STATISTICAL MECHANICS - EXERCISE 2

1. Show that in the high temperature case (i.e. small enough β), for any $A \subset \mathbb{Z}^d$, the correlation function $\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}}$ converges as we take $\Lambda \to \mathbb{Z}^d$ (take the limit along cubes for simplicity: let $\Lambda = \{-L..., L\}^d$ and $L \to \infty$).

Hint: Proceed as when proving uniqueness of the limit: estimate $|\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}}|$ by duplicating the summation variable and show that we are dealing with a Cauchy sequence.

Solution: Let Λ and Λ' be cubes and $\Lambda \subset \Lambda'$. Let us write $\Omega_\Lambda = \{-1, 1\}^\Lambda$ and similarly for $\Omega_{\Lambda'}$. We have

(1) $$\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}} = \frac{\sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega_{\Lambda'}} (\sigma_A - \sigma'_A) e^{-\beta \mathcal{H}_\Lambda^{\bar{\sigma}}(\sigma) - \beta \mathcal{H}_{\Lambda'}^{\bar{\sigma}}(\sigma')}}{\sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega_{\Lambda'}} e^{-\beta \mathcal{H}_\Lambda^{\bar{\sigma}}(\sigma) - \beta \mathcal{H}_{\Lambda'}^{\bar{\sigma}}(\sigma')}}.$$

As in the lectures, we write $e^{\beta(\sigma_x \sigma_y + \sigma'_x \sigma'_y)} = e^{-2\beta}(1 + f_{x,y})$. Write $\bar{\mathcal{B}}_{\Lambda'}$ for the set of bonds b intersecting Λ'. While we can't directly use the same expansion as in the case of the uniqueness of the limit, we can make use of it after a small trick: in the correlation function let us just multiply the numerator and denominator by the terms $e^{-\beta \bar{\sigma}_x \bar{\sigma}_y}$ so that $\mathcal{H}_\Lambda^{\bar{\sigma}}(\sigma)$ is formally extended to Λ' so that outside of Λ σ is defined to be $\bar{\sigma}$. Call the modified Hamiltonian $\tilde{\mathcal{H}}_{\Lambda'}^{\bar{\sigma}}(\sigma)$. We then have

(2) $$\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}} = \frac{\sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega_{\Lambda'}} (\sigma_A - \sigma'_A) e^{-\beta \tilde{\mathcal{H}}_{\Lambda'}^{\bar{\sigma}}(\sigma) - \beta \mathcal{H}_{\Lambda'}^{\bar{\sigma}}(\sigma')}}{\sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega_{\Lambda'}} e^{-\beta \tilde{\mathcal{H}}_{\Lambda'}^{\bar{\sigma}}(\sigma) - \beta \mathcal{H}_{\Lambda'}^{\bar{\sigma}}(\sigma')}}.$$

We use the same notation $f_{x,y}$ for the extended Hamiltonian with the understanding that outside of Λ, we replace σ by $\bar{\sigma}$. We can now expand:

(3) $$\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}} = \frac{\sum_{B \subset \bar{\mathcal{B}}_{\Lambda'}} \sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega_{\Lambda'}} (\sigma_A - \sigma'_A) \prod_{b \in B} f_b}{\sum_{\sigma \in \Omega_\Lambda, \sigma' \in \Omega'_\Lambda} \prod_{b \in \bar{\mathcal{B}}_{\Lambda'}} (1 + f_b)}.$$

Consider now some $B \subset \bar{\mathcal{B}}_{\Lambda'}$ and let $B_1 \subset B$ be the collection of bonds connected to A. If this collection does not intersect Λ^C, $(\sigma_A - \sigma'_A) \prod_{b \in B_1} f_b$ factors out of the σ, σ' sum and is antisymmetric under the relabelling of the the summation variables $\sigma \leftrightarrow \sigma'$ and we see that the sum over the spins vanishes. Thus such sets B don't contribute. Following the same reasoning as in the lecture notes, we find that

(4) $$|\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}}| \leq 2 \sum_{x \in A} \sum_{y \in \bar{\Lambda}' \setminus \Lambda} \sum_{P : x \to y} (4\beta)^{|P|}$$

(5) $$= 2 \sum_{x \in A} \sum_{y \in \bar{\Lambda}' \setminus \Lambda} \sum_{n = |x-y|}^{\infty} (4\beta)^n |\{P : x \to y | |P| = n\}|$$

(6) $$\leq 2 \sum_{x \in A} \sum_{y \in \bar{\Lambda}' \setminus \Lambda} \sum_{n = |x-y|}^{\infty} (4\beta)^n |\{P : x \to \text{ anywhere } | |P| = n\}|.$$

Consider now a path starting at some fixed point x. At each step in the path, it has $2d$ directions it can take. Thus there are $(2d)^n$ paths of length n so

(7) $$\sum_{P : x \to y} (4\beta)^{|P|} \leq \sum_{n = |x-y|}^{\infty} (8d\beta)^n \leq C e^{-\alpha |x-y|}$$

So we conclude that for small enough β

(8) $$|\langle \sigma_A \rangle_\Lambda^{\bar{\sigma}} - \langle \sigma_A \rangle_{\Lambda'}^{\bar{\sigma}}| \leq 2 \sum_{x \in A} \sum_{y \in \bar{\Lambda}' \setminus \Lambda} C e^{-\alpha |x-y|} \leq 2C e^{-\alpha \mathrm{dist}(X, \Lambda^C)} | \leq 2|A| \sum_{y \in \bar{\Lambda}' \setminus \Lambda} C e^{-\alpha \mathrm{dist}(y, A)}.$$

To check that this indeed is a Cauchy sequnce, note that this can be estimated from above by the integral

(9) $$C' \int_{L \leq |y| \leq L'} e^{-\alpha' |y|} d^d y.$$

Switching to spherical coordinates one this bound becomes

$$\tilde{C}\int_L^{L'} r^{d-1}e^{-\alpha' r}dr \tag{10}$$

and it is clear we are dealing with a Cauchy sequence.

2. Prove clustering in the high temperature case: for small enough β and for any $X, Y \subset \mathbb{Z}^d$

$$|\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma}| \leq Ce^{-\alpha \mathrm{dist}(X,Y)} \tag{11}$$

for some $\alpha, C > 0$ independent of Λ and $\bar\sigma$.

Solution: Since we are dealing with a given Λ and $\bar\sigma$, let us suppress these indices in the Hamiltonian. We note first that we can write

$$\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{\sum_{\sigma,\sigma'}\sigma_X\sigma_Y e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}{\sum_{\sigma,\sigma'}e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}, \tag{12}$$

since the sums over σ' just cancel. On the other hand,

$$\langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{\sum_{\sigma,\sigma'}\sigma_X\sigma'_Y e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}{\sum_{\sigma,\sigma'}e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}} \tag{13}$$

so

$$\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{\sum_{\sigma,\sigma'}(\sigma_X\sigma_Y - \sigma_X\sigma'_Y)e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}{\sum_{\sigma,\sigma'}e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}. \tag{14}$$

Just by relabelling the summation indices, we can write this as

$$\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{\sum_{\sigma,\sigma'}(\sigma'_X\sigma'_Y - \sigma'_X\sigma_Y)e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}{\sum_{\sigma,\sigma'}e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}} \tag{15}$$

so by adding these two expressions, we find

$$\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{1}{2}\frac{\sum_{\sigma,\sigma'}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}{\sum_{\sigma,\sigma'}e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))}}. \tag{16}$$

Let us now proceed as in the lectures: write $e^{\beta(\sigma_x\sigma_y+\sigma'_x\sigma'_y)} = e^{-2\beta}(1+f_{x,y})$ and let $\bar{\mathcal{B}}_\Lambda$ be the collection of bonds b intersecting Λ. Then

$$e^{-\beta(\mathcal{H}(\sigma)+\mathcal{H}(\sigma'))} = \prod_{b\in\bar{\mathcal{B}}_\Lambda} e^{-2\beta}(1+f_b) = e^{-2\beta|\bar{\mathcal{B}}_\Lambda|}\sum_{B\subset\bar{\mathcal{B}}_\Lambda}\prod_{b\in B} f_b. \tag{17}$$

Thus

$$\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma} = \frac{1}{2}\frac{\sum_{B\subset\bar{\mathcal{B}}_\Lambda}\sum_{\sigma,\sigma'}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)\prod_{b\in B}f_b}{\sum_{\sigma,\sigma'}\prod_{b\in\bar{\mathcal{B}}_\Lambda}(1+f_b)} \tag{18}$$

Consider now any $B \subset \bar{\mathcal{B}}_\Lambda$. Let $B_1 \subset B$ be the set of bonds that are not connected to both X and Y (we also interpret a bond to be connected to both X and Y if it is connected to say X and the boundary and if there exists a path from the boundary to Y). The sum over the σ and σ' now factorizes:

$$\sum_{\sigma,\sigma'}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)\prod_{b\in B} f_b = \sum_{\sigma_x,\sigma'_x : x\in B_1\cup X\cup Y}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)\prod_{b\in B_1} f_b \sum_{\sigma_x,\sigma'_x : x\in B\setminus B_1}\prod_{b\in B\setminus B_1} f_b. \tag{19}$$

Let us assume for definiteness, that in this case, it is the set X that is not connected to the boundary. Then the sum factorizes further, we can sum over the spins in the set X and those connected to the set X. Since the boundary conditions don't play a role and we aren't connected to Y, we can independently switch the names of the summation variables $\sigma \leftrightarrow \sigma'$ at these points. Since f_b is symmetric with respect to σ and σ', this just changes the total sign of the term in the sum. So we conclude that for such a B

$$\sum_{\sigma,\sigma'}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)\prod_{b\in B} f_b = -\sum_{\sigma,\sigma'}(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)\prod_{b\in B} f_b = 0. \tag{20}$$

So in the expansion, we only have to worry about sets B that contain paths from X to Y or paths from X to the boundary and from Y to the boundary.

Noting that $|(\sigma_X - \sigma'_X)(\sigma_Y - \sigma'_Y)| \leq 4$, we can proceed exactly as in the lecture notes to find

$$|\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma}| \leq 2\sum_P (4\beta)^{|P|}, \tag{21}$$

where P is either a path from X to Y or it has two components: one from X to the boundary and another from Y to the boundary. So we see that

$$\sum_P (4\beta)^{|P|} = \sum_{x\in X}\sum_{y\in Y}\sum_{P:x\to y}(4\beta)^{|P|} + \sum_{x\in X}\sum_{y\in Y}\sum_{z_1\in\partial\Lambda}\sum_{z_2\in\partial\Lambda}\sum_{P_1:x\to z_1}\sum_{P_2:y\to z_2}(4\beta)^{|P_1|+|P_2|}. \tag{22}$$

To estimate the sums over the paths, we note that

$$\sum_{P:x\to y}(4\beta)^{|P|} = \sum_{n=|x-y|}^{\infty}(4\beta)^n|\{P : x\to y||P| = n\}| \leq \sum_{n=|x-y|}^{\infty}(4\beta)^n|\{P : x\to \text{ anywhere }||P| = n\}|. \tag{23}$$

Consider now a path starting at some fixed point x. At each step in the path, it has $2d$ directions it can take. Thus there are $(2d)^n$ paths of length n so

$$\sum_{P:x\to y}(4\beta)^{|P|} \leq \sum_{n=|x-y|}^{\infty}(8d\beta)^n \leq Ce^{-\alpha|x-y|} \tag{24}$$

for small enough β. So we conclude that

$$\sum_{x\in X}\sum_{y\in Y}\sum_{P:x\to y}(4\beta)^{|P|} \leq C|X||Y|e^{-\alpha \text{dist}(X,Y)} \tag{25}$$

and

$$\sum_{x\in X}\sum_{y\in Y}\sum_{z_1\in\partial\Lambda}\sum_{z_2\in\partial\Lambda}\sum_{P_1:x\to z_1}\sum_{P_2:y\to z_2}(4\beta)^{|P_1|+|P_2|} \leq C^2|X||Y||\partial\Lambda|^2 e^{-\alpha(\text{dist}(X,\partial\Lambda)+\text{dist}(Y,\partial\Lambda))}. \tag{26}$$

For large L, $\text{dist}(X,\partial\Lambda) \sim L$ and $|\partial\Lambda| \sim L^{d-1}$ so we see that for large enough L,

$$|\partial\Lambda|^2 e^{-\alpha(\text{dist}(X,\partial\Lambda)+\text{dist}(Y,\partial\Lambda))} \leq e^{-\tilde\alpha \text{dist}(X,Y)} \tag{27}$$

and

$$|\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma}| \leq \tilde C|X||Y|e^{-\tilde\alpha \text{dist}(X,Y)}. \tag{28}$$

3. Let $\Lambda \subset \mathbb{Z}^d$ and $x \in \Lambda$. Let us assume that we have some configuration of contours in Λ. For a path P from x to Λ^C, define $N(P)$ to be the number of times the path crosses a contour. Show that for any two such paths P and P', $N(P) - N(P')$ is an even number.

Solution: Let us have two such paths P and P'. Let $y, y' \in \Lambda^C$ and consider a path $P'' : y \to y'$ so that P'' travels only in Λ^C. Let us then form the loop $\mathcal{L} = P \cup P' \cup P''$. By the definition of a contour, $\mathcal{L}$ can cross each contour only an even amount of times. Thus $N(\mathcal{L})$ is an even number. On the other hand, $N(\mathcal{L}) = N(P) + N(P') + N(P'')$. Since P'' is a path in Λ^C, $N(P'') = 0$. Thus $N(P) + N(P')$ is an even number as is $N(P) - N(P') = N(P) + N(P') - 2N(P')$.

4. What is the appropriate way to define a contour when $d \geq 3$? Prove Lemma 4.4 and relations (4.7) and (4.8) from the lecture notes in the case that $d \geq 3$.

Solution: The idea behind the definition is that we wish to construct the equivalents of loops (some sort of $d-1$ dimensional closed surfaces) without any internal "walls" or external legs. For $d=2$ the condition was that a contour is a connected union of dual bonds so that each point in the dual lattice belongs to only an even number of dual bonds. For example, if a point belonged to only one dual bond, it would be at the tip of a bond which would mean that we aren't dealing with a loop. If a point belonged to three bonds, it would mean that there would be an internal wall (drawing pictures might help).

Let us try to think what the condition that we are dealing with loops (closed surfaces) in three dimension should be. The simplest object we could form from plaquettes is just connecting one after another in a single direction. So that we could modify this into a closed surface, we must at least attach the two end edges to each other. This is similar to the idea that we can form a loop from a segment by attaching the end points. With this analogue it would seem natural to try define a contour as a connected collection of plaquettes so that each edge belongs to only an even number of plaquettes. Indeed after drawing some pictures, it becomes clear that the condition that no edge belongs to a single plaquette means that there are no external plaquettes hanging from our contour or holes in it. The condition that no edge belongs to three plaquettes implies that there are no internal walls.

For higher dimensions, drawing pictures becomes hard, but the idea remains the same. We are interested in connected collections of $d-1$ dimensional plaquettes (unit hypersquares or however you wish to call them). The regularity condition is that no $d-2$ dimensional "edge" should belong to an odd number of plaquettes.

The statement of Lemma 4.4 was that if for a spin configuration σ we define $C(\sigma)$ to be the collection of bonds b intersecting Λ so that if $b=\{x,y\}$, then $\sigma_x \neq \sigma_y$, then the dual object $C^*(\sigma)$ which consists of the plaquettes dual to these bonds, can be decomposed into a unique disjoint union of contours. Conversely, any disjoint collection of contours defines a spin configuration.

The proof of this is almost the same as in the 2-d case: consider a fixed edge in a connected component of the union and form a loop around it where the loop consists of the bonds in the original lattice. Take a product of all spins of the bonds. Just as in the 2-d case, the product is 1 so the power of -1 is an even number and this number corresponds precisely to how many plaquettes the edge is adjacent to. The other direction is identical to the 2-d case.

Formulae (4.7) and (4.8) follow immediately from the lemma.