STATISTICAL MECHANICS - EXERCISE 7

1. Consider a block spin argument for a system with Hamiltonian $H(\phi) = (\phi, (-\Delta + \Delta^2)\phi)$. Show that this model converges to the same fixed point as $(\phi, -\Delta\phi)$.

2. Calculate $D_0 = G_0 C^T G_1^{-1}$ explicitly and show that $(D_0)_{x,y}$ decays exponentially in $|x-y|$. Recall that C is the block spin map and $G_1 = CGC^T$. Moreover G_0 is $(-\Delta)^{-1}$ with periodic boundary conditions. Recall that the way we interpreted this is that we remove the $p = 0$ mode from the Fourier expansion of G_0.

3. The aim of this problem is to consider the central limit theorem from the point of view of the renormalization group. We shall not consider the CLT in it's full generality to make our RG approach simpler.

Consider 2^m independent identically distributed random variables Q_i. Let us assume that the distribution of Q_i has a density $\rho(q)$ and we assume that $\rho(q)$ has 'enough' regularity - if you are interested in the regularity assumptions we need, think about what regularity we need at each step. A speciffic thing we require from ρ is that $\int q^2\rho(q)dq - \left(\int \rho(q)qdq\right)^2 = 1$ (and that both of the terms are finite).

The central limit theorem states that $2^{-\frac{m}{2}}\sum_i Q_i$ converges to a standard Gaussian. Consider then the distribution of $\sum_i Q_i$. This is given by a convolution of the distributions of Q_i. Following the philosophy of the RG, one would like to calculate this convolution in steps - at the first step, pair up the distributions and calculate convolutions in these pairs. At the next step, pair up again and calculate convolutions in each pair. Following this idea, introduce the mapping on probability densities

$$(\mathcal{T}_\lambda \rho)(q) = \lambda \int \rho(q')\rho(\lambda q - q')dq' \tag{1}$$

(note that this is a d-dimensional integral). Morally, we would like to show that $\mathcal{T}_\lambda^n \rho$ converges to a Gaussian distribution as $n \to \infty$ for a suitable λ.

a) Let $\hat{\rho}(k)$ be the Fourier transform of $\rho(q)$ and $w(k) = \log \hat{\rho}(k)$. Lift $\mathcal{T}_\lambda$ to the functions w, i.e. define

$$\mathcal{T}_\lambda w = \log \widehat{\mathcal{T}_\lambda \rho}. \tag{2}$$

Show that acting on w, $\mathcal{T}_\lambda$ is linear:

$$(\mathcal{T}_\lambda w)(k) = 2w\left(\frac{k}{\lambda}\right). \tag{3}$$

b) Assuming that we can expand w as a series around $k = 0$, describe the fixed points of $\mathcal{T}_\lambda$ (when acting on functions of the form w).

c) Study the eigenfunctions and eigenvalues of $\mathcal{T}_\lambda$ (when acting on functions of the form w) and try to discuss the stability of different eigenfunctions and how they are related to the central limit theorem.

4. Let N be an integer and for $x, y \in \mathbb{Z}^2$ define the equivalence relation $x \sim_N y$ if $x - y \in (N\mathbb{Z})^2$. Let τ be an exponentially distributed random variable with parameter $\frac{1}{N^2}$ and let $\{w_m\}_{m=0}^\infty$ be a simple random walk on $\mathbb{Z}^2$ (check Wikipedia or something if you don't know what a random walk is). Define

$$F(x,y) = E^x\left(\sum_{m=0}^{\tau} \mathbf{1}(w_m \sim_N y)\right). \tag{4}$$

Here E^x means averaging over simple random walks starting form x and the exponential random variable τ (which we take independent of the random walk).

Prove that $F = (-\Delta + N^{-2})^{-1}$, where Δ is the two dimensional lattice Laplacian with periodic boundary conditions.