STATISTICAL MECHANICS - EXERCISE 4

1. Consider the space $\Omega = \{-1,1\}^{\mathbb{Z}^d}$ as a topological space where the topology is given by the product topology and the topology of $\{-1,1\}$ is the discrete topology. Show that this topology is metrizable and a compatible metric is

$$d(\sigma,\sigma') = \sum_{x\in\mathbb{Z}^d} 2^{-|x|}|\sigma_x - \sigma'_x|. \tag{1}$$

Moreover, show that Ω is compact in this topology.

2. Let Ω be as in the previous problem and define

$$C_0(\Omega) = \{f : \Omega \to \mathbb{R}|\ f \text{ depends on only finitely many } \sigma_x\}. \tag{2}$$

Show that $C_0(\Omega)$ is dense in $C(\Omega)$ (the space of real valued continuous functions defined on Ω) if we equip $C(\Omega)$ with the topology given by the sup-norm.

3. Consider a bounded spin model with a finite range potential on $\Lambda \subset \mathbb{Z}$ (if you are unsure about your skills in functional analysis, take the spins to be Ising spins). Show that $\langle\sigma_x\rangle_\Lambda^{\bar\sigma}$ converges as $\Lambda \to \mathbb{Z}$ and the limit is independent of the boundary conditions. Show also that the two point function $\langle\sigma_x\sigma_y\rangle_\Lambda^{\bar\sigma} - \langle\sigma_x\rangle_\Lambda^{\bar\sigma}\langle\sigma_y\rangle_\Lambda^{\bar\sigma}$ converges, is independent of boundary conditions and decays exponentially.

Hint: The approach is basically the same as in the case of the 1-d Ising model. The generalization to a finite range potential makes the transfer matrix larger, but it is still a useful quantity. You will also need some version of Perron-Frobenius.

4. Consider the so called mean field Ising model:

$$H = -\frac{J}{N}\sum_{x,y}\sigma_x\sigma_y, \tag{3}$$

where N is the number of spins we are considering (if you want take this to be in 1-d and consider an interval of length N) and J is a constant you can set to 1 if you don't want to carry it around and the σ_x are Ising spins and we consider free boundary conditions. Note that we are not summing over nearest neighbours but all x and y.

a) What do you think the term mean field refers to?

b) With a brief calculation, show that

$$H = -\frac{J}{2N}\left(\left(\sum_x \sigma_x\right)^2 - N\right). \tag{4}$$

c) Let $M = \frac{1}{N}\sum_x \sigma_x$ be the average magnetization (spatial average - not average with respect to the Gibbs measure). Show that the number of spin configurations with average magnetization M is given by

$$W(M) = \frac{N!}{\left(\frac{N}{2}(1+M)\right)!\left(\frac{N}{2}(1-M)\right)!}. \tag{5}$$

d) Show that the partition function can be written as

$$Z = \sum_M W(M) e^{-\beta H(M)}. \tag{6}$$

e) Estimate the partition function in the limit of large N by using Stirling's approximation. Show that the main contribution to the sum comes from its largest term and this term comes from maximizing the function

$$-F(M) = JM^2 - \frac{1}{\beta}((1+M)\log(1+M) + (1-M)\log(1-M)). \tag{7}$$

Do you know what the physical quantity F is?

f) By studying the maximum of $-F(M)$, prove that there is a phase transition in the model and try to find the critical temperature.

g) Expand $F(M)$ around $M = 0$ and try to evaluate few of the lowest order coefficients as functions of the temperature.

h) Think a bit about what changes and what doesn't change if we couple the spins to a uniform magnetic field (i.e. add a term $-h \sum_x \sigma_x$ to the Hamiltonian).