STATISTICAL MECHANICS - EXERCISE 10

1. a) Let $z_1, ..., z_n$ be centered jointly Gaussian random variables with a translation invariant covariance Γ_t, where $t \in [0, \infty)$ is a parameter. Let us assume that Γ_t is a smooth function of t. Let $\mu_{\Gamma_t}(dz)$ be the law of the random vector $z = (z_1, ..., z_n)$.

For any smooth enough function $F(z)$ calculate

$$\frac{d}{dt} \int F(z) \mu_{\Gamma_t}(dz). \tag{1}$$

b) How would you try to extend this to the case where z is a Gaussian field on $\mathbb{R}^d$?

Hint: In this and the next problem you might have to think about stuff like differentiating functions of functions. Take a physicist's approach and don't worry too much about the details of defining functional derivatives. Just try to think of $\frac{\delta F}{\delta z(x)}$ as the natural generalization of a partial derivative $\frac{\partial F}{\partial z_i}$.

2. In this problem we shall do our renormalization continuously in stead of in steps as we have done so far. Don't worry too much about the rigorous details. As usual, there might be some errors in the statement of the problem.

Consider the covariance G in Fourier space:

$$\hat{G}(p) = \frac{\chi(p)}{p^2}, \tag{2}$$

where $\chi(p) = e^{-p^2}$. As usual in the renormalization industry, we split the covariance into two parts:

$$\hat{G}(p) = \frac{\chi(e^s p)}{p^2} + \frac{\chi(p) - \chi(e^s p)}{p^2} = e^{2s} \hat{G}(e^s p) + \Gamma_s. \tag{3}$$

We then start with some interaction potential V_0 and define

$$e^{-V_s(\phi)} = \int e^{-V_0(e^{-s(d-2)}\phi(e^{-s}\cdot)+z)} \mu_{\Gamma_s}(dz). \tag{4}$$

a) Show that we have a semigroup property:

$$e^{-V_{s+t}(\phi)} = \int e^{-V_s(e^{-t(d-2)}\phi(e^{-t}\cdot)+z)} \mu_{\Gamma_t}(dz). \tag{5}$$

b) Using this, calculate $\frac{d}{ds} V_s = \frac{d}{dt} V_{s+t}\big|_{t=0}$.

Remark: Again, don't worry too much about the functional differentiation, try to think of it like all the rules of finite dimensional calculus hold. You might wind up with an answer like

$$\frac{d}{ds} V_s = \int dx (-(d-s)\phi(x) - x \cdot \nabla \phi(x)) \frac{\delta V}{\delta \phi(x)} + \frac{1}{2} \int dx dy \frac{d}{ds} \Gamma_s(x-y) \left(\frac{\delta^2}{\delta \phi(x) \delta \phi(y)} V_s - \frac{\delta V_s}{\delta \phi(x)} \frac{\delta V_s}{\delta \phi(y)} \right). \tag{6}$$

c) If you have time and interest, try to linearize around $V = 0$ and show that $: \phi^4 :$ is an eigenvalue.

3. Show that in perturbation theory, the tadpole diagrams don't contribute when calculating $(RV)_n$.