STATISTICAL MECHANICS - EXERCISE 1

1. Let us consider a probability measure p on a finite state space Ω (take for example $\Omega = \{-1,1\}^\Lambda$ for some finite set Λ if you wish). We define the entropy of the probability measure p by

$$S(p) = -\sum_{\sigma\in\Omega} p(\sigma)\log p(\sigma). \tag{1}$$

Let us also assume that we have an energy function $E : \Omega \to \mathbb{R}$ and that the average energy of the system is fixed, i.e.

$$\sum_{\sigma\in\Omega} E(\sigma)p(\sigma) = e \tag{2}$$

for some fixed $e \in [\min_\sigma E(\sigma), \max_\sigma E(\sigma)]$.

Show that the unique probability measure that maximizes the entropy under the constraint (2) is the Gibbs measure

$$p(\sigma) = \frac{e^{-\beta E(\sigma)}}{\sum_{\sigma\in\Omega} e^{-\beta E(\sigma)}} \tag{3}$$

for some unique value of β (there is one exception to the uniqueness of β - what is this?).

2. Consider the 1-dim Ising model on $\{-L, ..., L\}$ with arbitrary boundary conditions $\bar{\sigma}$ and a magnetic field h:

$$\mathcal{H}^{\bar{\sigma}}_{L,h} = -\sum_{x=-L}^{L+1} \sigma_{x-1}\sigma_x - h\sum_{x=-L}^{L} \sigma_x, \tag{4}$$

where $\sigma_{-L-1} = \bar{\sigma}_{-L-1}$ and $\sigma_{L+1} = \bar{\sigma}_{L+1}$.

a) Show that the magnetization

$$\langle\sigma_x\rangle^{\bar{\sigma}}_{L,h} = \frac{\sum_\sigma \sigma_x e^{-\beta\mathcal{H}^{\bar{\sigma}}_{L,h}(\sigma)}}{Z^{\bar{\sigma}}_{L,h}} \tag{5}$$

has a non-zero limit as $L \to \infty$ and that the limit independent of $\bar{\sigma}$. Show that as $h \to 0$, also the magnetization vanishes.

Also show that the two point function $\langle \sigma_x \sigma_y \rangle_{L,h}^{\bar{\sigma}}$ has a limit as $L \to \infty$ and that the limit is independent of $\bar{\sigma}$ and that correlations $\langle \sigma_x \sigma_y \rangle_h - \langle \sigma_x \rangle_h \langle \sigma_y \rangle_h$ decay exponentially in $|x - y|$.

Note: If you are having trouble with the problem, try doing it only in the case of periodic boundary conditions. Some things might look a bit more symmetric in this case.

b) For a vanishing magnetic field, $h = 0$, calculate an arbitrary correlation function $\langle \sigma_A \rangle$ in the limit $L \to \infty$ (recall that $\sigma_A = \prod_{x \in A} \sigma_x$).

3. Consider now a $d+1$ dimensional Ising model on $\{-M, ..., M\}^d \times \{-L, ..., L\}$ with no magnetic field and arbitrary boundary conditions $\bar{\sigma}$ in the L-direction and periodic boundary conditions in the other directions (so we are on a cylinder of length L). Show that in the limit $L \to \infty$ (while M is kept fixed), the magnetization vanishes (for any boundary condition). Calculate also the limit of the two point function $\langle \sigma_x \sigma_y \rangle$ and show that it is independent of the boundary conditions in the L direction and decays exponentially in $|x - y|$.

Hint: You might find some use in the Perron-Frobenius theorem.