STATISTICAL MECHANICS - EXERCISE 3

1. Go over the Peierls argument for the q-state Potts model, i.e. show that if $\mathbb{P}^{(i)}$ is the probability measure obtained from the limit of the q-state Potts model with boundary conditions i (outside of Λ each spin has value i), then $\lim_{\beta\to\infty}\mathbb{P}^{(i)}(\sigma_0 \neq i) = 0$.

Solution: We proceed exactly as in the Ising model. If the spin at the origin is different than that on the boundary, there must be a contour surrounding the origin. Also as in the case of the Ising model, we can simply estimate the ratio of the sums over the rest of the contours upwards to one (check the lecture notes). Thus we conclude that

$$\mathbb{P}^i(\sigma_0 \neq i) \leq \sum_{\gamma:\ \gamma \text{ surrounds } 0} e^{-\beta|\gamma|} = \sum_{n=2d}^{\infty} e^{-\beta n}|\{\gamma:\ \gamma \text{ surrounds } 0 \text{ and } |\gamma| = n\}| \tag{1}$$

As in the lecture notes, we estimate the number of contours of a given length surrounding zero by considering the number of dual bonds at a distance less than n from the origin and multiplying this by the number of connected sets of dual bonds of length n that contain the dual bond we are considering. We again get a bound for this number that is of the form Ce^{cn} (where c and C are independent of β). So we find

$$\mathbb{P}^i(\sigma_0 \neq i) \leq C\sum_{n=2d}^{\infty} e^{-n(\beta-c)} = Ce^{-2d(\beta-c)}\frac{1}{1-e^{-(\beta-c)}}. \tag{2}$$

This goes to zero as $\beta\to\infty$ uniformly in Λ.

2. Percolation is an idealized model for studying the problem that if you pour a liquid into a porous material, will it flow through the sample and come out on the other side. For the model, we consider a graph (G, E) and state that each edge $e \in E$ is open with probability p and closed with probability $1-p$. Moreover, this assignment of an edge being open or closed is independent of all the other edges. A typical question would be that if the graph is the set $\{0, ..., n\}^2$, with what probability does there exist a connected path of open edges from $\{0\}\times\{0, ..., n\}$ to $\{n\}\times\{0, ..., n\}$ as $n\to\infty$ or does there exist a cluster of open edges connecting 0 to infinity.

In the lectures it was remarked that using the FK random cluster representation, the $q = 1$-state Potts model can be interpreted as percolation. Try to motivate this remark a bit on a level of partition functions: try to define a partition function for percolation and show that it looks like the partition function for a random cluster model corresponding to a Potts model. Assuming this identification between the models, how would you interpret the $q\to 1$ limit of the Potts model correlation function $\langle\delta_{\sigma_x,a}\delta_{\sigma_y,a}\rangle - \langle\delta_{\sigma_x,a}\delta_{\sigma_y,b}\rangle$ for percolation?

Solution: A state in the percolation model is described by the bonds in the graph that are open. We can identify the collection of open bonds with a unique subgraph of the original graph. So we can index the states by subgraphs. The weight we give a state comes from the probability of the given configuration. Each open edge comes with a weight of p and each closed edge comes with a weight of $1-p$. Let us write $E(G')$ for the set of edges in G'. Thus the weight of a state we identify with a subgraph G' is

$$P(G') = p^{|E(G')|}(1-p)^{|E(G)|-|E(G')|} \tag{3}$$

and the partition function for percolation is

(4) $$Z_{perc,G} = \sum_{G'\subset G} p^{|E(G')|}(1-p)^{|E(G)|-|E(G')|}.$$

Recall that for the FK-random cluster model the partition funciton was given by

(5) $$Z_{FK,G} = \sum_{G'\subset G} q^{C(G')}(e^{\beta}-1)^{|E(G')|},$$

where $C(G')$ is the number of connected components of G'. Writing $p = 1-e^{-\beta}$ a short calculation shows that we can write this as

(6) $$Z_{FK,G} = \frac{1}{(1-p)^{|E(G)|}} \sum_{G'\subset G} q^{C(G')}(1-p)^{|E(G)|-|E(G')|}p^{E(G')}.$$

Constant multiples of partition functions don't generally change the model since they simply cancel in all correlation functions. So if we ignore the constant and take $q \to 1$, we recover the partition function of percolation. Note that the partition function for percolation has particularly uninteresting behavior in p: writing $1 = (p+(1-p)) = (p+(1-p))^{|E(G)|}$ and the same old expansion for such powers, we see that $Z_{perc,G} = 1$.

For the correlation function, let us write

$$\begin{aligned}\langle \delta_{\sigma_x,a}\delta_{\sigma_y,a}\rangle - \langle \delta_{\sigma_x,a}\delta_{\sigma_y,b}\rangle &= \langle \mathbf{1}\{x\sim y, \sigma_x = a\} + \mathbf{1}\{x\nsim y, \sigma_x = a, \sigma_y = a\}\rangle \\ &\quad - \langle \mathbf{1}\{x\nsim y, \sigma_x = a, \sigma_y = b\}\rangle \\ &= \mathbb{P}(x\sim y, \sigma_x = a) + \mathbb{P}(x\nsim y, \sigma_x = a, \sigma_y = a) - \mathbb{P}(x\nsim y, \sigma_x = a, \sigma_y = b),\end{aligned}$$

where we have writte $x\sim y$ for the relation that x and y are in the same cluster. Now my the permutation symmetry in the Potts model, the probabilities where the points are in different clusters cancel and we have

(7) $$\langle \delta_{\sigma_x,a}\delta_{\sigma_y,a}\rangle - \langle \delta_{\sigma_x,a}\delta_{\sigma_y,b}\rangle = \mathbb{P}(x\sim y, \sigma_x = a).$$

Taking the percolation limit, we see that the correlation function is simply the probability that the two points are in the same cluster.

3.

Consider again the Ising model in high temperature with no external field and for simplicity, free boundary conditions.

a) In the last exercise session, we proved that the correlation function $\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle$ decays exponentially in $\mathrm{dist}(X,Y)$. Let us do this again, but this time using the polymer expansion. Extrapolating the arguments from the lectures, one can write

(8) $$\langle\sigma_X\rangle_\Lambda = \sum_{B:\partial B = X} \rho(B)\exp\left(\sum_{C:C\cap B\neq\emptyset} f(C)\right)$$

Using this try to describe a diagrammatic expansion for $\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle$ and estimate the correlation function using this expansion. Note that you can use all the regularity results on f and so on.

b) Consider $\langle\sigma_x\sigma_y\sigma_z\sigma_w\rangle - \langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_w\rangle - \langle\sigma_x\sigma_z\rangle\langle\sigma_y\sigma_w\rangle - \langle\sigma_x\sigma_w\rangle\langle\sigma_y\sigma_z\rangle$. Use the polymer expansion on this and try to describe a diagrammatic expansion using the polymer expansion. How does the correlation function decay?

c) Define the cumulant in the following manner:

$$\langle\sigma_A\rangle^c = \sum_{\pi\in\mathrm{Part}(A)} (-1)^{|\pi|+1} \prod_{B\in\pi} \langle\sigma_B\rangle, \tag{9}$$

where Part(A) is the set of partitions of A.

What kind of diagrammatic expansion would you expect the cumulant to have in the polymer expansion?

Solution:

a)

Remark: The problem could have been a bit more explicit. The sum over B in the polymer expansion is not over arbitrary B. Recall that the point of the polymer expansion is to get rid of vacuum bubbles - that is loops that are not connected to any of the fixed points (in the case of the 2-point function it was the points x and y and for us it is the points in X or Y). For the 2-point function, getting rid of vacuum bubbles resulted in B being a connected set: the only graphs B with $\partial B = \{x, y\}$ without and vacuum bubbles are either connected graphs connecting x to y or graphs with two components, one containting x and the other containing y. In each of the components, one could make use of the $\mathbb{Z}^2$ symmetry to notice that they sum to 0 so we are only left with connected graphs. In the general case, we can't restric to connected graphs. We can still have graphs with several components if each component is connected to an even amount of points. But what the polymer expansion guarrantees is that we don't have to take into account vacuum bubbles. We shall suppress these constraints in the summation symbols.

Using the polymer expansion, we have (we suppress the index Λ)

$$\begin{aligned}\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle = &\sum_{B:\partial B = X\cup Y} \rho(B) e^{\sum_{C:C\cap B\neq\emptyset} f(C)} \\ &- \sum_{B_1,B_2:\partial B_1 = X, \partial B_2 = Y} \rho(B_1)\rho(B_2) e^{\sum_{C_1:C_1\cap B_1\neq\emptyset} f(C_1) + \sum_{C_2:C_2\cap B_2\neq\emptyset} f(C_2)}.\end{aligned}$$

Let us now examine what the sets B with $\partial B = X\cup Y$ look like. One possibility is that $B = B_1\cup B_2$ where B_1 and B_2 are disjoint and $\partial B_1 = X$ and $\partial B_2 = Y$. If this is not the case, B has a component connecting a point in X to a point in Y. To simplify notation, let us write $X \sim_B Y$ for such a case. So we can write the first sum as

$$\sum_{B_1,B_2:\partial B_1 = X,\partial B_2 = Y, B_1\cap B_2 = \emptyset} \rho(B_1\cup B_2) e^{\sum_{C:C\cap(B_1\cup B_2)\neq\emptyset} f(C)} + \sum_{B:\partial B = X\cup Y, X\sim_B Y} \rho(B) e^{\sum_{C:C\cap B\neq\emptyset} f(C)}. \tag{10}$$

Note that since B_1 and B_2 are disjoint in the first sum, $\rho(B_1\cup B_2) = \rho(B_1)\rho(B_2)$ and $\sum_{C:C\cap(B_1\cup B_2)\neq\emptyset} = \sum_{C:C\cap B_1\neq\emptyset} + \sum_{C:C\cap B_2\neq\emptyset}$.

On the other hand, we can split the $\langle\sigma_X\rangle\langle\sigma_Y\rangle$ sum into two parts: one where the B_i are disjoint and another where they overlap. The disjoint part cancels with a term from the $\langle\sigma_X\sigma_Y\rangle$ sum so we have

$$\begin{aligned}\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle = &\sum_{B:\partial B = X\cup Y, X\sim_B Y} \rho(B) e^{\sum_{C:C\cap B\neq\emptyset} f(C)} \\ &- \sum_{B_1,B_2:\partial B_1 = X,\partial B_2 = Y, B_1\cap B_2\neq\emptyset} \rho(B_1)\rho(B_2) e^{\sum_{C_1:C_1\cap B_1\neq\emptyset} f(C_1) + \sum_{C_2:C_2\cap B_2\neq\emptyset} f(C_2)}.\end{aligned}$$

Inside the second sum, let us insert the number one disguised in a slightly diguised form

$$\sum_{B_1,B_2:\partial B_1=X,\partial B_2=Y,B_1\cap B_2\neq\emptyset}\sum_{B:\partial B=X\cup Y,X\sim_B Y}\mathbf{1}(B_! \cup B_2=B)\rho(B_1)\rho(B_2)e^{\sum_{C_1:C_1\cap B_1\neq\emptyset}f(C_1)+\sum_{C_2:C_2\cap B_2\neq\emptyset}f(C_2)}.$$

Under the condition $B_1\cup B_2=B$, we have $\rho(B_1)\rho(B_2)=(\tanh\beta)^{|B|}(\tanh\beta)^{|B_1\cap B_2|}$ and

$$\sum_{C_1:C_1\cap B_1\neq\emptyset}f(C_1)+\sum_{C_2:C_2\cap B_2\neq\emptyset}f(C_2)=\sum_{C:C\cap B\neq\emptyset}f(C)-\sum_{C':C'\cap(B_1\cap B_2)\neq\emptyset}f(C'). \tag{11}$$

So we find that

$$\langle\sigma_X\sigma_Y\rangle-\langle\sigma_X\rangle\langle\sigma_Y\rangle=\sum_{B:\partial B=X\cup Y,X\sim_B Y}\rho(B)e^{\sum_{C:C\cap B\neq\emptyset}f(C)}g(B,X,Y), \tag{12}$$

where

$$g(B,X,Y)=\left(1-\sum_{B_1,B_2:\partial B_1=X,\partial B_2=Y,B_1\cap B_2\neq\emptyset}\mathbf{1}(B=B_1\cup B_2)\rho(B_1\cap B_2)e^{\sum_{C':C'\cap(B_1\cap B_2)\neq\emptyset}f(C')}\right). \tag{13}$$

So we have deduced an expansion for $\langle\sigma_X\sigma_Y\rangle-\langle\sigma_X\rangle\langle\sigma_Y\rangle$ where we sum over all graphs which have $X\cup Y$ as a boundary, no vacuum bubbles and contain a path connecting a point in X to a point in Y. Each graph B is then weighted by the term $\rho(B)e^{\sum_{C:C\cap B\neq\emptyset}f(C)}g(B,X,Y)$.

To estimate this, we make some very crude estimates. First of all, we use Lemma 4.5 from the lecture notes and the resoning below equation (4.15) to see that $|\rho(B_1\cap B_2)e^{\sum_{C':C'\cap(B_1\cap B_2)\neq\emptyset}f(C')}|\leq(2\tanh\beta)^{|B_1\cap B_2|}\leq 2\tanh\beta$. Then we note that since we are summing over two subsets of B, there are at most $2^{|B|}\times 2^{|B|}$ terms in the sum so $|g(B,X,Y)|\leq 2\tanh\beta 4^{|B|}$. Thus we find that for small enough β

$$\begin{aligned}|\langle\sigma_X\sigma_Y\rangle-\langle\sigma_X\rangle\langle\sigma_Y\rangle|&\leq\sum_{B:\partial B=X\cup Y,X\sim_B Y}(2\tanh\beta)^{|B|}4^{|B|}\\&\leq\sum_{n=\mathrm{dist}(X,Y)}^{\infty}(8\tanh\beta)^n|\{B:|B|=n,\partial B=X\cup Y,X\sim_B Y\}|.\end{aligned}$$

To estimate the number of such sets B, note that since we have no vacuum bubbles, each point in X is connected to a point in X or to one in Y. Moreover, since $|B|=n$, the length of each path must be less than n. So we conclude that

(14)
$$|\{B:|B|=n,\partial B=X\cup Y,X\sim_B Y\}|\leq\prod_{a\in X\cup Y}\sum_{b\in X\cup Y}|\{P:a\to b:|P|\leq n\}|\leq(C(|X|+|Y|)(2d)^n)^{|X|+|Y|},$$

where we used the familiar estimate for a number of paths from a point to another. What this estimate gives is that the size of the set is some number (depending on X and Y) to power n. Thus by making β small enough, we find that the correlation function decays exponentially in $\mathrm{dist}(X,Y)$.

b) Using the polymer expansion we write

$$\langle\sigma_x\sigma_y\sigma_z\sigma_w\rangle=\sum_{B:\partial B=\{x,y,z,w\}}\rho(B)e^{\sum_{C:C\cap B\neq\emptyset}f(C)}. \tag{15}$$

The $\mathbb{Z}^2$ symmetry implies that the only types of B that don't sum to zero are the following: either B is connected or it has two components B_1 and B_2 which are disjoint and ∂B_i contains two of the points x, y, z, w (∂B_1 and ∂B_2 are disjoint). Thus we find

$$\begin{aligned}\langle \sigma_x \sigma_y \sigma_z \sigma_w \rangle = & \sum_{B:\partial B=\{x,y,z,w\},B \text{ connected}} \rho(B) e^{\sum_{C:C\cap B\neq\emptyset} f(C)} + \\ & + \sum_{B_1,B_2:\partial B_1=\{x,y\}\partial B_2=\{z,w\},B_1\cap B_2=\emptyset,B_i \text{ connnected}} \rho(B_1)\rho(B_2) e^{\sum_{C_1:C_1\cap B_1\neq\emptyset} f(C_1)+\sum_{C_2:C_2\cap B_2\neq\emptyset} f(C_2)} \\ & + \text{other possible pairings.}\end{aligned}$$

With similar reasoning,

$$\begin{aligned}\langle \sigma_x \sigma_y \rangle \langle \sigma_z \sigma_w \rangle = & \sum_{B_1,B_2:\partial B_1=\{x,y\}\partial B_2=\{z,w\},B_1\cap B_2=\emptyset,B_i \text{ connnected}} \rho(B_1)\rho(B_2) e^{\sum_{C_1:C_1\cap B_1\neq\emptyset} f(C_1)+\sum_{C_2:C_2\cap B_2\neq\emptyset} f(C_2)} \\ & + \sum_{B_1,B_2:\partial B_1=\{x,y\}\partial B_2=\{z,w\},B_1\cap B_2\neq\emptyset,B_i \text{ connnected}} \rho(B_1)\rho(B_2) e^{\sum_{C_1:C_1\cap B_1\neq\emptyset} f(C_1)+\sum_{C_2:C_2\cap B_2\neq\emptyset} f(C_2)}.\end{aligned}$$

So we see that in the correlation function we are interested in, the terms with B_1 and B_2 disjoint cancel. As in a), we combine the remaining sums into one and are left with a sum

$$\sum_{B:\partial B=\{x,y,z,w\},B \text{ connected}} \rho(B) e^{\sum_{C:C\cap B\neq\emptyset} f(C)} g(B,x,y,z,w). \tag{16}$$

Writing things out in detail and following the reasoning in a) estimate g, we see that the decay of the correlations is controlled by the minimum distance between the points in $\{x, y, z, w\}$. So we see that correlations decay exponentially in this minimum distance (though it might be that using finer estimates we could get faster decay, since our arguments only give an upper bound).

c) Based on the arguments used in a) and b) it should be at least plausable that a cancellation occurse and the only graphs that contribute are connected ones.

Remark. I'll try to write out a proper proof for this fact at some point.