STATISTICAL MECHANICS - EXERCISE 3

1. Go over the Peierls argument for the q-state Potts model, i.e. show that if $\mathbb{P}^{(i)}$ is the probability measure obtained from the limit of the q-state Potts model with boundary conditions i (outside of Λ each spin has value i), then $\lim_{\beta\to\infty} \mathbb{P}^{(i)}(\sigma_0 \neq i) = 0$.

2. Percolation is an idealized model for studying the problem that if you pour a liquid into a porous material, will it flow through the sample and come out on the other side. For the model, we consider a graph (G, E) and state that each edge $e \in E$ is open with probability p and closed with probability $1-p$. Moreover, this assignment of an edge being open or closed is independent of all the other edges. A typical question would be that if the graph is the set $\{0, ..., n\}^2$, with what probability does there exist a connected path of open edges from $\{0\} \times \{0, ..., n\}$ to $\{n\} \times \{0, ..., n\}$ as $n \to \infty$ or does there exist a cluster of open edges connecting 0 to infinity.

In the lectures it was remarked that using the FK random cluster representation, the $q = 1$-state Potts model can be interpreted as percolation. Try to motivate this remark a bit on a level of partition functions: try to define a partition function for percolation and show that it looks like the partition function for a random cluster model corresponding to a Potts model. Assuming this identification between the models, how would you interpret the $q \to 1$ limit of the Potts model correlation function $\langle \delta_{\sigma_x,a}\delta_{\sigma_y,a}\rangle - \langle \delta_{\sigma_x,a}\delta_{\sigma_y,b}\rangle$ for percolation?

3.

Consider again the Ising model in high temperature with no external field and for simplicity, free boundary conditions.

a) In the last exercise session, we proved that the correlation function $\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle$ decays exponentially in $\mathrm{dist}(X, Y)$. Let us do this again, but this time using the polymer expansion. Extrapolating the arguments from the lectures, one can write

$$\langle\sigma_X\rangle_\Lambda = \sum_{B:\partial B = X} \rho(B) \exp\left(\sum_{C: C\cap B\neq\emptyset} f(C)\right) \tag{1}$$

Using this try to describe a diagrammatic expansion for $\langle\sigma_X\sigma_Y\rangle - \langle\sigma_X\rangle\langle\sigma_Y\rangle$ and estimate the correlation function using this expansion. Note that you can use all the regularity results on f and so on.

b) Consider $\langle\sigma_x\sigma_y\sigma_z\sigma_w\rangle - \langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_w\rangle - \langle\sigma_x\sigma_z\rangle\langle\sigma_y\sigma_w\rangle - \langle\sigma_x\sigma_w\rangle\langle\sigma_y\sigma_z\rangle$. Use the polymer expansion on this and try to describe a diagrammatic expansion using the polymer expansion. How does the correlation function decay?

c) Define the cumulant in the following manner:

$$\langle\sigma_A\rangle^c = \sum_{\pi\in\mathrm{Part}(A)} (-1)^{|\pi|+1} \prod_{B\in\pi} \langle\sigma_B\rangle, \tag{2}$$

where $\mathrm{Part}(A)$ is the set of partitions of A.

What kind of diagrammatic expansion would you expect the cumulant to have in the polymer expansion?