_x \varphi_y \varphi_z^4 \rangle_{\lambda=0}$.

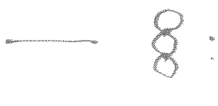
The weight of  is the number of times we can pair "x to y and z to itself twice." Perhaps the proper way to think of this is to think of the four z as distinct points z_1, z_2, z_3, z_4 and the weight is then the number of pairings of these points, which is 3.

For the weight of 

, we note that we can connect a z to x in 4 ways and out of the remaining 3, we can connect one to y in 3 ways so the weight is 12.

At order λ^2, the reasoning is similar, but the numbers get bigger.

 : $3 \cdot 3 = 9$

 : $6 \cdot 2 \cdot 6 = 72$

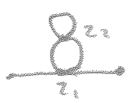
number of ways of forming a pair of z_1 vertices

number of ways of connecting leftovers

 : $4 \cdot 3 \cdot 2 = 24$

 : $2 \cdot 12 \cdot 3 = 72$

interchange of $z_1 \leftrightarrow z_2$



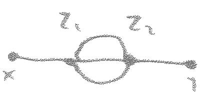
: $2 \cdot 6 \cdot 12 \cdot 2 = 288$

z_2 pairs

connecting z_1 to z_2

$z_1 \leftrightarrow z_2$ symmetry

connecting z_1 to $\{x, y\}$



: $2 \cdot 4 \cdot 4 \cdot 6 = 192$

z_1 to y

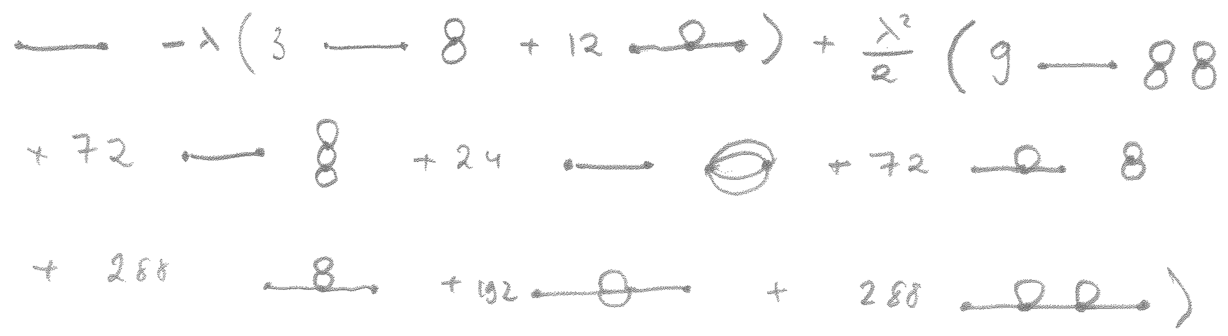
$z_1 \leftrightarrow z_2$

z_1 to x

z_1 to z_2

 : $2 \cdot 12 \cdot 12 = 288$

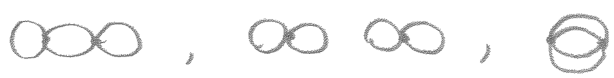
So we conclude that the numerator has the expansion



For the denominator:

λ^0 : No diagrams

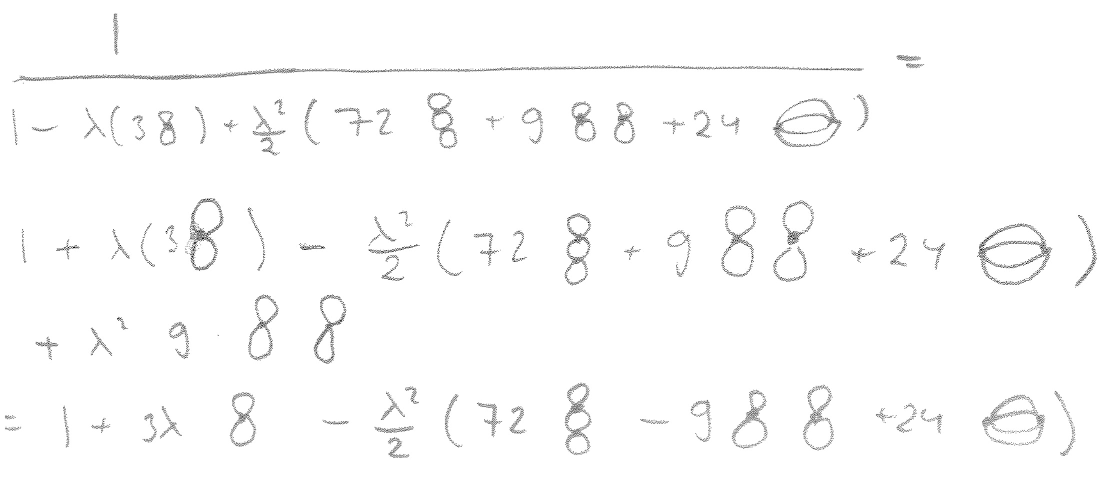
λ^1 : 

λ^2 : 

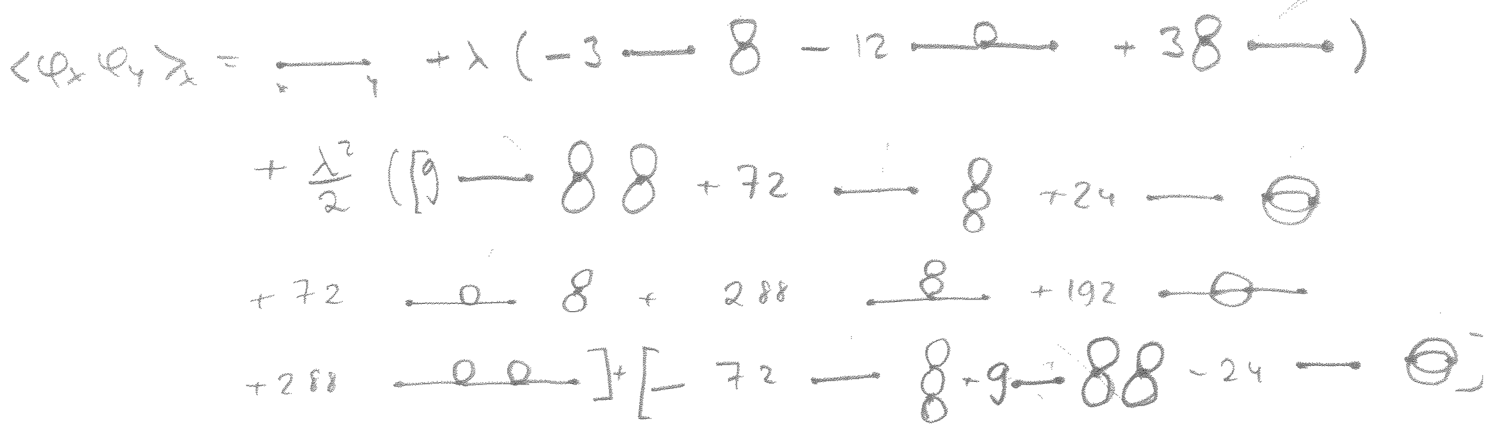
Symmetry factors.

So for the denominator, we have the expansion

We then use the expansion of a geometric series for the denominator (to order λ^2):

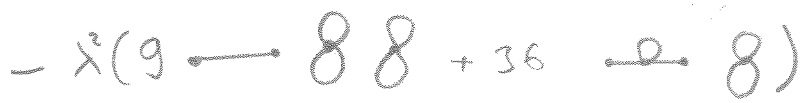


Multiplying this with the numerator and keeping terms up to second order, we find:

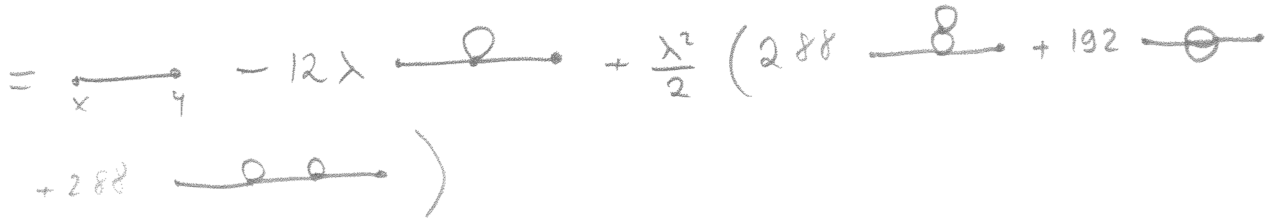


λ^2 term in numerator

λ^1 term from $\frac{1}{\text{denominator}}$



λ^1 term in numerator $\cdot$ λ^1 term in $\frac{1}{\text{denominator}}$



Only connected diagrams!

2. For $V(\varphi) = \sum_x \varphi_x^4$, show that

$$\frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \log \int e^{-\lambda V(\varphi)} d\mu_G(\varphi) = \langle V; V; \ldots; V \rangle$$

Solution Consider a Taylor polynomial of $\langle F \rangle_\lambda$, where F is some random variable.

We have

$$\langle F \rangle_\lambda = \frac{\sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} \langle F V^k \rangle_{\lambda=0} + \varepsilon_1^N(\lambda)}{1 + \sum_{k=1}^{N} \frac{(-\lambda)^k}{k!} \langle V^k \rangle_{\lambda=0} + \varepsilon_2^N(\lambda)}$$

where $\varepsilon_i^N(\lambda)$ are some higher order terms in λ.

Then expanding as a geometric series, we find

$$\langle F\rangle_\lambda = \sum_{k=0}^{N}\frac{(-\lambda)^k}{k!}\langle FV^k\rangle_{\lambda=0}\sum_{n=0}^{\infty}\Big(-\sum_{l=1}^{N}\frac{(-\lambda)^l}{l!}\langle V^l\rangle_{\lambda=0}\Big)^n + \tilde{\varepsilon}^N(\lambda)$$

$$= \sum_{n,k=0}^{N}(-1)^n\cdot\frac{(-\lambda)^k}{k!}\langle FV^k\rangle_{\lambda=0}\sum_{k_1=1}^{N}\cdots\sum_{k_n=1}^{N}\frac{(-\lambda)^{k_1}}{k_1!}\cdots\frac{(-\lambda)^{k_n}}{k_n!}\langle V^{k_1}\rangle_{\lambda=0}\cdots$$

$$+ \hat{\varepsilon}^N(\lambda)$$

Picking out the term of order m, we have

$$\frac{(-\lambda)^m}{m!}\sum_{i=0}^{m}\sum_{k_0=0}^{m}\sum_{k_1=1}^{m}\cdots\sum_{k_i=1}^{m}(-1)^i\frac{m!}{k_0!\cdots k_i!}\langle FV^{k_0}\rangle_{\lambda=0}\prod_{j=1}^{i}\langle V^{k_j}\rangle_{\lambda=0}$$
$$\mathbb{1}\Big(\sum_{j=0}^{i}k_j = m\Big)$$

Define now in general

$$\langle F_1; F_2; \ldots; F_n\rangle = \sum_{\pi}(-1)^{|\pi|-1}(|\pi|-1)!\prod_{i\in\pi}\langle\prod_{j\in\pi_i}F_j\rangle \quad (\text{sum over partitions of } \{1,\ldots,n\})$$

Split the sum according to the size of π:

$$\langle F_1; \ldots; F_n\rangle = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!\sum_{\pi:|\pi|=k}\prod_{i\in\pi}\langle\prod_{j\in\pi_i}F_j\rangle$$

Taking $F_1=F$, $F_i=V$ and taking into account all the symmetries once we split the sum over π into sums of sizes of the subsets, we find that the term of order λ^m in the Taylor expansion of $\langle F\rangle_\lambda$ is

$$-\frac{(-\lambda)^m}{m!}\sum_{\pi}(-1)^{|\pi|-1}(|\pi|-1)!\,\langle FV^{|\pi_1|-1}\rangle\prod_{j\neq 1}\langle V^{|\pi_j|}\rangle$$

Setting $F=V$, this becomes

$$-\frac{(-\lambda)^m}{m!}\underbrace{\langle V; V; \ldots; V\rangle}_{m+1\text{ factors}}$$

Let us now see what $\langle V \rangle_\lambda$ has to do with

$$\log \int e^{-\lambda V} \mu_C(d\varphi)$$

Note that

$$-\frac{d}{d\lambda} \log \int e^{-\lambda V} \mu_C(d\varphi) = \frac{\int V e^{-\lambda V} \mu_C(d\varphi)}{\int e^{-\lambda V} \mu_C(d\varphi)}$$

$$= \langle V \rangle_\lambda$$

So $\left(-\frac{d}{d\lambda}\right)^{n+1}\Big|_{\lambda=0} \log \int e^{-\lambda V} \mu_C(d\varphi)$

$$= \left(-\frac{d}{d\lambda}\right)^{n}\Big|_{\lambda=0} \langle V \rangle_\lambda$$

which is just $n!$ times the term we just calculated.

$$\therefore \left(-\frac{d}{d\lambda}\right)^{n+1}\Big|_{\lambda=0} \log \int e^{-\lambda V} \mu_C(d\varphi) = \underbrace{\langle V; V; \dots; V \rangle}_{n+1 \text{ times}}$$

To see that this contains only connected diagrams for $V = \sum_x \varphi_x^4$, one can argue as in the lecture notes that the expansion of $\langle V \rangle_\lambda$ has only connected graphs.

3. Prove that $G_4^c(x_1,x_2,x_3,x_4) = G_4(x_1,x_2,x_3,x_4) - G_2(x_1,x_2)G_2(x_3,x_4)$ (7)
$-$ (permutations)

has only connected diagrams in its expansion.

Solution: Consider an expansion of G_4 up to order N:

$$G_4(x_1,x_2,x_3,x_4) = \sum_{n=0}^{N} \frac{(-\lambda)^n}{n!} \sum_{g\in\Gamma_4(n)} n(g) I(g) + \mathcal{O}(\lambda^{N+1}),$$

where $\Gamma_4(n)$ is the set of graphs with n vertices and 4 external legs, $n(g)$ is the symmetry factor of a graph g and $I(g)$ the value of the graph.
Recall further that in the lectures it was argued that vacuum graphs don't contribute, i.e. each vertex in g is connected to at least one point x_i.
On the other hand, for each internal vertex, there are four edges, so no internal vertex in g can be connected to an odd amount of the x_i. So each internal vertex z_j is connected to either 2 or 4 of the x_i.
If z_j is connected to all 4 x_i, then all internal vertices are connected to z_j (take a path through some x_i) and the graph is connected.

Consider now the situation where z_j is connected to only $\{x_1,x_2\}$. Let $V_1 \subset g$: $V_1 = \{z \mid z \text{ connected to } z_j\}$ and $V_2 = \{z \mid z \text{ not connected to } z_j\}$.
Now the points in V_2 are all connected to $\{x_3,x_4\}$ and those in V_1 are connected to $\{x_1,x_2\}$. So V_i are connected graphs and $V_1 \cup V_2 = g$, $V_1 \cap V_2 = \emptyset$.
So for such a graph g, we have

$$I(g) = \sum_{z_i\in g} \prod_{l\in \mathrm{Vert}(g)} C_l = \sum_{z_1\in V_1}\sum_{z_2\in V_2} \prod_{l_1\in \mathrm{Vert}(V_1)} C_{l_1} \prod_{l_2\in\mathrm{Vert}(V_2)} C_{l_2}$$

$$= \Big(\sum_{z_1\in V_1}\prod_{l_1\in\mathrm{Vert}(V_1)} C_{l_1}\Big)\Big(\sum_{z_2\in V_2}\prod_{l_2\in\mathrm{Vert}(V_2)} C_{l_2}\Big)$$

$$= I(V_1) I(V_2)$$

The number of way can do this splitting into two subgraphs of given sizes $|V_1|, |\mathcal{G}| - |V_1|$ is

$$\frac{|\mathcal{G}|!}{|V_1|!(|\mathcal{G}|-|V_1|)!} \quad \text{so} \quad n(\mathcal{G}) = \frac{|\mathcal{G}|!}{|V_1|!(|\mathcal{G}|-|V_1|)!} n(V_1)n(V_2).$$

we conclude that the contribution from such graphs is

$$\sum_{n=0}^{N} \frac{(-\lambda)^n}{n!} \sum_{n_1=0}^{n} \sum_{V_1 \in \Gamma_2(n_1)} \sum_{V_2 \in \Gamma_2(n-n_1)} \frac{n!}{n_1!(n-n_1)!} n(V_1)n(V_2)I(V_1)I(V_2)$$

Performing a change of variables in the summations, we find this to be

$$\left(\sum_{n_1=0}^{N} \frac{(-\lambda)^{n_1}}{n_1!} \sum_{V_1 \in \Gamma_2(n_1)} n(V_1)I(V_1)\right)\left(\sum_{n_2=0}^{N} \frac{(-\lambda)^{n_2}}{n_2!} \sum_{V_2 \in \Gamma_2(n_2)} n(V_2)I(V_2)\right)$$

$$= G_2(x_1,x_2)\, G_2(x_3,x_4)$$

Considering other such graphs where the x_i are permuted, we see that these terms cancel in G_4^c and only the connected graphs contribute

4. Show that $F(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\varphi^4 - \varphi^2} d\varphi$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and has an essential singularity at 0. Estimate the error in the Taylor expansion.

Solution. The integral is absolutely convergent when $\operatorname{Re}\lambda \geq 0$ so at least F is well defined for such λ

Let us first show that F is analytic in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$ and try to analytically continue.

Pick some $\lambda_0 \in \mathbb{C} : \operatorname{Re}\lambda_0 > 0$. For $\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \operatorname{Re}\lambda_0$, write

$$e^{-\lambda\varphi^4 - \varphi^2} = \sum_{n=0}^{\infty} \frac{(\lambda_0-\lambda)^n \varphi^{4n}}{n!} e^{-\lambda_0\varphi^4 - \varphi^2}$$

Using some fairly basic estimation and the definition of the Gamma function, one can check that after integrating term by term, one gets an absolutely convergent series and

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{(\lambda_0-\lambda)^n}{n!} \int_{-\infty}^{\infty} d\varphi\, \varphi^{4n} e^{-\lambda_0\varphi^4 - \varphi^2}$$

So F is analytic in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$.

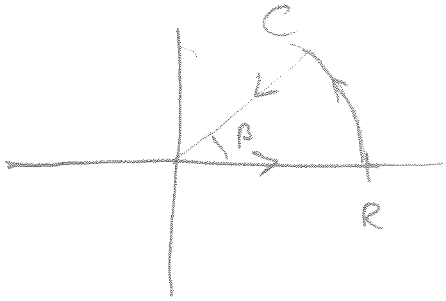
To analytically continue F to other values of λ, write

$$a_n(\lambda_0) = \int_{-\infty}^{\infty} d\varphi\, \varphi^{4n} e^{-\lambda_0\varphi^4-\varphi^2} \quad \text{for } Re(\lambda_0)>0.$$

By symmetry, $a_n(\lambda_0) = 2\lim_{R\to\infty}\int_0^R d\varphi\,\varphi^{4n}e^{-\lambda_0\varphi^4-\varphi^2} = 2\lim_{R\to\infty} I_R$

Let $\lambda_0 = re^{i\theta}$, with $|\theta|<\frac{\pi}{2}$, and $r>0$. Moreover, let $\beta = -\frac{\theta}{4}$

Consider the following contour, C:



By Cauchy's theorem,

$$\oint_C z^{4n}e^{-\lambda_0 z^4 - z^2} = 0$$

$$= I_R + \int_0^\beta iRe^{i\alpha}R^{4n}e^{i4n\alpha}e^{-re^{i\theta}R^4e^{i4\alpha}-R^2e^{i2\alpha}}\,d\alpha$$

$$+ \int_R^0 e^{i\beta}t^{4n}e^{i4n\beta}e^{-re^{i\theta}t^4e^{i4\beta}-t^2e^{i2\beta}}\,dt$$

Noting that $\theta+4\beta=0$,

$$I_R = \int_0^R e^{i\beta(4n+1)}t^{4n}e^{-rt^4-t^2e^{i2\beta}}\,dt$$

$$- iR^{4n+1}\int_0^\beta e^{i\alpha(4n+1)}e^{-rR^4e^{i(\theta+4\alpha)}-R^2e^{i2\alpha}}\,d\alpha$$

For the α-integral, we have

$$\left|\int_0^\beta \cdot\, d\alpha\right| \le \int_0^\beta e^{-rR^4\cos(\theta+4\alpha)-R^2\cos 2\alpha}\,d\alpha.$$

$|\theta+4\alpha| = 4|\beta-\alpha| \le 4|\beta| < \frac{\pi}{2}$ so $\cos(\theta+4\alpha)>0$

Also $|2\alpha|\le|2\beta|<\frac{\pi}{4} \Rightarrow \cos(2\alpha)>0$

$$\Rightarrow \left|\int_0^\beta\cdot\, d\alpha\right| \le e^{-rR^4\cos(4|\beta|)}\,|\beta|$$

$$\Rightarrow \lim_{R\to\infty}\left(iR^{4n+1}\int_0^\beta\cdot\, d\alpha\right)=0$$

And $a_n(\lambda_0) = 2e^{i\beta(4n+1)}\int_0^\infty dt\, t^{4n}e^{-rt^4-t^2e^{2i\beta}}$, $\beta=-\frac{\theta}{4}$.

Motivated by this, for any $\lambda_0 \in \mathbb{C}\setminus\{0\}$, $\lambda_0 = re^{i\theta}$, $r>0$, $|\theta| \le \pi$. (10)

$$a_n(\lambda_0) = 2\, e^{-i\frac{\theta}{4}(4n+1)} \int_0^\infty t^{4n} e^{-rt^4 - t^2 e^{-i\frac{\theta}{2}}} dt$$

Now using the definition of the Γ-function,

$$|a_n(\lambda_0)| \le 2\int_0^\infty t^{4n} e^{-rt^4} dt = \frac{\Gamma(n+\frac{1}{4})}{2r^{n+\frac{1}{4}}} \le \frac{n!}{2r^{n+\frac{1}{4}}}$$

So we see that

$$\sum_{n=0}^\infty \frac{|\lambda-\lambda_0|^n}{n!} |a_n(\lambda_0)| \le \frac{1}{2r^{\frac{1}{4}}} \sum_{n=0}^\infty \left(\frac{|\lambda-\lambda_0|}{r}\right)^n$$

converges when $|\lambda-\lambda_0| < r = |\lambda_0|$

Let us see what $\displaystyle\sum_{n=0}^\infty \frac{(\lambda-\lambda_0)^n}{n!} a_n(\lambda_0)$ looks like.

$$\int_0^\infty dt \sum_{n=0}^\infty \frac{(\lambda_0-\lambda)^n}{n!}\, 2e^{-i\frac{\theta}{4}}\, e^{-in\theta}\, t^n\, e^{-rt^4 - t^2 e^{-i\frac{\theta}{2}}}$$

$$= \int_0^\infty dt\, 2e^{-i\frac{\theta}{4}} e^{-rt^4 - t^2 e^{-i\frac{\theta}{2}}} \sum_{n=0}^\infty \frac{\left((\lambda_0-\lambda)t^4 e^{-i\theta}\right)^n}{n!}$$

Put in $\lambda_0 = re^{i\theta}$

$$= \int_0^\infty dt\, 2e^{-i\frac{\theta}{4} - rt^4 - t^2 e^{-i\frac{\theta}{2}}}\, e^{-\lambda t^4 e^{-i\theta} + rt^4}$$

$$= 2e^{-i\frac{\theta}{4}} \int_0^\infty e^{-t^2 e^{-i\frac{\theta}{2}} - \lambda t^4 e^{-i\theta}} dt$$

Write $\lambda = r' e^{i\theta'}$

$$= 2e^{-i\frac{\theta}{4}} \int_0^\infty e^{-t^2 e^{-i\frac{\theta}{2}} - r' e^{-i(\theta-\theta')} t^4} dt$$

[substitute $z = te^{i\frac{\theta'-\theta}{4}}$

$$= 2\int_0^\infty dz\, e^{-i\frac{\theta'}{4}} e^{-r' z^4 - z^2 e^{-i\frac{\theta'}{2}}}$$

– this is properly justified by Cauchy's theorem]

We have constructed a proper analytic continuation of F to $\mathbb{C}\setminus[-\infty,0]$. Note that due to the condition $|\lambda-\lambda_0| < |\lambda_0| \Rightarrow |\theta-\theta'| < \frac{\pi}{2}$ and we don't know about analyticity on $[-\infty, 0]$

To show that 0 is an essential singularity, we show that there is a branch cut along $(-\infty, 0]$.

Let us write F for our continuations as well. We have

$$F(re^{\pm i\pi}) = 2e^{\mp i\frac{\pi}{4}} \int_0^\infty dt\, e^{-rt^4 - t^2 e^{\mp i\frac{\pi}{2}}}$$
$$= 2e^{\mp i\frac{\pi}{4}} \int_0^\infty dt\, e^{-rt^4 \pm it^2}$$
$$= 2\int_0^\infty dt\, e^{-rt^4} \left(\cos\left(t^2 - \frac{\pi}{4}\right) \pm i \sin\left(t^2 - \frac{\pi}{4}\right)\right)$$

So we see that the real part is continuous across $(-\infty, 0]$, but

$$F(re^{i\pi}) - F(re^{-i\pi}) = 4i\int_0^\infty dt\, e^{-rt^4} \underbrace{\sin\left(t^2 - \frac{\pi}{4}\right)}_{= \frac{1}{\sqrt{2}}\left(\sin(t^2) - \cos(t^2)\right)}$$

$$\neq 0$$

So F is discontinuous across $(-\infty, 0]$ and there is a branch cut, and 0 is an essential singularity.

By Taylor's theorem, at order n, we make an error

$$R_n(\lambda) = \frac{F^{(n+1)}(\xi)\, \lambda^{n+1}}{(n+1)!}$$ when approximating by the Taylor polynomial $(0 \le \xi \le \lambda)$

$$|F^{(n+1)}(\xi)| = \left| \int_{-\infty}^{\infty} \varphi^{4(n+1)} e^{-\xi\varphi^4} e^{-\varphi^2} d\varphi \right|$$
$$\le \int_{-\infty}^{\infty} \varphi^{4(n+1)} e^{-\varphi^2} d\varphi$$
$$= \Gamma\left(2n + \frac{5}{2}\right)$$
$$\le (2(n+1))!$$

Using Stirling's formula,

$$R_n(\lambda) \approx \lambda^{n+1} \frac{\sqrt{2\pi \cdot 2(n+1)} \cdot \left(\frac{2(n+1)}{e}\right)^{2(n+1)}}{\sqrt{2\pi \cdot (n+1)} \left(\frac{n+1}{e}\right)^{n+1}}$$

$$\sim \lambda^{n+1} \, 2^{2(n+1)} \left(\frac{n+1}{e}\right)^{n+1}$$

$$= \left(\frac{4\lambda (n+1)}{e}\right)^{n+1}.$$

This clearly blows up as $n \to \infty$ for any $\lambda > 0$

But say that $\lambda \sim \frac{1}{100}$ and $n \sim 20$.

Then the error we make would be

$\approx 3{,}5 \cdot 10^{-11}$, which is spectacular precision.

5. Derive graphical rules in ~~the~~ case that φ^4 is replaced by φ^n.

$n = 0, 1, 2$ reduces to the case of a Gaussian measure and we don't need perturbation theory.

Just as in the case $n=4$, vacuum diagrams cancel (argue as in Problem 1) for all correlation functions

The basic rules for calculating

$\langle \prod_{i=1}^{k} \varphi(x_i) \rangle_\lambda$ at order ~~m~~ one

Sum over all graphs with m internal vertices and k external legs, so that at each vertex, there are n edges. Each graph is given a value as in the $n=4$ case and it also has a combinatorial weight one can calculate with symmetry arguments