STATISTICAL MECHANICS - EXERCISE 9

1. Let ϕ be a centered Gaussian random varible with respect to both the measures μ and ν. Let $E_\mu(\phi^2) = \sigma_\mu^2$ and $E_\nu(\phi^2) = \sigma_\nu^2$. Denote by $: :_\mu$ normal ordering with respect to the measure μ. Show that

$$: \phi^n :_\mu = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^m m!(n-2m)!} : \phi^{n-2m} :_\nu (\sigma_\nu^2 - \sigma_\mu^2)^m. \tag{1}$$

2. Let ϕ and ψ be centered Gaussian random variables. Show that

$$E(: \phi^n :: \psi^m :) = \delta_{nm} n! E(\phi\psi)^n. \tag{2}$$

3. Consider the model where the Fourier transform of the covariance is $\frac{\chi(\frac{p}{\Lambda})}{p^2+r}$ and $\chi(p) = e^{-p^2}$. Consider the Feynman graph with four external legs, two vertices and one internal loop (see page 63 in the lecture notes - the graph labeled by $2m = 4$, $N = 2$). Show that for $d < 4$, the value of this graph is bounded as $\Lambda \to \infty$, for $d = 4$, it diverges logarithmically - its value goes like $\log \Lambda$ and for $d > 4$ it behaves like Λ^{d-4}.

4. Consider a translation invariant kernel $K(x_1, x_2, x_3, x_4)$ and the potential

$$V = \int K(x_1, x_2, x_3, x_4) : \prod_{i=1}^{4} \phi(x_i) : \prod_{i=1}^{d} dx_i, \tag{3}$$

where K is such that $V \in \mathcal{K}_\lambda$. Show that we can write this as $a \int : \phi(x)^4 : dx + \tilde{V}$, where $\tilde{V} \in \mathcal{K}_{\frac{\lambda}{2}}$ is irrelevant.