STATISTICAL MECHANICS - EXERCISE 10

1. a) Let $z_1, ..., z_n$ be centered jointly Gaussian random variables with a translation invariant covariance Γ_t, where $t \in [0, \infty)$ is a parameter. Let us assume that Γ_t is a smooth function of t. Let $\mu_{\Gamma_t}(dz)$ be the law of the random vector $z = (z_1, ..., z_n)$.

For any smooth enough function F, calculate

$$\frac{d}{dt} \int F(z) \mu_{\Gamma_t}(dz) \tag{1}$$

b) How would you try to extend this to the case where z is a Gaussian field on $\mathbb{R}^d$?

Hint: In this and the next problem you might have to think about stuff like differentiating functions of functions. Take a physicist's approach and don't worry too much about the details of defining functional derivatices, just try to think of $\frac{\delta F}{\delta z(x)}$ as the natural generalization of a partial derivative $\frac{\partial F}{\partial z_x}$.

Solution: Let us write

$$I_t = \int F(z) \mu_{\Gamma_t}(dz) = \int F(z) \frac{e^{-\frac{1}{2} z^T \Gamma_t^{-1} z}}{\sqrt{(2\pi)^n \det(\Gamma_t)}} dz. \tag{2}$$

Diagonalizing, we note that there is a positive definite symmetric matrix C_t so that $C_t \cdot C_t = \Gamma_t$. Changing the integration variable to $w = C_t^{-1} z$, we find

$$I_t = \int F(C_t w) \mu_{Id}(dw), \tag{3}$$

where μ_{Id} is the Gaussian measure with the identity matrix as the covariance matrix. Differentiating, we find

$$\begin{aligned}
\frac{d}{dt} I_t &= \int \sum_i (\partial_i F)(C_t w) \sum_j \left(\frac{d}{dt} C_t \right)_{ij} w_j \mu_{Id}(dw) \\
&= \sum_{i,j} \left(\frac{d}{dt} C_t \right)_{ij} \int (\partial_i F)(C_t w) \left(-\frac{\partial}{\partial w_j} \right) \mu_{Id}(dw) \\
&= \sum_{i,j} \left(\frac{d}{dt} C_t \right)_{ij} \int \frac{\partial}{\partial w_j} (\partial_i F)(C_t w) \mu_{Id}(dw) \\
&= \sum_{i,j} \left(\frac{d}{dt} C_t \right)_{ij} \int \sum_k (\partial_k \partial_i F)(C_t w) \left(\frac{\partial}{\partial w_j} (C_t w)_k \right) \mu_{Id}(dw) \\
&= \sum_{i,j} \left(\frac{d}{dt} C_t \right)_{ij} \int \sum_k (\partial_k \partial_i F)(C_t w) \left(\frac{\partial}{\partial w_j} \sum_l (C_t)_{k,l} w_l \right) \mu_{Id}(dw) \\
&= \sum_{i,k} \left(\left(\frac{d}{dt} C_t \right) C_t \right)_{i,k} \int (\partial_k \partial_i F)(C_t w) \mu_{Id}(dw) \\
&= \sum_{i,k} \left(\left(\frac{d}{dt} C_t \right) C_t \right)_{i,k} \int \partial_k \partial_i F(z) \mu_{\Gamma_t}(dz).
\end{aligned}$$

When C_t commutes with $\frac{d}{dt} C_t$, we see that $C_t \frac{d}{dt} C_t = \frac{1}{2} \frac{d}{dt} \Gamma_t$ and we have in fact

(4) $$\frac{d}{dt}I_t = \frac{1}{2}\sum_{i,k}\left(\frac{d}{dt}\Gamma_t\right)(i-k)\int \partial_k\partial_i F(z).\mu_{\Gamma_t}(dz).$$

b) Taking the square root of Γ_t can be done in the continuum case as well in some generality, but a rigorous definition requires some functional analysis and we will skip it now. Given this piece of information, the argument goes through pretty much in the same manner and for the case that the square root commutes with its derivative, the following formula should be believable at least at a heuristic level:

(5) $$\frac{d}{dt}I_t = \frac{1}{2}\int dxdy\left(\frac{d}{dt}\Gamma_t\right)(x-y)\int \frac{\delta}{\delta z(x)}\frac{\delta}{\delta z(y)}F(z)\mu_{\Gamma_t}(dz).$$

If you are unfamiliar with functional differentiation, check Wikipedia.

2. In this problem we shall do our renormalization continuously instead of in steps as we have done so far. Don't worry too much about the rigorous details. As usual, there might be some errors in the statement of the problem.

Consider the covariance G in Fourier space:

(6) $$\hat{G}(p) = \frac{\chi(p)}{p^2},$$

where $\chi(p) = e^{-p^2}$. As usual in the renormalization industry, we split the covariance into two parts:

(7) $$\hat{G}(p) = \frac{\chi(e^s p)}{p^2} + \frac{\chi(p)-\chi(e^s p)}{p^2} = e^{2s}\hat{G}(e^s p) + \Gamma_s.$$

We then start with some interaction potential V_0 and define

(8) $$e^{-V_s(\phi)} = \int e^{-V_0(e^{-s(d-2)}\phi(e^{-s}\cdot)+z)}\mu_{\Gamma_s}(dz).$$

a) Show that we have a semigroup property:

(9) $$e^{-V_{s+t}(\phi)} = \int e^{-V_s(e^{-t(d-2)}\Phi(e^{-t}\cdot)+z)}\mu_{\Gamma_t}(dz).$$

b) Using this, calculate

(10) $$\frac{d}{ds}V_s = \left.\frac{d}{dt}V_{s+t}\right|_{t=0}.$$

Remark: Again, don't worry too much about the functional differentiation. Try to think of it like all the rules of finite dimensional calculus hold.

c) Linearize around $V = 0$ and show that $:\phi^4:$ is an eigenvector.

Solution: a) By the definition of V_s, we have

(11) $$\int e^{-V_s(e^{-t(d-2)}\phi(e^{-t}\cdot)+z)}\mu_{\Gamma_t}(dz) = \int\int e^{-V_0(e^{-s(d-2)}(e^{-t(d-2)}\phi(e^{-s}e^{-t}\cdot)+z_1)+z_2)}\mu_{\Gamma_t}(dz_1)\mu_{\Gamma_s}(dz_2).$$

We then note that

$$\hat{\Gamma}_{t+s}(p) = \frac{\chi(p) - \chi(e^{t+s}p)}{p^2} = \hat{\Gamma}_t(p) + \frac{\chi(e^t p) - \chi(e^{t+s}p)}{p^2} = \hat{\Gamma}_t(p) + e^{2t}\hat{\Gamma}_s(e^t p). \tag{12}$$

Thus in real space,

$$\Gamma_{t+s}(x) = \Gamma_t(x) + \int dp e^{ip\cdot x} e^{2t}\hat{\Gamma}_s(e^t p) = \Gamma_t(x) + e^{-(d-2)t}\Gamma_s(e^{-t}x). \tag{13}$$

Plugging this in gives the semigroup property.

b) We have now (you can check that indeed $C_t^2 = \Gamma_t$ with a simple calculation)

$$C_t(x-y) = \int dp e^{ip\cdot(x-y)}\sqrt{\frac{\chi(p) - \chi(e^t p)}{p^2}}. \tag{14}$$

From this representation, it is a straightforward calculation to check that C_t commutes with $\partial_t C_t$. Thus we can use (4) from problem 1. Using the semigroup property, we have

$$\begin{aligned}\frac{\partial}{\partial t}V_{s+t}\Big|_{t=0} &= -\frac{1}{e^{-V_s}}\left(e^{-V_s(\phi)}\int dx \frac{\delta V_s}{\delta\phi(x)}(-(d-2)\phi(x) - x\cdot\nabla\phi(x)) + \partial_t \int e^{-V_s(\phi+z)}\mu_{\Gamma_t}(dz)\Big|_{t=0}\right) \\ &= \int dx \frac{\delta V_s}{\delta\phi(x)}((d-2)\phi(x) + x\cdot\nabla\phi(x)) + \frac{1}{2}\int dxdy\left(\frac{d}{dt}\Gamma_t\Big|_{t=0}\right)(x-y)\frac{\delta V_s(\phi)}{\delta\phi(x)}\frac{\delta V_s(\phi)}{\delta\phi(y)}\end{aligned}$$

c) The linearized version of the renormalization map is

$$L_s V = \int \mu_{\Gamma_s}(dz) V(e^{-s(d-2)}\phi(e^{-s}\cdot) + z). \tag{15}$$

We wish to show that $V = \int dx : \phi(x)^4 :$ is an eigenvector for this mapping. In a similar manner as in the lecture notes and the next problem, write $\Psi(x) = e^{-s(d-2)}\phi(e^{-s}x)$. Let C be the covariance of Ψ. By the definition of normal ordering (note that we normal order with respect to the covariance $C + \Gamma_s$)

$$\begin{aligned}: (\Psi(x) + z(x))^4 : &= (\Psi(x) + z(x))^4 - 6(C(0) + \Gamma_s(0))(\Psi(x) + z(x))^2 + 3(C(0) + \Gamma_s(0))^2 \\ &= \Psi(x)^4 + 4\Psi(x)z(x)^3 + 6\Psi(x)^2 z(x)^2 + 4\Psi(x)^3 z(x) + z(x)^4 \\ &\quad - 6(C(0) + \Gamma_s(0))(\Psi(x)^2 + 2\Psi(x)z(x) + z(x)^2) + 3(C(0) + \Gamma_s(0))^2.\end{aligned}$$

Integrating this with respect to μ_{Γ_s} (and noting that $\int \mu_{\Gamma_s}(dz) z(x)^2 = \Gamma_s(0)$, $\int \mu_{\Gamma_s}(dz) z(x)^4 = 3\Gamma_s(0)^2$ and integrals of odd powers are zero), we find for $V = \int : \phi^4 :$

$$\begin{aligned}L_s V &= \int dx\left(\Psi(x)^4 + 6\Gamma_s(0)\Psi(x)^2 + 3\Gamma_s(0)^2 - 6(C(0) + \Gamma_s(0))(\Psi(x)^2 + \Gamma_s(0)) + 3(C(0) + \Gamma_s(0))^2\right) \\ &= \int dx(\Psi(x)^4 - 6C(0)\Psi(x)^2 + 3C(0)^2) \\ &= \int dx : \Psi(x)^4 : \\ &= \int dx : (e^{-s(d-2)}\phi(e^{-s}x))^4 : \\ &= e^{ds}e^{-4s(d-2)}\int dx : \phi(x)^4 : \\ &= e^{-3ds+8s}V.\end{aligned}$$

So indeed V is an eigenvector.

Remark: The formulation of the problem was incorrect once again. It is fixed here.

3. Show that if we Wick order the interaction: $V = \int dx : \phi(x)^4 :$, then there are no tadpole diagrams in the expansion of RV (tadpole diagrams being those containing loops coming from Z lines).

Solution: A loop coming from a Z-line corresponds to a factor of $\Gamma(0)$ so let us show that such terms don't emerge in the expansion of RV. The general philosophy is that Wick ordering removes loops and considering cumulants leads to connected graphs so it might be reasonable to expect that even a correlation function of the form (we are using the notation of the lecture notes - see page 100 and onwards)

$$\left\langle \prod_{i=1}^{m} : (\Psi(x_i) + Z(x_i))^4 : \right\rangle_Z . \tag{16}$$

would not contain terms of the form $\Gamma(0)$. Note that here we wick order with respect to the whole covariance $\Sigma_{x,y} = E((\Psi(x) + Z(x))(\Psi(y) + Z(y))) = \Gamma(x - y) + C(x - y)$ (let us also write $C(x - y) = E(\Psi(x)\Psi(y))$), but we only integrate out the field Z (the terms appearing in the expansion of RV are linear combinations of such correlation functions).

The generating functional of such correlation functions is

$$\left\langle : \prod_{i=1}^{m} e^{(J_i, \Psi+Z)} : \right\rangle_Z . \tag{17}$$

Using the definition of Wick ordering, we see that

$$: \prod_{i=1}^{m} e^{(J_i,\Psi+Z)} := \prod_{i=1}^{m} e^{(J_i,\Psi+Z)} e^{-(J_i,(C+\Gamma)J_i)} = e^{\left(\sum_i J_i,\Psi\right) - \sum_i (J_i,(C+\Gamma)J_i) + \left(\sum_i J_i, Z\right)}. \tag{18}$$

Then by the basic identity of Gaussian integrals we find

$$\begin{aligned} \left\langle : \prod_{i=1}^{m} e^{(J_i,\Psi+Z)} : \right\rangle_Z &= e^{\left(\sum_i J_i,\Psi\right) - \frac{1}{2}\sum_i (J_i,(C+\Gamma)J_i)} e^{-\frac{1}{2}\sum_{i,j}(J_i,\Gamma J_j)} \\ &= e^{\left(\sum_i J_i,\Psi\right) - \frac{1}{2}\sum_i (J_i,CJ_i)} e^{-\frac{1}{2}\sum_{i\neq j}(J_i,\Gamma J_j)} \end{aligned}$$

Since in the $(J_i, \Gamma J_j)$ term we always have $i \neq j$, we only get terms of the form $\Gamma(x_i - x_j)$ with $i \neq j$ in the correlation functions (which are given by functional derivatives of the generating functional). Thus there are no $\Gamma(0)$ terms in even this type of correlation functions so certainly there are none in the expansion of RV.