STATISTICAL MECHANICS - EXERCISE 8

1. Let ϕ be a centered Gaussian random variable of variance σ^2.

a) Write $:\phi^n:$ in terms of ϕ and σ.
b) Try to invert the formula you got, i.e. write ϕ^n in terms of $:\phi^m:$ and σ.

2. Continuing from the previous problem; if $\sigma = 1$, show that $:\phi^n: = H_n(\phi)$, where H_n is the nth Hermite Polynomial.

3. Let $\Lambda_{L,N} = \{-\frac{L^N-1}{2}, ..., \frac{L^N-1}{2}\}^2$ (we assume that L is odd). Let $\mathcal{C}_L^N = \{B_{L^n}(L^n z) | n \in \mathbb{Z}_+, z \in \mathbb{Z}^2, B_{L^n}(L^n z) \subset \Lambda_{N,L}\}$, where $B_L(z)$ is a square of side lenght L centered at z (we assume L is odd). For each $C \in \mathcal{C}_L^N$, introduce a standard Gaussian random variable ζ_C which are independent for different boxes C. The hierarchical model was defined by the fields

$$\phi_N(x) = \sum_{C \in \mathcal{C}_L^N : x \in C} L^{-n(C)\frac{a}{2}} \zeta_C. \tag{1}$$

Here $n(C)$ is the unique number for which $C = B_{L^{n(C)}}(L^{n(C)} z)$ for some $z \in \mathbb{Z}^2$.

a) Calculate the covariance of this field.

b) Show that after a suitable deterministic normalization, the partition function

$$Z_{\beta,N} = \sum_{z \in \Lambda_{N,L}} e^{-\beta \phi_N(z)} \tag{2}$$

is a martingale when considered as a function of N.

c) Consider a modification of the Hierarchical field: define $\mathcal{D}_L^N = \{B_{L^n}(z) | n \in \{0, ..., N\}, z \in \mathbb{Z}^2, B_{L^n}(z) \cap \Lambda_{N,L} \neq \emptyset\}$ and to each $D \in \mathcal{D}_L^N$ associate ξ_D - a centered Gaussian of variance $L^{-2n(D)}$ ($n(D)$ defined in the obvious way). We then define

$$\psi_N(x) = \sum_{D \in \mathcal{D}_L^N : x \in D} L^{-n(D)\frac{a}{2}} \xi_D. \tag{3}$$

Try to compare the fields ψ_N and ϕ_N - what are the similarities and differences. Try to calculate the covariance of ψ_N and write out an asymptotic estimate for it (estimate $G_{x,y}^N$ for large N and $d(x,y)$.