STATISTICAL MECHANICS - EXERCISE 1

1. Let us consider a probability measure p on a finite state space Ω (take for example $\Omega = \{-1,1\}^\Lambda$ for some finite set Λ if you wish). We define the entropy of the probability measure p by

$$S(p) = -\sum_{\sigma\in\Omega} p(\sigma)\log p(\sigma). \tag{1}$$

Let us also assume that we have an energy function $E:\Omega\to\mathbb{R}$ and that the average energy of the system is fixed, i.e.

$$\sum_{\sigma\in\Omega} E(\sigma)p(\sigma) = e \tag{2}$$

for some fixed $e\in[\min_\sigma E(\sigma), \max_\sigma E(\sigma)]$.

Show that the unique probability measure that maximizes the entropy under the constraint (2) is the Gibbs measure

$$p(\sigma) = \frac{e^{-\beta E(\sigma)}}{\sum_{\sigma\in\Omega} e^{-\beta E(\sigma)}} \tag{3}$$

for some unique value of β (there is one exception to the uniqueness of β - what is this?).

Solution: To maximize a function under constraints, we use the method of Lagrange multipliers. Note that in addition to the energy constraint, we have an implicit constraint that the measure is a probability measure, i.e. that $\sum_\sigma p(\sigma)=1$ and that $p(\sigma)\geq 0$ and $p(\sigma)\leq 1$. The $p(\sigma)\leq 1$ constraint is baked into the two other constraints ($p(\sigma)\geq 0$ and $\sum p(\sigma)=1$) but we do need to deal with the inequality constraint $p(\sigma)\geq 0$. If you are unfamiliar with the method of Lagrangian multipliers with inequality constraints, you might look at some book on optimization or just Google it. The basic idea is that for an inequality constraint $f(p)\geq 0$, we add a term $\lambda f(p)$ to the function we wish to optimize and at the extremal point, one must have $\lambda f(p)=0$ so either $f(p)=0$ or $\lambda=0$. So we define

$$S_{\lambda,\mu}(p) = -\sum_\sigma p(\sigma)\log p(\sigma) - \lambda\left(\sum_\sigma E(\sigma)p(\sigma)-e\right) - \mu\left(\sum_\sigma p(\sigma)-1\right) - \sum_\sigma \nu_\sigma p(\sigma) \tag{4}$$

We then have

$$\frac{\partial S_{\lambda,\mu}(p)}{\partial p(\sigma)} = -\log p(\sigma) - 1 - \lambda E(\sigma) - \mu - \nu_\sigma \tag{5}$$

and this vanishes when

$$p(\sigma) = e^{-1-\mu-\lambda E(\sigma)-\nu_\sigma}. \tag{6}$$

For the inequality constraints, we had the condition that $\nu_\sigma p(\sigma)$ must be zero at the critical point so we see that this is only possible if $\nu_\sigma = 0$ for all σ.

Note that this is the unique critical point in our domain and it is simple to check that the Hessian matrix is negative definite so this indeed is unique the global maximum for the entropy.

Using the constraint that we are considering a probability measure, we see that for such a $p(\sigma)$ we must have

$$e^{1+\mu} = \sum e^{-\lambda E(\sigma)} \tag{7}$$

so

$$p(\sigma) = \frac{e^{-\lambda E(\sigma)}}{\sum_\sigma e^{-\lambda E(\sigma)}}. \tag{8}$$

The value of λ is determined by the energy constraint:

$$(9) \qquad \frac{1}{\sum_\sigma e^{-\lambda E(\sigma)}} \sum_\sigma E(\sigma) e^{-\lambda E(\sigma)} = e.$$

To see that this equation has a unique solution, let us consider

$$(10) \qquad f(\lambda) = \frac{1}{\sum_\sigma e^{-\lambda E(\sigma)}} \sum_\sigma E(\sigma) e^{-\lambda E(\sigma)}.$$

Differentiating, we find

$$\begin{aligned} f'(\lambda) &= \frac{-\sum_\sigma E(\sigma)^2 e^{-\lambda E(\sigma)} \sum_\sigma e^{-\lambda E(\sigma)} + \left(\sum_\sigma E(\sigma) e^{-\lambda E(\sigma)}\right)^2}{\left(\sum_\sigma e^{-\lambda E(\sigma)}\right)^2} \\ &= -\langle E(\sigma)^2 \rangle + \langle E(\sigma) \rangle^2. \end{aligned}$$

Here we have denoted by $\langle \cdot \rangle$ the expectation with respect to the measure p. So we see that f' is just the negative of the variance of $E(\sigma)$ with respect to our probability measure so it's a strictly negative quantity unless $E(\sigma)$ is constant. So at least if $E(\sigma)$ is not constant, there is at most one solution to the equation for λ. On the other hand, $f(\infty) = \min_\sigma E(\sigma)$ and $f(-\infty) = \max_\sigma E(\sigma)$ so by continuity there is at least one solution in $[-\infty, \infty]$.

If the $E(\sigma)$ don't depend on σ, then the Gibbs measure is just the uniform measure and we can choose λ (or β) to have any value.

2. Consider the 1-dim Ising model on $\{-L, ..., L\}$ with arbitrary boundary conditions $\bar\sigma$ and a magnetic field h:

$$(11) \qquad \mathcal{H}^{\bar\sigma}_{L,h} = -\sum_{x=-L}^{L+1} \sigma_{x-1}\sigma_x - h \sum_{x=-L}^{L} \sigma_x,$$

where $\sigma_{-L-1} = \bar\sigma_{-L-1}$ and $\sigma_{L+1} = \bar\sigma_{L+1}$.

a) Show that the magnetization

$$(12) \qquad \langle \sigma_x \rangle^{\bar\sigma}_{L,h} = \frac{\sum_\sigma \sigma_x e^{-\beta \mathcal{H}^{\bar\sigma}_{L,h}(\sigma)}}{Z^{\bar\sigma}_{L,h}}$$

has a non-zero limit as $L \to \infty$ and that the limit independent of $\bar\sigma$. Show that as $h \to 0$, also the magnetization vanishes.

Also show that the two point function $\langle \sigma_x \sigma_y \rangle^{\bar\sigma}_{L,h}$ has a limit as $L \to \infty$ and that the limit is independent of $\bar\sigma$ and that correlations $\langle \sigma_x \sigma_y \rangle_h - \langle \sigma_x \rangle_h \langle \sigma_y \rangle_h$ decay exponentially in $|x - y|$.

Note: If you are having trouble with the problem, try doing it only in the case of periodic boundary conditions. Some things might look a bit more symmetric in this case.

b) For a vanishing magnetic field, $h = 0$, calculate an arbitrary correlation function $\langle \sigma_A \rangle$ in the limit $L \to \infty$ (recall that $\sigma_A = \prod_{x \in A} \sigma_x$).

Solution:

Remark: One could make use of the Perron-Frobenius theorem in the solution of this problem as we do in problem 3, but here we shall give a constructive proof with most of the gory details.

a) Let us begin by calculating the partition function. We have by definition

$$Z^{\bar\sigma}_{L,h} = \sum_{\sigma_{-L}} \cdots \sum_{\sigma_L} e^{\beta \bar\sigma_{-L-1}\sigma_{-L}} e^{h\beta\sigma_{-L}} \left(\prod_{x=-L+1}^{L} e^{\beta\sigma_{x-1}\sigma_x + \beta h \sigma_x} \right) e^{\beta \sigma_L \bar\sigma_{L+1}}$$

As in the lectures, we introduce the transfer matrix:

$$(13) \qquad T_{\sigma,\sigma'} = e^{\beta\sigma\sigma' + \beta h \sigma'}.$$

To clarify, these are the entries of a 2x2 matrix. We let σ and σ' run over -1 and 1 and the matrix notation later on will be such that in the upper left corner both indeces are $+1$, upper right corner the second index is positive and the first index is negative and so on.

Let us now consider for example the sum over σ_{-L+1}. The only terms depending on σ_{-L+1} are the first two terms of the product. So we can just factor everything else out and focus first on doing the sum

$$\sum_{\sigma_{-L+1}} e^{\beta\sigma_{-L}\sigma_{-L+1}+\beta h\sigma_{-L+1}} e^{\beta\sigma_{-L+1}\sigma_{-L+2}+\beta h\sigma_{-L+2}} = \sum_{\sigma_{-L+1}} T_{\sigma_{-L},\sigma_{-L+1}} T_{\sigma_{-L+1},\sigma_{-L+2}} = (T^2)_{\sigma_{-L},\sigma_{-L+2}}, \tag{14}$$

where in the last step we used the definition of matrix multiplication. Now we can continue in a recursive manner. We consider the sum over σ_{-L+2}. The only terms left that are depending on it are $(T^2)_{\sigma_{-L},\sigma_{-L+2}}$ and $e^{\beta\sigma_{-L+2}\sigma_{-L+3}+\beta h\sigma_{-L+3}} = T_{\sigma_{-L+2},\sigma_{-L+3}}$. So summing over σ_{-L+2} and using the definition of matrix multiplication gives $(T^3)_{\sigma_{-L},\sigma_{-L+3}}$. Continuing in this manner and doing all the intermediate sums we finally obtain

$$Z^{\bar{\sigma}}_{L,h} = \sum_{\sigma_{-L}}\sum_{\sigma_L} e^{\beta\bar{\sigma}_{-L-1}\sigma_{-L}} e^{h\beta\sigma_{-L}} (T^{2L})_{\sigma_{-L},\sigma_L} e^{\beta\sigma_L\bar{\sigma}_{L+1}}. \tag{15}$$

As in the lectures, if we introduce the vectors f^- and f^+, where

$$f^-_\sigma = e^{\beta\bar{\sigma}_{-L-1}\sigma+h\beta\sigma} \tag{16}$$

and

$$f^+_\sigma = e^{\beta\sigma\bar{\sigma}_{L+1}}, \tag{17}$$

then again simply by the definition of matrix multiplication, we find that

$$Z^{\bar{\sigma}}_{L,h} = (f^-, T^{2L} f^+), \tag{18}$$

where $(\cdot,\cdot)$ is just the standard inner product: $(f,g) = \sum_\sigma f_\sigma g_\sigma$.

Now that we have a magnetic field, the transfer matrix is no longer symmetric: $T_{\sigma,\sigma'} \neq T_{\sigma',\sigma}$, but we can still try to diagonalize the matrix. After some elementary but tedious calculations (or after using Mathematica), one finds that $T = ADA^{-1}$, where

$$A = \begin{pmatrix} a-b & a+b \\ 1 & 1 \end{pmatrix}, \tag{19}$$

$$D = \begin{pmatrix} c-d & 0 \\ 0 & c+d \end{pmatrix} \tag{20}$$

and

$$A^{-1} = \frac{1}{2b}\begin{pmatrix} -1 & a+b \\ 1 & b-a \end{pmatrix}. \tag{21}$$

Here we have introduced the following terms

$$a = \frac{1}{2}(-e^{2\beta} + e^{2\beta+2\beta h}), \tag{22}$$

$$b = \frac{1}{2}\sqrt{e^{4\beta} + 4e^{2\beta h} - 2e^{2\beta h+4\beta} + e^{4\beta h+4\beta}} \tag{23}$$

$$c = e^\beta \cosh(\beta h) \tag{24}$$

and

$$d = e^{-\beta-h\beta} b. \tag{25}$$

We thus find that

$$Z^{\bar\sigma}_{L,h} = \left(A^T f^-, \begin{pmatrix} (c-d)^{2L} & 0 \\ 0 & (c+d)^{2L} \end{pmatrix} A^{-1} f^+\right). \tag{26}$$

For the vectors $A^T f^-$ and $A^{-1} f^+$ we find

$$A^T f^- = \begin{pmatrix} a-b & 1 \\ a+b & 1 \end{pmatrix} \begin{pmatrix} e^{\beta\bar\sigma_{-L-1}+\beta h} \\ e^{-\beta\bar\sigma_{-L-1}-\beta h} \end{pmatrix} = \begin{pmatrix} (a-b)e^{\beta\bar\sigma_{-L-1}+\beta h} + e^{-\beta\bar\sigma_{-L-1}-\beta h} \\ (a+b)e^{\beta\bar\sigma_{-L-1}+\beta h} + e^{-\beta\bar\sigma_{-L-1}-\beta h} \end{pmatrix} =: \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{27}$$

and

$$A^{-1} f^+ = \frac{1}{2b} \begin{pmatrix} -1 & a+b \\ 1 & b-a \end{pmatrix} \begin{pmatrix} e^{\beta\bar\sigma_{L+1}} \\ e^{-\beta\bar\sigma_{L+1}} \end{pmatrix} = \frac{1}{2b} \begin{pmatrix} -e^{\beta\bar\sigma_{L+1}} + (a+b)e^{-\beta\bar\sigma_{L+1}} \\ e^{\beta\bar\sigma_{L+1}} + (b-a)e^{-\beta\bar\sigma_{L+1}} \end{pmatrix} =: \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{28}$$

So we find that

$$Z^{\bar\sigma}_{L,h} = (c-d)^{2L} u_1 v_1 + (c+d)^{2L} u_2 v_2. \tag{29}$$

The thing to note here is that $|c+d| > |c-d|$ and $|b| > |a|$ so $u_2 v_2 > 0$ and we see that

$$Z^{\bar\sigma}_{L,h} = (c+d)^{2L} u_2 v_2 + o((c+d)^{2L}). \tag{30}$$

Let us now turn to the magnetization and proceed as we did in the case where there was no field. We find that

$$\sum_{\sigma_{-L}} \cdots \sum_{\sigma_L} \sigma_x e^{-\beta \mathcal{H}^{\bar\sigma}_{L,h}} = \sum_{\sigma_{-L}} \sum_{\sigma_x} \sum_{\sigma_L} e^{\beta\bar\sigma_{-L-l}\sigma_{-L}} e^{h\beta\sigma_{-L}} (T^{L+x})_{\sigma_{-L},\sigma_x} \sigma_x (T^{L-x})_{\sigma_x\sigma_L} e^{\beta\sigma_L\bar\sigma_{L+1}} \tag{31}$$

Again simply by the definition of matrix multiplication we have

$$\sum_{\sigma_x} (T^{L+x})_{\sigma_{-L},\sigma_x} \sigma_x (T^{L-x})_{\sigma_x\sigma_L} = \left(T^{L+x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-x}\right)_{\sigma_{-L},\sigma_L}. \tag{32}$$

Thus we can write the magnetization as

$$\langle\sigma_x\rangle^{\bar\sigma}_{L,h} = \frac{\left(f^-, T^{L+x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-x} f^+\right)}{(f^-, T^{2L} f^+)} = \frac{\left(A^T f^-, D^{L+x} A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A D^{L-x} A^{-1} f^+\right)}{(c+d)^{2L} u_2 v_2 + o((c+d)^{2L})} \tag{33}$$

To simplify this, we first note that

$$A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = -\frac{1}{b} \begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix}. \tag{34}$$

Thus

$$D^{L+x} A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A D^{L-x} = \frac{1}{b} \begin{pmatrix} -a(c-d)^{2L} & -(a+b)(c-d)^{L+x}(c+d)^{L-x} \\ (a-b)(c-d)^{L-x}(c+d)^{L+x} & a(c+d)^{2L} \end{pmatrix}. \tag{35}$$

Using the fact that $|c+d| > |c-d|$ we see that in the $L \to \infty$ limit, we have

$$\lim_{L\to\infty} \langle\sigma_x\rangle^{\bar\sigma}_{h,L} = \frac{1}{u_2 v_2} \left(A^T f^-, \frac{a}{b} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} f^+\right) = \frac{a}{b}. \tag{36}$$

Let us see if this result makes any sense at all. First of all, it is independent of the boundary conditions. A second thing to note is that it is independent of the point x. This makes sense since in the model the interactions and the field do not depend on the point x so there is no real reason to expect that the magnetization would depend on it. In fact the model is translation invariant in the limit (but before taking the limit, the boundary conditions break the translation invariance). We further note that $|a| < |b|$ so the magnetization is a number of absolute value less than one. This makes sense since the spin takes values ± 1 so its average value shouldn't be able to have values outside of $[-1, 1]$. Finally we note that a changes its sign at $h = 0$ which again makes sense: as we reverse the direction of the magnetic field, the spins should also flip so the direction of the magnetization should reverse too. Also $a = 0$ when $h = 0$ so this is at least consistent with our previous result that there is no magnetization when we don't have a magnetic field.

Remark: Things are a bit simpler if we take periodic boundary conditions. In this case, one can make use of the periodicity to write the transfer matrix as a symmetric matrix and diagonalization and everything related to it is simpler. Moreover with periodic boundary conditions, the model is translation invariant already before we take the limit. So we know that $\langle\sigma_x\rangle$ is independent of x. Thus we can write

$$\langle\sigma_x\rangle_{L,h} = \frac{\sum_x \langle\sigma_x\rangle_{L,h}}{2L} = \frac{1}{2L}\partial_h \log Z_{L,h} \tag{37}$$

and life is a bit simpler (very short extra exercise: check that with periodic boundary conditions the magnetization is given by the partial derivative of the free energy with respect to the magnetic field as claimed above).

Returning to the 2-point function for general boundary conditions we follow the same old arguments. After writing out the definition, we find that for $x < y$

$$\langle\sigma_x\sigma_y\rangle^{\bar{\sigma}}_{L,h} = \frac{\left(f^-, T^{L+x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{y-x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-y} f^+\right)}{Z^{\bar{\sigma}}_{L,h}} \tag{38}$$

Inserting our diagonalization decomposition for T and using our previous result that

$$A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = -\frac{1}{b}\begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix} \tag{39}$$

we find

$$\langle\sigma_x\sigma_y\rangle^{\bar{\sigma}}_{L,h} = \frac{1}{b^2 Z^{\bar{\sigma}}_{L,h}}\left(A^T f^-, D^{L+x}\begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix} D^{y-x}\begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix} D^{L-y} A^{-1} f^+\right).$$

For the matrix product we find

$$D^{L+x}\begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix} D^{y-x}\begin{pmatrix} a & a+b \\ b-a & -a \end{pmatrix} D^{L-y} =$$
$$\begin{pmatrix} (b^2-a^2)(c+d)^{y-x}(c-d)^{2L-y+x} + a^2(c-d)^{2L} & a(a+b)((c-d)^{L+y}(c+d)^{L-y} - (c-d)^{L+x}(c+d)^{L-x}) \\ a(b-a)((c-d)^{L-x}(c+d)^{L+x} - (c-d)^{L-y}(c+d)^{L+y}) & (b^2-a^2)(c+d)^{2L-y+x}(c-d)^{y-x} + a^2(c+d)^{2L} \end{pmatrix}.$$

Again we use the fact that $\left|\frac{c-d}{c+d}\right| < 1$ so in the limit we have

$$\begin{aligned}\lim_{L\to\infty}\langle\sigma_x\sigma_y\rangle^{\bar{\sigma}}_{L,h} &= \frac{1}{b^2}\frac{\left(A^T f^-, \begin{pmatrix} 0 & 0 \\ 0 & (b^2-a^2)(c+d)^{x-y}(c-d)^{y-x} + a^2 \end{pmatrix} A^{-1} f^+\right)}{u_2 v_2} \\ &= \frac{b^2-a^2}{b^2}\left(\frac{c-d}{c+d}\right)^{y-x} + \frac{a^2}{b^2}.\end{aligned}$$

Thus for the limit of the correlation function we find

$$\langle\sigma_x\sigma_y\rangle_h - \langle\sigma_x\rangle_h\langle\sigma_y\rangle_h = \frac{b^2-a^2}{b^2}\left(\frac{c-d}{c+d}\right)^{y-x}. \tag{40}$$

So we have found that the correlation function has a limit which is independent of the boundary conditions. Moreover, it decays exponentially as claimed. If setting $h = 0$ in our result gives the result derived in the lectures, the risk that we made some mistakes in our calculation is at least slightly smaller so let us check this. Recall that for $h = 0$, $a = 0$, $b = 1$, $c = e^{\beta}$ and $d = e^{-\beta}$. Thus

$$\langle\sigma_x\sigma_y\rangle_{h=0} - \langle\sigma_x\rangle_{h=0}\langle\sigma_y\rangle_{h=0} = \left(\frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}\right)^{|y-x|} = (\tanh\beta)^{|y-x|} = e^{(\log\tanh\beta)|y-x|} \tag{41}$$

which is the same as in the lecture notes.

b) The approach to the probelm should be clear by now. Let us write $A \subset \{-L, ..., L\}$ as $A = \{x_1, ..., x_n\}$ where $x_i < x_{i+1}$ Then

(42)
$$\langle \sigma_A \rangle_L^{\bar{\sigma}} = \left\langle \prod_{i=1}^{n} \sigma_{x_i} \right\rangle_L^{\bar{\sigma}} = \frac{\left(f^-, T^{L+x_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{x_2-x_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{x_3-x_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-x_n} f^+\right)}{Z_L^{\bar{\sigma}}}.$$

Again we plug in the decomposition for T and we notice we'll be dealing with product of diagonal matrices and matrices of the form

$$A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A. \tag{43}$$

In the case where there is no magnetic field, the matrix A is much simpler: $a = 0$ and $b = 1$. We find that

$$A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{44}$$

Thus our correlation function is

(45)
$$\frac{\left(A^T f^-, D^{L+x_1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D^{x_2-x_1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D^{x_3-x_2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdots D^{x_n-x_{n-1}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D^{L-x_n} A^{-1} f^+\right)}{Z_L^{\bar{\sigma}}}.$$

To calculate this, we first calculate

$$D^m \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(e^\beta - e^{-\beta})^m \\ -(e^\beta + e^{-\beta})^m & 0 \end{pmatrix}. \tag{46}$$

Let us first consider the case where n is even: $n = 2k$. Then we can pair consecutive matrices of this form in the product and we see that

$$\begin{aligned} &D^{L+x_1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D^{x_2-x_1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D^{x_3-x_2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdots D^{x_n-x_{n-1}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \prod_{i=1}^k (e^\beta + e^{-\beta})^{x_{2i}-x_{2i-1}} (e^\beta - e^{-\beta})^{x_{2i-1}-x_{2i-2}} & 0 \\ 0 & \prod_{i=1}^k (e^\beta - e^{-\beta})^{x_{2i}-x_{2i-1}} (e^\beta + e^{-\beta})^{x_{2i-1}-x_{2i-2}} \end{pmatrix}, \end{aligned}$$

where we have written $x_0 = -L$. The vectors $A^T f^-$ and $A^{-1} f^+$ are much simpler now that we don't have a field:

$$A^T f^- = 2 \begin{pmatrix} -\sinh(\beta\bar{\sigma}_{-L-1}) \\ \cosh(\beta\bar{\sigma}_{-L-1}) \end{pmatrix} \quad A^{-1} f^+ = \begin{pmatrix} -\sinh(\beta\bar{\sigma}_{L+1}) \\ \cosh(\beta\bar{\sigma}_{L+1}) \end{pmatrix}. \tag{47}$$

Thus for even n we find

$$\begin{aligned} \langle \sigma_A \rangle_L^{\bar{\sigma}} = &\frac{2\sinh(\beta\bar{\sigma}_{-L-1})\sinh(\beta\bar{\sigma}_{L+1})(e^\beta - e^{-\beta})^{L-x_n} \prod_{i=1}^k (e^\beta + e^{-\beta})^{x_{2i}-x_{2i-1}} (e^\beta - e^{-\beta})^{x_{2i-1}-x_{2i-2}}}{2\sinh(\beta\bar{\sigma}_{-L-1})\sinh(\beta\bar{\sigma}_{L+1})(e^\beta - e^{-\beta})^{2L} + 2\cosh(\beta\bar{\sigma}_{-L-1})\cosh(\beta\bar{\sigma}_{L+1})(e^\beta + e^{-\beta})^{2L}} \\ &+ \frac{2\cosh(\beta\bar{\sigma}_{-L-1})\cosh(\beta\bar{\sigma}_{L+1})(e^\beta + e^{-\beta})^{L-x_n} \prod_{i=1}^k (e^\beta - e^{-\beta})^{x_{2i}-x_{2i-1}} (e^\beta + e^{-\beta})^{x_{2i-1}-x_{2i-2}}}{2\sinh(\beta\bar{\sigma}_{-L-1})\sinh(\beta\bar{\sigma}_{L+1})(e^\beta - e^{-\beta})^{2L} + 2\cosh(\beta\bar{\sigma}_{-L-1})\cosh(\beta\bar{\sigma}_{L+1})(e^\beta + e^{-\beta})^{2L}}. \end{aligned}$$

Noting that $\frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} < 1$, we see that the only thing that survives the $L \to \infty$ limit is

$$\lim_{L\to\infty} \langle \sigma_A \rangle_L^{\bar{\sigma}} = (e^\beta + e^{-\beta})^{-x_n} (e^\beta - e^{-\beta})^{x_2-x_1} (e^\beta + e^{-\beta})^{x_1} \prod_{i=2}^{k} (e^\beta - e^{-\beta})^{x_{2i}-x_{2i-1}} (e^\beta + e^{-\beta})^{x_{2i-1}-x_{2i-2}}. \tag{48}$$

To simplify this a bit, we note that

$$(49)\quad (e^\beta+e^{-\beta})^{-x_n+x_1}\prod_{i=2}^{k}(e^\beta+e^{-\beta})^{x_{2i-1}-x_{2i-2}}=\prod_{i=2}^{2k}(e^\beta+e^{-\beta})^{-x_i+x_{i-1}}\prod_{i=2}^{k}(e^\beta+e^{-\beta})^{x_{2i-1}-x_{2i-2}}=\prod_{i=1}^{k}(e^\beta+e^{-\beta})^{-x_{2i}+x_{2i-1}}$$

so for the correlation function we find

$$\left\langle \prod_{i=1}^{2k}\sigma_{x_i}\right\rangle=\prod_{i=1}^{k}(\tanh\beta)^{x_{2i}-x_{2i-1}}. \tag{50}$$

Let us now turn to the case where n is odd: $n=2k+1$. The approach is similar, but now we will not be able to pair all the matrices as before. We shall still do the pairing for the first $2k$ matrices. After doing the pairing, we are left with the expression

$$\langle\sigma_A\rangle_L^{\bar\sigma}=\frac{(A^Tf^-,MA^{-1}f^+)}{Z_L^{\bar\sigma}}, \tag{51}$$

where

$$\begin{aligned}M&=\begin{pmatrix}\prod_{i=1}^k(e^\beta+e^{-\beta})^{x_{2i}-x_{2i-1}}(e^\beta-e^{-\beta})^{x_{2i-1}-x_{2i-2}} & 0\\ 0 & \prod_{i=1}^k(e^\beta-e^{-\beta})^{x_{2i}-x_{2i-1}}(e^\beta+e^{-\beta})^{x_{2i-1}-x_{2i-2}}\end{pmatrix}\\ &\times\begin{pmatrix}0 & -(e^\beta-e^{-\beta})^{x^{2k+1}-x^{2k}}\\ -(e^\beta+e^{-\beta})^{x^{2k+1}-x^{2k}} & 0\end{pmatrix}\begin{pmatrix}(e^\beta-e^{-\beta})^{L-x^{2k+1}} & 0\\ 0 & (e^\beta+e^{-\beta})^{L-x_{2k+1}}\end{pmatrix}\\ &=\begin{pmatrix}0 & p\\ q & 0\end{pmatrix}\end{aligned}$$

where

$$p=-(e^\beta-e^{-\beta})^{x^{2k+1}-x^{2k}}(e^\beta+e^{-\beta})^{L-x_{2k+1}}\prod_{i=1}^{k}(e^\beta+e^{-\beta})^{x_{2i}-x_{2i-1}}(e^\beta-e^{-\beta})^{x_{2i-1}-x_{2i-2}} \tag{52}$$

and

$$q=-(e^\beta+e^{-\beta})^{x^{2k+1}-x^{2k}}(e^\beta-e^{-\beta})^{L-x^{2k+1}}\prod_{i=1}^{k}(e^\beta-e^{-\beta})^{x_{2i}-x_{2i-1}}(e^\beta+e^{-\beta})^{x_{2i-1}-x_{2i-2}}. \tag{53}$$

Note that neither p nor q contains both of the terms $(e^\beta+e^{-\beta})^{L+x_1}$ and $(e^\beta+e^{-\beta})^{L-x_{2k+1}}$. Thus

$$\frac{1}{(e^\beta+e^{-\beta})^{2L}}M\to 0 \tag{54}$$

as $L\to\infty$ so we see that $\langle\sigma_A\rangle=0$ if A is a set with an odd amount of elements. As a conclusion, we have now calculated the limit of all possible correlation functions and showed that the limit is independent of the boundary conditions.

3. Consider now a $d+1$ dimensional Ising model on $\{-M,...,M\}^d\times\{-L,...,L\}$ with no magnetic field and arbitrary boundary conditions $\bar\sigma$ in the L-direction and periodic boundary conditions in the other directions (so we are on a cylinder of length L). Show that in the limit $L\to\infty$ (while M is kept fixed), the magnetization vanishes (for any boundary condition). Calculate also the limit of the two point function $\langle\sigma_x\sigma_y\rangle$ and show that it is independent of the boundary conditions in the L direction and decays exponentially in $|x-y|$.

Hint: You might find some use in the Perron-Frobenius theorem.

Solution: Our approach relies on the idea that this model shouldn't differ too much from the one dimensional Ising model so we'll try to make things look like it as much as possible. We begin by collecting the spins located at points with the same coordinate along the L-direction: for $x\in\{-L,...,L\}$ define $\mathbf{S}_x$ to be the $(2M+1)^d$-dimensional vector with entries $(\mathbf{S}_x)_y=\sigma_{(x,y)}$ for any $y\in\{-M,...,M\}^d$. The Hamiltonian in terms of these vectors is

$$\mathcal{H}_{L,M}^{\bar\sigma}=-\sum_{x=-L}^{L+1}\sum_{y\in\{-M,...,M\}^d}(\mathbf{S}_{x-1})_y(\mathbf{S}_x)_y-\sum_{x=-L}^{L}\sum_{y,y:\ |y-y'|=1}(\mathbf{S}_x)_y(\mathbf{S}_x)_{y'}, \tag{55}$$

where at $x = \pm(L+1)$ we need to use the boundary conditions in the L direction $\bar{\sigma}$ and in the last sum over y and y' we take into account the periodic boundary conditions in the M directions. Let us introduce the following notation to save us some writing: for any $(2M+1)^d$-dimensional vectors (indexed by the points $y \in \{-M,...,M\}^d$) $\mathbf{S}$ and $\mathbf{S}'$ we define

$$E(\mathbf{S},\mathbf{S}') = -\sum_{y\in\{-M,...,M\}^d} \mathbf{S}'_y \mathbf{S}_y \tag{56}$$

and

$$E(\mathbf{S}) = -\sum_{y,y:\ |y-y'|=1} \mathbf{S}_y \mathbf{S}_{y'}. \tag{57}$$

So now

$$\mathcal{H}^{\bar{\sigma}}_{L,M} = \sum_{x=-L}^{L+1} E(\mathbf{S}_x,\mathbf{S}_{x-1}) + \sum_{x=-L}^{L} E(\mathbf{S}_x). \tag{58}$$

We now introduce a matrix that will turn out to have a similar role as the transfer matrix in the 1-dimensional case. We define

$$T_{\mathbf{S},\mathbf{S}'} = e^{-\beta E(\mathbf{S},\mathbf{S}')-\beta E(\mathbf{S})}. \tag{59}$$

Let us now try to calculate the partition function in this notation:

$$Z^{\bar{\sigma}}_{L,M} = \sum_{\mathbf{S}_{-L}} \cdots \sum_{\mathbf{S}_L} e^{-\beta E(\mathbf{S}_{-L-1},\mathbf{S}_{-L})-\beta E(\mathbf{S}_{-L})} \left(\prod_{x=-L+1}^{L} e^{-\beta E(\mathbf{S}_{x-1},\mathbf{S}_x)-\beta E(\mathbf{S}_x)} \right) e^{-\beta E(\mathbf{S}_L,\mathbf{S}_{L+1})}. \tag{60}$$

As in the 1-dimensional case, we introduce vectors for the boundary conditions (these are now indexed by vectors $\mathbf{S}$):

$$f^-_{\mathbf{S}} = e^{-\beta E(\mathbf{S}_{-L-1},\mathbf{S})-\beta E(\mathbf{S})} \tag{61}$$

$$f^+_{\mathbf{S}} = e^{-\beta E(\mathbf{S}_{L+1},\mathbf{S})}. \tag{62}$$

Exactly as in the 1-dimensional case, we simply use the definition of matrix multiplication to write the partition function as a suitable inner product:

$$Z^{\bar{\sigma}}_{L,M} = (f^-, T^{2L} f^+). \tag{63}$$

One might hope to be able to do what we did in the 1-dimensional case, i.e. just diagonalize T and calculate everything relevant, but diagonalization is not as simple now that we are dealing with much larger matrices. In stead, we shall rely on some less constructive methods, namely we shall use the Perron-Frobenius theorem. Look it up on Wikipedia or some other source if you are unfamiliar with it.

The theorem states (among other things) that if A is a matrix with only positive entries, then there is a positive eigenvalue λ so that every other eigenvalue has absolute value stricly less than λ, the eigenspace corresponding to λ is one-dimensional and

$$\lim_{k\to\infty} \frac{1}{\lambda^k} A^k = vw^T, \tag{64}$$

where v is a positive (i.e. all of its entries are positive) right eigenvector corresponding to λ: $Av = rv$, w is a positive left eigenvector: $w^T A = rw^T$ and vw^T is a projection to the eigenspace corresponding to λ normalized so that $w^T v = 1$.

In our case, T is a matrix with strictly positive entries so we can make use of the Perron-Frobenius theorem. Let us write λ for the eigenvalue discussed in the theorem and v and w for the vectors. So using the theorem, we see that

$$\lim_{L\to\infty} \frac{Z^{\bar{\sigma}}_{L,M}}{\lambda^{2L}} = (f^-, vw^T f^+) = (f^-, v)(w, f^+). \tag{65}$$

Since v and w have strictly positive entries as do $f^{\pm}$, we see that this limit is positive.

Let us now turn to the magnetization. Let $x \in \{-L, ..., L\}$ and $y \in \{-M, ..., M\}^d$. Similarly to the 1-dimensional case, we have

$$\langle \sigma_{(x,y)} \rangle_{L,M}^{\bar{\sigma}} = \frac{1}{Z_{L,M}^{\bar{\sigma}}} \sum_{\mathbf{S}_{-L}} \sum_{\mathbf{S}_x} \sum_{\mathbf{S}_L} f^-_{\mathbf{S}_{-L}} (T^{L+x})_{\mathbf{S}_{-L},\mathbf{S}_x} (\mathbf{S}_x)_y (T^{L-x})_{\mathbf{S}_x,\mathbf{S}_L} f^+_{\mathbf{S}_L}. \tag{66}$$

We thus find that

$$\lim_{L\to\infty} \frac{\sum_{\mathbf{S}_{-L}} \sum_{\mathbf{S}_x} \sum_{\mathbf{S}_L} f^-_{\mathbf{S}_{-L}} (T^{L+x})_{\mathbf{S}_{-L},\mathbf{S}_x} (\mathbf{S}_x)_y (T^{L-x})_{\mathbf{S}_x,\mathbf{S}_L} f^+_{\mathbf{S}_L}}{\lambda^{2L}} = (f^-, v)(w, f^+) \sum_{\mathbf{S}_x} w_{\mathbf{S}_x} (\mathbf{S}_x)_y v_{\mathbf{S}_x}. \tag{67}$$

So we see that the magnetization has a limit and it is independent of the boundary conditions:

$$\langle \sigma_{(x,y)} \rangle_M = \sum_{\mathbf{S}_x} w_{\mathbf{S}_x} (\mathbf{S}_x)_y v_{\mathbf{S}_x}. \tag{68}$$

We still wish to show that this is zero. To do this, we first of all note that the expectation is linear - a property which is preserved in the limit: $\langle -\sigma_{(x,y)} \rangle_M = -\langle \sigma_{(x,y)} \rangle_M$. On the other hand, $E(-\mathbf{S}, -\mathbf{S}') = E(\mathbf{S}, \mathbf{S}')$ and $E(-\mathbf{S}) = E(\mathbf{S})$ so $T_{-\mathbf{S},-\mathbf{S}'} = T_{\mathbf{S},\mathbf{S}'}$. Let us now just rename all the dummy summation variables: $\mathbf{S} \to -\mathbf{S}$, we find

$$\langle -\sigma_{(x,y)} \rangle_{L,M}^{\bar{\sigma}} = \frac{1}{Z_{L,M}^{\bar{\sigma}}} \sum_{\mathbf{S}_{-L}} \sum_{\mathbf{S}_x} \sum_{\mathbf{S}_L} \hat{f}^-_{\mathbf{S}_{-L}} (T^{L+x})_{\mathbf{S}_{-L},\mathbf{S}_x} (\mathbf{S}_x)_y (T^{L-x})_{\mathbf{S}_x,\mathbf{S}_L} \hat{f}^+_{\mathbf{S}_L}, \tag{69}$$

where

$$\hat{f}^-_{\mathbf{S}} = e^{-\beta E(-\mathbf{S}_{-L-1}, \mathbf{S}_L) - \beta E(\mathbf{S})} \tag{70}$$

and

$$\hat{f}^+_{\mathbf{S}} = e^{-\beta E(-\mathbf{S}_{L+1}, \mathbf{S})}. \tag{71}$$

So calculating the negative of the magnetization just amounts to calculating the magnetization with flipped boundary conditions. But we have already showed that the magnetization converges and the limit is independent of the boundary conditions. Thus

$$\lim_{L\to\infty} \langle -\sigma_{(x,y)} \rangle_{L,M}^{\bar{\sigma}} = \langle \sigma_{(x,y)} \rangle_M. \tag{72}$$

So by linearity, we see that $\langle \sigma_{(x,y)} \rangle_M = -\langle \sigma_{(x,y)} \rangle_M$ and $\langle \sigma_{(x,y)} \rangle_M = 0$.

To calculate the 2-point function, let $x_1 \leq x_2$ and $y_1, y_2 \in \{-M, ..., M\}^d$. We have

$$\langle \sigma_{(x_1,y_1)} \sigma_{(x_2,y_2)} \rangle_{L,M}^{\bar{\sigma}} = \frac{1}{Z_{L,M}^{\bar{\sigma}}} \sum_{\mathbf{S}_{-L}} \sum_{\mathbf{S}_{x_1}} \sum_{\mathbf{S}_{x_2}} \sum_{\mathbf{S}_L} f^-_{\mathbf{S}_{-L}} (T^{L+x_1})_{\mathbf{S}_{-L},\mathbf{S}_{x_1}} (\mathbf{S}_{x_1})_{y_1} (T^{x_2-x_1})_{\mathbf{S}_{x_1},\mathbf{S}_{x_2}} (\mathbf{S}_{x_2})_{y_2} (T^{L-x_2})_{\mathbf{S}_{x_2},\mathbf{S}_L} f^+_{\mathbf{S}_L}. \tag{73}$$

Using Perron-Frobenius once again, we see that

$$\lim_{L\to\infty} \langle \sigma_{(x_1,y_1)} \sigma_{(x_2,y_2)} \rangle_{L,M}^{\bar{\sigma}} = \sum_{\mathbf{S}_{x_1}} \sum_{\mathbf{S}_{x_2}} w_{\mathbf{S}_{x_1}} (\mathbf{S}_{x_1})_{y_1} (T^{x_2-x_1})_{\mathbf{S}_{x_1},\mathbf{S}_{x_2}} (\mathbf{S}_{x_2})_{y_2} v_{\mathbf{S}_{x_2}} \lambda^{-(x_2-x_1)}. \tag{74}$$

So we see that the 2-point function converges and is independent of the boundary conditions. Using the Perron-Frobenius theorem, we can write

$$\lambda^{-(x_2-x_1)} T^{x_2-x_1} = v w^T + A(x_2 - x_1), \tag{75}$$

where $A(x) \to 0$ as $x \to \infty$. Thus

$$\langle \sigma_{(x_1,y_1)} \sigma_{(x_2,y_2)} \rangle_M = \sum_{\mathbf{S}_{x_1}} w_{\mathbf{S}_{x_1}} (\mathbf{S}_{x_1})_{y_1} v_{\mathbf{S}_{x_1}} \sum_{\mathbf{S}_{x_2}} w_{\mathbf{S}_{x_2}} (\mathbf{S}_{x_2})_{y_2} v_{\mathbf{S}_{x_2}} + \sum_{\mathbf{S}_{x_1}} \sum_{\mathbf{S}_{x_2}} w_{\mathbf{S}_{x_1}} (\mathbf{S}_{x_1})_{y_1} (A(x_2-x_1))_{\mathbf{S}_{x_1},\mathbf{S}_{x_2}} (\mathbf{S}_{x_2})_{y_2} v_{\mathbf{S}_{x_2}}. \tag{76}$$

We already proved (although indirectly) that

$$\langle \sigma_{(x,y)} \rangle_M = \sum_{\mathbf{S}_x} w_{\mathbf{S}_x} (\mathbf{S}_x)_y v_{\mathbf{S}_x} = 0 \tag{77}$$

so in fact,

$$\langle \sigma_{(x_1,y_1)} \sigma_{(x_2,y_2)} \rangle_M = \sum_{\mathbf{S}_{x_1}} \sum_{\mathbf{S}_{x_2}} w_{\mathbf{S}_{x_1}} (\mathbf{S}_{x_1})_{y_1} (A(x_2 - x_1))_{\mathbf{S}_{x_1},\mathbf{S}_{x_2}} (\mathbf{S}_{x_2})_{y_2} v_{\mathbf{S}_{x_2}}. \tag{78}$$

Thus to prove that the two-point function decays exponentially, we only need to check that A decays exponentially in some sense (for example if it decays exponentially in some norm, then using some standard estimates such as the triangle inequality, Cauchy-Scwarz inequality and the equivalence of norms in finite dimensional normed spaces, we see that the 2-point function decays exponentially).

Let us look at the definition of A a bit more carefully. Through a similarity transform, we can always bring T into Jordan normal form. Let $J_k = \lambda_k \mathbf{1}_k + N_k$ be the Jordan blocks, where λ_k is an eigenvalue, $\mathbf{1}_k$ is a unit matrix of suitable size and N_k is a nilpotent matrix. There is a unique block corresponding to λ and for all the other blocks, $|\lambda_k| < \lambda$. The Jordan blocks of $T^{x_2-x_1}$ are then $J_k^{x_2-x_1}$. Let us consider what $J_k^{x_2-x_1}$ looks like for large $x_2 - x_1$. Performing a binomial expansion (which is valid as a unit matrix commutes with everything),

$$J_k^{x_2-x_1} = \sum_{m=0}^{x_2-x_1} \frac{(x_2-x_1)!}{m!(x_2-x_1-m)!} \lambda_k^m N_k^{x_2-x_1-m} \tag{79}$$

For each k, there are only say α_k non-zero terms in this sum (α_k independent of $x_2 - x_1$) since N_k is nilpotent. Using some elementary estimation, we can then check that

$$||\lambda^{-(x_2-x_1)} J_k^{x_2-x_1}|| \leq C_k (x_2 - x_1)^{\alpha_k} \left(\frac{|\lambda_k|}{\lambda} \right)^{x_2-x_1} \leq \tilde{C}_k q_k^{x_2-x_2}, \tag{80}$$

where $0 < q_k < 1$, $\tilde{C}_k$ is independent of $x_2 - x_1$ and $\max_k \tilde{C}_k$ is some finite number since we have only some fixed finite number of Jordan blocks and similarly $q^* = \max_k q_k < 1$. We can use any matrix norm here we wish since they are all equivalent - different norms just give rise to a different $\tilde{C}_k$. We can thus write $A(x_2 - x_1)$ in terms of these blocks and its norm will be bounded by some constant times $q^{*x_2-x_1}$. By our previous remarks, we conclude that the 2-point functions decays exponentially.