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INTRODUCTION TO THE RENORMALIZATION GROUP

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1 Ising Model

We discuss first a concrete example of a spin system, the Ising model. This is a simple model for *ferromagnetism*, i.e. the phenomenon that certain materials (e.g. iron) are can stay permanently magnetized even in the absence of an external magnetic field. We can think of such materials consisting of elementary magnetic moments, residing in atoms located at a crystal lattice. A magnetic moment $m \in \mathbb{R}^3$ interacts with a magnetic field $B \in \mathbb{R}^3$ with energy

$$E = -m \cdot B$$

($\cdot$ is the scalar product). Energy is minimized by having m parallel to B.

In Ising model one simplifies by letting each m take only two values, parallel or antiparallel to the field: $m = \mu \frac{B}{|B|}\sigma$, $\sigma \in \{1, -1\}$ and $\mu > 0$ a constant. The variable σ is called "spin" where the terminology comes from the fact that the atomic magnetic moments often come from the spin degree of freedom of electrons.

The crystal lattice is modeled by a regular lattice which for definiteness we take to be $\mathbb{Z}^d$ i.e. $x = (x_1, \cdots, x_d) \in \mathbb{Z}^d$ with $x_i \in \mathbb{Z}$. To each $x \in \mathbb{Z}^d$, we associate a *spin variable* $\sigma_x \in \{-1, 1\}$. We call $\sigma = \{\{\sigma_x\} \mid x \in \mathbb{Z}^d\}$ a spin configuration on $\mathbb{Z}^d$. The set of all spin configurations is denoted by $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$.

Physical lattices are finite and so we consider also spin configurations on finite subsets $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ where $|\Lambda|$ be the number of elements in Λ. . We denote by $\Omega_\Lambda = \{-1, 1\}^\Lambda$ the spin configurations in Λ. In practice $|\Lambda|$ is very large ($> 10^{23}$) so we need to study the $|\Lambda| \to \infty$ limit, so-called *thermodynamic limit.* For simplicity we mostly take Λ a cube centered at origin: for $L \in \mathbb{N}$ let $\Lambda_L := \{x \in \mathbb{Z}^d \mid |x_i| \le L,\ i = 1, \cdots, d\}$ "cube of side $2L + 1$".

Definition 1.1 Let $|\Lambda| < \infty$. The *Ising Hamiltonian* in Λ is $\mathcal{H}_\Lambda : \Omega_\Lambda \to \mathbb{R}$ given by

$$\mathcal{H}_\Lambda(\sigma) = -J \sum_{\{x,y\} \in \mathcal{B}_\Lambda} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x \tag{1.1}$$

where $J > 0$, $h \in \mathbb{R}$ and $\mathcal{B}_\Lambda$ denotes the set of nearest neighbor pairs [1]

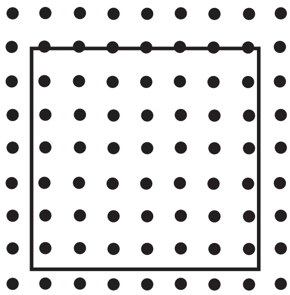
$$\mathcal{B}_\Lambda = \{\{x, y\} \mid x, y \in \Lambda,\ |x - y| = 1\}. \tag{1.2}$$

Remarks. 1. The constant h equals $\mu|B|$ in the previous discussion.

2. The nearest neighbor interaction favors spins being parallel: if $h = 0$, $\mathcal{H}$ takes its smallest value when $\sigma_x = 1$ for all x or $\sigma_x = -1$ for all x. This models ferromagnetism. If $h \neq 0$ then $\mathcal{H}$ has a unique minimum at $\sigma_x = \mathit{sign}\ h \equiv \frac{h}{|h|}$.

Since the spins interact with their neighbors it will be necessary to discuss *boundary conditions* for the model. Indeed, (1.1) is the so called *free boundary condition* Hamiltonian where only σ_x with $x \in \Lambda$ enter. Important examples of other boundary conditions are:

1. $+$ boundary condition. Take $\mathcal{H}$ as above, but demand that x *or* y is in Λ. For $x \notin \Lambda$ let $\sigma_x = 1$.



$\sigma = +1$

2. $-$ boundary condition is similar with $\sigma_x = -1$ for $x \notin \Lambda$.

3. Periodic boundary condition. Take $\Lambda = (\mathbb{Z}_L)^d$ where $\mathbb{Z}_L = \{0, 1, \cdots, L-1\}$ and let $\mathcal{B}_\Lambda$ be as in (1.2) where we replace $|x - y| = 1$ by $d_L(x, y) = 1$ where $d_L(x, y)$ is the periodic distance

$$d_L(x, y)^2 = \sum_i |x_i - y_i \bmod L|^2 \tag{1.3}$$

i.e. we view $\mathbb{Z}_L$ as the cyclic group of order L. We can also consider $\mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2} \times \cdots \times \mathbb{Z}_{L_d}$.

More generally we define:

Definition 1.2 Let $\bar{\sigma} \in \{1, -1\}^{\mathbb{Z}^d}$ be arbitrary. The Ising Hamiltonian with $\bar{\sigma}$ as boundary condition is defined as

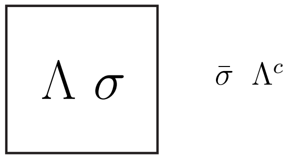
$$\mathcal{H}_\Lambda^{\bar{\sigma}}(\sigma) = -J \sum_{\{x,y\} \in \bar{\mathcal{B}}_\Lambda} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x \tag{1.4}$$

[1]we use the norm $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$

where

$$\bar{\mathcal{B}}_\Lambda = \{\{x, y\} \mid x \text{ or } y \in \Lambda,\ |x - y| = 1\} \tag{1.5}$$

and it is understood in the sum that $\sigma_x = \bar{\sigma}_x$ if $x \notin \Lambda$ (resp. y).



Note that the $\pm$ boundary conditions are special cases where $\bar{\sigma}_x = \pm 1$, $\forall x$. Also note that $\mathcal{H}^{\bar{\sigma}}$ depends on $\bar{\sigma}_x$ only for $x \in \partial\Lambda = \{y \in \Lambda^c | \text{dist}(y, \Lambda) = 1\}$. Indeed,

$$\sum_{\{x,y\} \in \bar{\mathcal{B}}_\Lambda} \sigma_x \sigma_y = \sum_{\{x,y\} \in \mathcal{B}_\Lambda} \sigma_x \sigma_y + \sum_{x \in \Lambda, y \in \Lambda^c, |x-y|=1} \sigma_x \bar{\sigma}_y.$$

Denote by $\mathcal{H}_\Lambda^{b.c.}(h, \sigma)$ the Ising Hamiltonian in Λ with given *b.c.* as above. We also set $J = 1$ since it will play no role below.

In statistical mechanics we view the the spin configurations $\sigma \in \Omega_\Lambda$ *random variables* whose probability distribution is determined by the Hamiltonian.

Remark. Recall some definitions from probability theory. A measure μ be of total mass one on a σ-algebra Σ of subsets of some set $\mathcal{M}$ is a *probability measure*. $A \in \Sigma$ is called an *event* and $\mu(A)$ is the probability of A. Let $f : \mathcal{M} \to \mathbb{R}$ be measurable, where $\mathbb{R}$ is equipped with the Borel σ-algebra, the smallest σ-algebra containing open sets in $\mathbb{R}$. We say that f is a *random variable*. The *distribution* of f is the probability measure ν on $\mathbb{R}$ with $\nu(B) = \mu(f^{-1}(B))$. For the *mean* of f we use the following notations

$$\int f d\mu \equiv \langle f \rangle \equiv \mathbb{E}\ f$$

and the *variance* of f is $\int f^2 d\mu - (\int f d\mu)^2$.

Definition 1.3 The *Ising measure* on Ω_Λ is the measure (note : Ω_Λ is a finite set, $|\Omega_\Lambda| = 2^{|\Lambda}$)

$$\mu_{\beta,h,\Lambda}^{b.c.}(\sigma) = \frac{1}{Z_{(\beta,h,\Lambda)}^{b.c.}} e^{-\beta \mathcal{H}_\Lambda^{b.c.}(h,\sigma)}$$

where $Z^{b.c.}$ is the *partition function*:

$$Z^{b.c.}(\beta, h, \Lambda) = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta \mathcal{H}_\Lambda^{b.c.}(h,\sigma)}$$

so μ is a *probability measure*, $\mu(\Omega_\Lambda) = \sum_{\sigma \in \Omega_\Lambda} \mu(\sigma) = 1$.

Remark. In physics, $\beta = \frac{1}{kT}$ where k = Boltzman's constant and T = temperature : β is the "inverse temperature". Note that for $\beta \to \infty$ $(T \to 0)$ the minima of $\mathcal{H}$ dominate : at low temperatures we expect magnetism.

In the real world $|\Lambda| \sim 10^{23}$, so we inquire

1° What happens as $\Lambda \to \mathbb{Z}^d$? Is there a limit measure $\mu = \lim\limits_{\Lambda \to \mathbb{Z}^d} \mu_\Lambda$ on Ω ? This is called the *thermodynamic limit.*

2° Does μ depend on *b.c.* ? How does it depend on β, h ?

We will study μ_Λ via its *correlation functions* :

Definition 1.4 Let $A \subset \mathbb{Z}^d, |A| < \infty$. Denote

$$\sigma_A = \prod_{x \in A} \sigma_x.$$

The *correlation function* of spins in A is (let $A \subset \Lambda$)

$$\langle \sigma_A \rangle^{b.c.}_{\beta,h,\Lambda} := \sum_{\sigma \in \Omega_\Lambda} \sigma_A \mu^{b.c.}_{\beta,h,\Lambda}(\sigma) = \frac{1}{Z} \sum_{\sigma \in \Omega_\Lambda} e^{-\beta \mathcal{H}^{b.c.}_\Lambda(h,\sigma)} \sigma_A.$$

We often drop β, h in the notation and write $\langle \sigma_A \rangle^{b.c.}_\Lambda$.

Examples. For $A = \{x\}$, $\langle \sigma_x \rangle^{b.c.}_\Lambda$ is the *magnetization* at x. For $A = \{x, y\}$, $\langle \sigma_x \sigma_y \rangle^{b.c.}_\Lambda$ is the *pair correlation* or *2-point function.*

2 Infinite volume limit

Suppose we succeed proving that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma_A \rangle^{b.c.}_{\beta,h,\Lambda}$ exists for all finite sets $A \subset \mathbb{Z}^d$ where the limit is taken e.g. along the sequence of cubes Λ_L. It is then natural to inquire whether these numbers actually are correlation functions of some probability measure $\mu^{b.c.}_{\beta,h}$ on (some sigma-algebra on) Ω. Unlike Ω_Λ, Ω is not a finite set (indeed, it is uncountable). Thus we need some measure theory to describe it. Intuitively the "density" of $\mu^{b.c.}_{\beta,h}$ is proportional to $\exp(-\beta \mathcal{H}^{b.c.}_{\mathbb{Z}^d}(h, \sigma))$ but this factor is ill defined since $\mathcal{H}^{b.c.}_\Lambda(h, \sigma)$ is proportional to $|\Lambda|$. We will later explain how to make sense of this intuition, but for the time being we will just show that the limits of correlation functions indeed are moments of some measure.

Let us drop *b.c.*, β and h from the notation and suppose

$$\lim_{L \to \infty} \langle \sigma_A \rangle_{\Lambda_L} \tag{2.1}$$

exists for all $A \subset \mathbb{Z}^d$, $|A| < \infty$. Let

$$C_0(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ depends on finitely many } \sigma_x\}.$$

Every $f \in C_0(\Omega)$ is a linear combination of the functions σ_A with $|A| < \infty$ with the convention $\sigma_\emptyset = 1$. Indeed, if $g : \{1, -1\} \to \mathbb{R}$ we may write $g(\sigma) = \frac{1}{2}(g(1) + g(-1)) + \frac{1}{2}\sigma(g(1) - g(-1))$. Letting $P_x^\pm f := \frac{1}{2}(f\big|_{\sigma_x=1} \pm f\big|_{\sigma_x=-1})$ the desired representation follows by expanding the product in

$$f = \prod_{x \in A} (P_x^+ + \sigma_x P_x^-) f$$

where A is the set of x s.t. f depends on σ_x. From this and (2.2) we conclude

$$\ell(f) := \lim_{L \to \infty} \langle f \rangle_{\Lambda_L} \tag{2.2}$$

exists for all $f \in C_0(\Omega)$. $C_0(\Omega)$ is a vector space and ℓ defines a linear map $C_0(\Omega) \to \mathbb{R}$. Clearly we have

$$\ell(1) = 1 \tag{2.3}$$

and

$$\ell(f) \geq 0 \;\text{ for all } f \in C_0(\Omega),\; f \geq 0. \tag{2.4}$$

Moreover, since $|\langle f \rangle_\Lambda| \leq \sup_{\sigma \in \Omega} |f(\sigma)|$ (where we take Λ to be large enough to include the support A of f; the sup is actually a max) we conclude

$$|\ell(f)| \leq \sup_{\sigma \in \Omega} |f(\sigma)| := \|f\| \tag{2.5}$$

where the RHS defines a norm in the vector space $C_0(\Omega)$.

Generally, let M be a compact metric space, and $C(M) = \{f : M \to \mathbb{R} |\, f \text{ continuous}\}$. $C(M)$ is a Banach space (= vector space with norm which is complete i.e. Cauchy sequences have limits), with the norm:

$$\|f\| = \sup_{x \in M} |f(x)|.$$

A linear map $\ell : C(M) \to \mathbb{R}$ is called a *state* on $C(M)$ if (2.2)-(2.5) hold. To get into this setup we need to discuss the topology of Ω.

Ω is a compact metric space: $\{-1, 1\}$ is compact, so $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ is also compact in the product topology by the Tychonov theorem (see [33] chap. 4 or [24] chap. 5). A metric compatible with this topology is e.g.

$$d(\sigma, \sigma') = \sum_{x \in \mathbb{Z}^d} 2^{-|x|} |\sigma_x - \sigma'_x|$$

i.e. two spin configurations are close if they agree in a big box around origin. Using the Stone-Weierstrass theorem (see [33] chap. 4) one then shows that $C_0(\Omega)$ is dense in $C(\Omega)$. Thus our linear functional ℓ extends to $C(\Omega)$ and defines a state there.

Homework : Check all this !

We may now use the basic real analysis to get our infinite volume measure. Recall that on a compact metric space we may consider the σ-algebra of Borel sets of M i.e. the smallest σ-algebra containing the open sets and Borel measures which are measures defined on this algebra. Given a Borel probability measure μ on M the linear map

$$\ell_\mu(f) = \int f d\mu$$

on $C(M)$ satisfies clearly the conditions (2.2)-(2.5) and defines a a state on $C(M)$. Conversely:

Theorem 2.1 (Riesz Representation Theorem) *Given a state ℓ on $C(M)$ there is a Borel probability measure μ on M such that*

$$\ell(f) = \int f d\mu$$

Proof See Rudin, Real and Complex Analysis, chap 2 [35] or Reed-Simon, vol. 1, chap. 4 [33] □

Exercise Prove this directly for $M = \Omega_\Lambda$!

We have thus obtained the following

Corollary 2.2 *Suppose that for all finite $A \subset \mathbb{Z}^d$ the limit $\lim_{L\to\infty}\langle\sigma_A\rangle^{\bar\sigma}_{\Lambda_L}$ exists. Then there is a probability measure $\mu^{\bar\sigma}$ such that this limit equals*

$$\langle\sigma_A\rangle^{\bar\sigma} := \int \sigma_A d\mu^{\bar\sigma}.$$

We call $\langle - \rangle^{\bar\sigma}$ (or $\mu^{\bar\sigma}$) *infinite volume state* and the question of thermodynamic limit is : find all infinite-volume states by taking different *b.c.* We'll see that for all *b.c.* the $\Lambda \to \mathbb{Z}^d$ limit exists (at least, through subsequences), but there can be many different $\mu' s$.

3 $d = 1$ Ising Model : Transfer Matrix

Consider the Ising model above for $d = 1$. Let first $h = 0$ and consider $\mathcal{H}^{\bar\sigma}$

$$\beta\mathcal{H}^{\bar\sigma}_L := -\beta \sum_{i=-L}^{L+1} \sigma_{i-1}\sigma_i$$

where $\sigma_{-L-1} = \bar\sigma_{-L-1} \equiv \sigma^-$ and $\sigma_{L+1} = \bar\sigma_{L+1} \equiv \sigma^+$ (note that these may vary with L).

Consider first the partition function:

$$Z^{\bar\sigma}_L := \sum_{\sigma_{-L},\cdots,\sigma_L=\pm1} \prod_{i=-L}^{L+1} e^{\beta\sigma_{i-1}\sigma_i}.$$

Let $T = (T_{\sigma\sigma'})_{\sigma,\sigma'=\pm 1}$ be the 2×2 matrix:

$$(T_{\sigma\sigma'}) = (e^{\beta\sigma\sigma'}) = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

Then,

$$Z_L^{\bar{\sigma}} = \sum_{\sigma_{-L},\sigma_L=\pm 1} e^{\beta\sigma_{-L-1}\sigma_{-L}} (T^{2L})_{\sigma_{-L}\sigma_L} e^{\beta\sigma_L\sigma_{L+1}} = (f^-, T^{2L} f^+)$$

where we use the scalar product $(f,g) \equiv \sum_\sigma f_\sigma g_\sigma$ and $f^\pm$ are the vectors $(e^{\beta\sigma^\pm}, e^{-\beta\sigma^\pm})$.

T is a symmetric matrix with has eigenvalues the roots of $(e^\beta - \lambda)^2 - e^{-2\beta}$ i.e. $2\cosh\beta := \lambda_1$ and $2\sinh\beta := \lambda_2$. The corresponding orthonormal eigenvectors are $\frac{1}{\sqrt{2}}(1, \pm 1)$ and introducing the corresponding orthogonal projections

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

we may write

$$T = \lambda_1 P_1 + \lambda_2 P_2.$$

Since $P_1 P_2 = 0$ and $P_i^2 = P_i$

$$T^{2L} = (2\cosh\beta)^{2L}(P_1 + (\tanh\,\beta)^{2L} P_2)$$

and

$$Z_L^{\bar{\sigma}} = (2\cosh\beta)^{2L}\left[(f^-, P_1 f^+) + (\tanh\,\beta)^{2L}(f^-, P_2 f^+)\right]. \tag{3.1}$$

We have $(f^-, P_1 f^+) = (f^-, e_1)(f^+, e_1)$. Since

$$(f^\pm, e_1) = \frac{1}{2}(e^\beta \pm e^{-\beta}) > 0$$

for all $\beta > 0$ the leaden term is non vanishing and we get

$$Z_L^{\bar{\sigma}} = \exp 2L\left[\log(2\cosh\beta) + \mathcal{O}\left(\frac{1}{L}\right) + \mathcal{O}(e^{-\alpha L})\right]$$

where $\alpha = -\frac{1}{2}\log(\tanh\,\beta) > 0$ for all β.

Definition 3.1 The free energy in volume Λ is

$$F_\Lambda^{\bar{\sigma}} = -\frac{1}{\beta|\Lambda|}\log Z_\Lambda^{b.c.}.$$

We get for the $1d$ Ising:

$$F_{2L}^{\bar{\sigma}} = \log(2\cosh\beta) + \mathcal{O}(\frac{1}{L}) + \mathcal{O}(e^{-\alpha L}).$$

It has the $L \to \infty$ limit

$$F^{\bar{\sigma}} = \log(2\cosh\beta)$$

which is *independent* on $\bar{\sigma}$. Note that $\log Z_{2L}$ is *extensive* i.e., once divided by $|\Lambda|$, it has the thermodynamic limit.

Let us next consider the magnetization:

$$\langle\sigma_x\rangle_L^{\bar{\sigma}} = \frac{1}{Z_{2L}^{\bar{\sigma}}}\sum_{\sigma} e^{-\beta\mathcal{H}_{2L}^{\bar{\sigma}}(\sigma)}\sigma_x.$$

Proceeding as with the partition function we have

$$\begin{aligned}\sum_{\sigma} e^{-\beta\mathcal{H}_{2L}^{\bar{\sigma}}(\sigma)}\sigma_x &= \sum_{\sigma_{-L},\sigma,\sigma_L} f^-_{\sigma_{-L}}(T^{L+x})_{\sigma_{-L}\sigma} \quad \sigma(T^{L-x})_{\sigma\sigma_L} \quad f^+_{\sigma_L} \\ &= (2\cosh\beta)^{2L}\left((f^-, P_1\sigma P_1 f^+) + \mathcal{O}(e^{-\alpha(L+x)}) + \mathcal{O}(e^{-\alpha(L-x)})\right)\end{aligned}$$

where in the last formula σ denotes the diagonal matrix $\sigma\delta_{\sigma\sigma'}$. But $P_1\sigma P_1 = 0$ so the first term in the numerator vanishes and combining with (3.1) we obtain

$$|\langle\sigma_x\rangle_{2L}^{\bar{\sigma}}| \le C(e^{-\alpha(L+x)} + e^{-\alpha(L-x)}) \to 0 \quad \text{as } L\to\infty.$$

Hence the magnetization vanishes.

For the 2-point function with $x < y$ we get in the same way

$$\begin{aligned}\langle\sigma_x\sigma_y\rangle_L^{\bar{\sigma}} &= \frac{1}{Z_L^+}(f^-, T^{L+x}\sigma T^{y-x}\sigma T^{L-y} f^+) \\ &= \frac{(f^-, (P_1 + e^{-\alpha(L+x)}P_2)\sigma(P_1 + e^{-\alpha(y-x)}P_2)\sigma(P_1 + e^{-\alpha(L-y)}P_2) f^+)}{(f^-, P_1 f^+) + \mathcal{O}(e^{-\alpha L})}. \end{aligned} \tag{3.2}$$

Using again $P_1\sigma P_1 = 0$ we get

$$\langle\sigma_x\sigma_y\rangle_L^{\bar{\sigma}} = \frac{(f^-, P_1\sigma P_2\sigma P_1 f^+)}{(f^-, P_1 f^+)}e^{-\alpha(y-x)} + \mathcal{O}(e^{-\alpha(L+x)}) + \mathcal{O}(e^{-\alpha(L-y)}) + \mathcal{O}(e^{-\alpha L})$$

Since $\sigma P_2 \sigma = P_1$ and $P_1^3 = P_1$ we obtain

$$\lim_{L\to\infty}\langle\sigma_x\sigma_y\rangle_L^{\bar{\sigma}} \equiv \langle\sigma_x\sigma_y\rangle = e^{-\alpha|y-x|} \qquad \alpha = -\log\tanh\beta > 0 \qquad \beta < \infty.$$

The limit is again independent on $\bar{\sigma}$.

We thus obtained

$$\langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle\langle\sigma_y\rangle = e^{-\alpha|x-y]} \xrightarrow[|x-y|\to\infty]{} 0$$

i.e. *2-point correlation function decaya exponentially* (here we had $\langle\sigma_x\rangle = 0$). $\xi = \frac{1}{\alpha}$ is called the *correlation length.* Note that $\xi \to \infty$ as $\beta \to \infty$. At low temperatures, the $1d$

model gets more correlated.

It is easy to show now (exercise) : let $x_1 < x_2 < \cdots < x_{2n}$. Then

$$\langle \prod_{i=1}^{2n} \sigma_{x_i} \rangle_L^{\bar{\sigma}} \quad \underset{L\to\infty}{\longrightarrow} \exp\left(-\alpha \sum_{i=1}^{n} (x_{2i} - x_{2i-1})\right)$$

i.e. all the correlation functions have a limit which is independent on $\bar{\sigma}$ and equals

$$\langle \prod_{i=1}^{2n} \sigma_{x_i} \rangle = \prod_{i=1}^{n} \langle \sigma_{x_{2i-1}} \sigma_{x_{2i}} \rangle. \tag{3.3}$$

We get

Theorem 3.2 *For the* 1d *Ising model the thermodynamic limit exists and is independent on b.c. (for all* $\beta \in [0, \infty)$*). The correlation functions of the* ∞*-volume state are given by* (3.3) *with*

$$\langle \sigma_x \sigma_y \rangle = (\tanh\, \beta)^{|x-y|}$$

Exercise. Calculate $Z_L^{b.c.}(\beta, h)$ and $F(\beta, h) = -\lim_{L\to\infty} \frac{1}{\beta L} \log Z_L^{b.c.}(\beta, h)$. Show the limit is independent of *b.c.* and *analytic* in β, h.

Remark. T is called the *transfer matrix*. What was important above was that $(f^{\bar{\sigma}}, e_1) > 0$. This is a general fact, following from the Perron-Frobenius theorem (see e.g. [37] and [21], Theorem 3.3.2, for an extension to compact operators).

Exercise. Consider the d-dimensional Ising model in the cylinder $[-L, L] \times \mathbb{Z}_\ell^{d-1}$ i.e. we have periodic boundary conditions in $d-1$ dimensions with period ℓ and $\bar{\sigma}$ boundary conditions in the two ends of the cylinder. Write the partition function and the correlations in terms of a transfer matrix and using the Perron-Frobenius theorem show the limits exist as $L \to \infty$ (but ℓ fixed) and are independent on $\bar{\sigma}$. We'll see later the last statement is not true in low temperatures if we took $\ell = L$.

Remark : What is shown above means that the 1d Ising model has no phase transition i.e. there is a unique infinite volume state. We'll discuss this later.

4 $d \geq 2$. High Temperature Uniqueness

Ising model in two dimensions. Onsager succeeded calculating the free energy $F(\beta)$ in closed form in 1944. The result is not an analytic function of β. At the positive real axis $F(\beta)$ has a point of non analyticity at $\beta = \beta_c$ given by $\tanh(2\beta_c) = 1/\sqrt{2}$ where the second derivative of F (the specific heat) diverges logarithmically $\partial_\beta^2 F(\beta) \sim$ const. $\log|\beta - \beta_c|$. Subsequently he showed the existence of phase transition (see below) by computing the magnetization $m(\beta) = \langle\sigma_0\rangle^+$ in closed form. It vanishes if $\beta \le \beta_0$ and is nonzero for $\beta > \beta_0$. Furthermore as $\beta \downarrow \beta_c$ $m(\beta) \sim (\beta - \beta_c)^{1/8}$. He also calculated the 2-point correlation function which has exponential decay for $\beta \neq \beta_c$ and at β_c $\langle\sigma_0\sigma_x\rangle \sim |x|^{1/4}$ as $x \to \infty$. These results were revolutionary. It was the first proof of phase transitions from first principles. The *critical exponents* (1/8 and 1/4 above and others) were different form naive expectations (mean field theory) and called for explanation. This was finally achieved with Wilson's renormalization group theory 30 years later.

Other exactly solvable models exist in $d = 2$ (see Baxter's book [4]). However, this is very exceptional and, in $d > 2$, there are practically none so other methods are called for. We will first develop methods to study $\langle\sigma_A\rangle$ for β small (high temperature) and β large (low temperature). These methods work for all d and for much more general models than the Ising model. The high and low temperature expansions are still the best ways to numerically to study the critical point.

Recall our setup: $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ and the boundary condition on the boundary of $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, is given by a configuration $\bar{\sigma} \in \Omega$.

Note that to say $\bar{\sigma} = +1$ means that we have on the boundary of Λ a positive magnetic field that tends to force σ_x to be $+1$. The question is : can this result in a positive $\langle\sigma_x\rangle$ as $\Lambda \uparrow \mathbb{Z}^d$?

Theorem 4.1 (a) *There exists $\beta_0 > 0$ such that if $\beta < \beta_0$ then $\langle\sigma_A\rangle^{\bar{\sigma}}_{\beta,h,\Lambda}$ has a limit as $\Lambda \to \mathbb{Z}^d$ (via cubes say), independent on $\bar{\sigma}$.*
(b) *Let $h = 0$, $d \ge 2$. There exists $\beta_1 < \infty$ such that if $\beta > \beta_1$ then there are at least two different infinite-volume (pure) states $\langle - \rangle^+ \neq \langle - \rangle^-$.*

Remarks. 1. (a) means uniqueness of the Gibbs state, see below.

2. (b) means phase transition. See next section.

3. In $d = 2$ there are only 2 (pure) Gibbs states [1], in $d \ge 3$ there are many more [12].

4. Finally, it can be proved, see [3] that $\beta_0 = \beta_1$.

We prove (a) in this section and (b) in section 4.3. The result is due to Peierls, Dobrushin and Griffiths [31, 9, 22].

4.1 High Temperature Expansion

We use an expansion due to M. Fischer [16]. When $\beta = 0$, $\mu^{b.c.}_\Lambda$ is a product measure

$$\mu^{b.c.}_\Lambda(\sigma) = \prod_{x\in\Lambda} \frac{e^{h\sigma_x}}{e^h + e^{-h}} \equiv \prod_{x\in\Lambda} \nu_h(\sigma_x).$$

Thus

$$\langle \sigma_A \rangle_\Lambda^{b.c.} = \left(\sum_{\sigma=\pm 1} \sigma \nu_h(\sigma) \right)^{|A|} = (\tanh\, h)^{|A|}$$

factorizes and is Λ-independent. We wish to show that for small β our measure approximately factorizes. Let $h = 0$ for simplicity. Write (since $\sigma_x \sigma_y \in \{-1, 1\}$)

$$e^{\beta \sigma_x \sigma_y} = \cosh \beta + \sigma_x \sigma_y \sinh \beta = \cosh \beta (1 + \sigma_x \sigma_y \tanh\, \beta).$$

Note that for β small $\tanh\, \beta$ is small. Consider e.g.

$$\langle \sigma_x \sigma_y \rangle_\Lambda = \frac{\displaystyle\sum_\sigma \sigma_x \sigma_y e^{-\beta \mathcal{H}}}{\displaystyle\sum_\sigma e^{-\beta \mathcal{H}}}.$$

Notation. A *bond* is a nearest neighbour pair $\{x, y\}, x, y \in \mathbb{Z}^d, |x - y| = 1$. We may picture a bond by a line between the points 

. It is a special subset of $\mathbb{Z}^d$.

Let us first discuss the *free b.c.* case. With this notation

$$\mathcal{H}_\Lambda^{free}(\sigma) = -\sum_{b \in \mathcal{B}_\Lambda} \sigma_b$$

where we recall $\mathcal{B}_\Lambda$ denotes the set of all bonds $b \subset \Lambda$. Thus (with $|\mathcal{B}_\Lambda|$ the cardinality of $\mathcal{B}_\Lambda$)

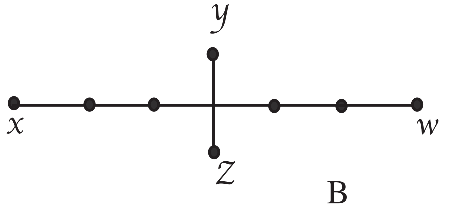
$$e^{-\mathcal{H}_\Lambda^{free}(\sigma)} = (\cosh \beta)^{|\mathcal{B}_\Lambda|} \prod_{b \in \mathcal{B}_\Lambda} (1 + \sigma_b \tanh \beta).$$

Insert this in $\langle \sigma_x \sigma_y \rangle_\Lambda$, cancel $(\cosh \beta)^{|\mathcal{B}_\Lambda|}$ in numerator and denominator and expand the product over b:

$$\langle \sigma_x \sigma_y \rangle_\Lambda^{free} = \frac{\displaystyle\sum_{\sigma \in \Omega_\Lambda} \sum_{B \subset \mathcal{B}_\Lambda} \sigma_x \sigma_y \prod_{b \in B} \sigma_b (\tanh \beta)^{|B|}}{\displaystyle\sum_{\sigma \in \Omega_\Lambda} \sum_{B \subset \mathcal{B}_\Lambda} \prod_{b \in B} \sigma_b (\tanh \beta)^{|B|}}.$$

Definition. Given $B \in \mathcal{B}_\Lambda$, a family of bonds, let ∂B denote the set of sites $x \in \Lambda$ that occur an *odd* number of times in the bonds in B.

Example.



B

$$\partial B = \{x, y, z, w\}$$

Since $\sum_{\sigma=\pm} \sigma^n = 0$ if n is odd, we get

$$\langle \sigma_x \sigma_y \rangle_\Lambda^{free} = \frac{\sum_{B:\partial B=\{x,y\}} (\tanh \beta)^{|B|}}{\sum_{B:\partial B=\emptyset} (\tanh \beta)^{|B|}}.$$

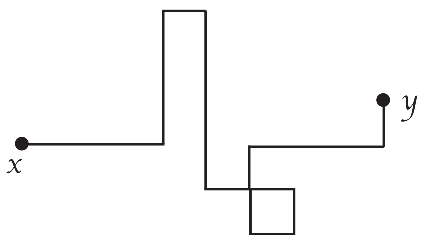
This is the starting point of the high temperature expansion. We use it to prove the existence of a finite correlation length.

Theorem 4.2 *There exists $\beta_0 > 0$ such that for $\beta < \beta_0$*

$$|\langle \sigma_x \sigma_y \rangle_\Lambda^{free}| \leq C e^{-|x-y|/\xi}$$

uniformly in Λ (i.e. C, $\xi < \infty$ are Λ-independent).

Proof Each B with $\partial B = \{x, y\}$ contains a subset of bonds $P = \{b_i\}_{i=1}^n$ $b_i = \{x_i, x_{i+1}\}, x_1 = x, x_{n+1} = y$ (if not, $\partial B \neq \{x, y\}$, show this !). We call P a (connected) path joining x and y.



Thus every B in the numerator can be decomposed as $B = P \bigcup B'$ where P is some set as above and $\partial B' = \emptyset$. Given B, P is not unique, but let us choose it arbitarily, and call the choice $P(B)$.
We may then rewrite the numerator as

$$\sum_P (\tanh \beta)^{|P|} \sum_{B' \in \mathcal{B}(P)} (\tanh \beta)^{|B'|}$$

where the sum over P runs through the paths joining x and y and $\mathcal{B}(P)$ consists of sets $B' \subset \mathcal{B}_\Lambda$ such that $\partial B' = \emptyset$ and $P(P \bigcup B') = P$. Since $\mathcal{B}(P) \subset \{B \in \mathcal{B}_\Lambda | \partial B = \emptyset\}$ each term in this sum occurs in the denominator and we have

$$\frac{\sum_{B' \in \mathcal{B}(P)} (\tanh \beta)^{|B'|}}{\sum_{B:\partial B=\emptyset} (\tanh \beta)^{|B|}} \leq 1$$

and so

$$0 \le \langle \sigma_x \sigma_y \rangle_\Lambda \le \sum_P (\tanh \beta)^{|P|}. \tag{4.1}$$

To control the sum over P write as above $P = \{(x_i, x_{i+1})\}_{i=1}^n$, $x_1 = x$, $x_{n+1} = y$. Given x_i there are at most $2d$ nearest neighbours x_{i+1} and n is at least $|x-y|$. Thus

$$(4.1) \le \sum_{n \ge |x-y|} (2d \tanh \beta)^n \le C e^{-|x-y|/\xi}$$

if $2d \tanh \beta < 1$ $\square$.

<u>Remarks.</u> 1. We could have chosen P *self-avoiding* and so could have had only $2d-1$ choices : $x_{i+1} \neq x_{i-1}$. Thus $(2d-1)\tanh\beta < 1$ suffices.

2. In $d = 2$, $\tanh \beta_c = \sqrt{2} - 1 = 0,4142....$ We got $\tanh \beta < \frac{1}{3}$ which is not that bad. As $d \to \infty$, our estimate $\tanh \beta_c \sim 1/(2d-1)$ becomes exact by combining [16] and [18], see [6].

We now slightly modify the previous argument to prove part A) of Theorem 4.1.

Proof of (A) We prove uniqueness (independence on b.c) of the limit. Existence is analogous. Consider $\bar\sigma, \bar\sigma' \in \Omega$ and

$$\begin{aligned}
\langle \sigma_A \rangle_\Lambda^{\bar\sigma} - \langle \sigma_A \rangle_\Lambda^{\bar\sigma'} &= \frac{\displaystyle\sum_\sigma e^{-\beta \mathcal{H}_\Lambda^{\bar\sigma}(\sigma)} \sigma_A}{\displaystyle\sum_\sigma e^{-\beta \mathcal{H}_\Lambda^{\bar\sigma}(\sigma)}} - \text{(same with primes)} \\
&= \frac{\displaystyle\sum_{\sigma,\sigma'} e^{-\beta \mathcal{H}_\Lambda^{\bar\sigma}(\sigma) - \beta \mathcal{H}_\Lambda^{\bar\sigma'}(\sigma')} (\sigma_A - \sigma'_A)}{\displaystyle\sum_{\sigma,\sigma'} e^{-\beta \mathcal{H}_\Lambda^{\bar\sigma}(\sigma) - \beta \mathcal{H}_\Lambda^{\bar\sigma'}(\sigma')}}.
\end{aligned} \tag{4.2}$$

We expand slightly differently (to understand why, see Remark after the Proof).

$$e^{\beta(\sigma_x\sigma_y + \sigma'_x\sigma'_y)} = e^{-2\beta}(1 + f_{xy})$$

$$f_{xy} = e^{\beta(\sigma_x\sigma_y + \sigma'_x\sigma'_y + 2)} - 1.$$

Note that

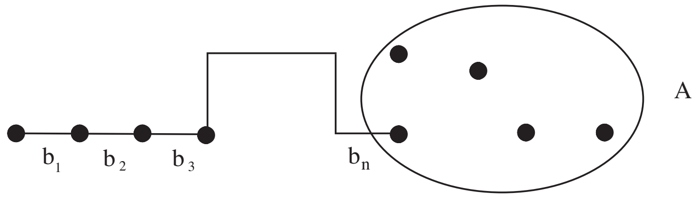
$$0 \le f_{xy} \le 4\beta(1 + f_{xy}) \tag{4.3}$$

using $0 \le e^z - 1 \le z e^z$ for $z \ge 0$, and $0 \le \sigma_x\sigma_y + \sigma'_x\sigma'_y + 2 \le 4$. Expand in powers of f as before, but only in the numerator : let $\bar{\mathcal{B}}_\Lambda$ be the set of bonds intersecting Λ (i.e.

$b = \{x, y\}$, either x or y is in Λ), then

$$\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma'} = \frac{\sum_{B\subset\bar{\mathcal{B}}_\Lambda}\sum_{\sigma\sigma'}(\sigma_A - \sigma'_A)\prod_{b\in B} f_b}{\sum_{\sigma\sigma'}\prod_{b\in\bar{\mathcal{B}}_\Lambda}(1+f_b)} \tag{4.4}$$

Let us say that $b \in B$ is connected in B to A if $\exists\{b_i\}_{i=1}^n$, $b_i \in B$ $i = 1, \cdots, n$ such that $b_i = \{x_i, x_{i+1}\}, b_1 = b$ and $x_{n+1} \in A$, see the picture.



Let $B_0 \subset B$ be the set of b in B connected to A. Suppose that no bond in B_0 intersects Λ^c (i.e. all are in Λ i.e. all f_b depend on σ, σ' and not on $\bar\sigma, \bar\sigma'$). Then that term in (4.4) vanishes. Indeed, the sum over σ_x, σ'_x with $x \in \cup_{b\in B_0} b \cup A$ factorizes out and vanishes, since $\prod_{b\in B_0} f_b$ is symmetric under the interchange of σ and σ' whereas $\sigma_A - \sigma'_A$ is antisymmetric.
Hence, the nonvanishing terms in (4.4) have B's such that B includes a *connected path P of bonds joining A to Λ^c*.
Bound the numerator as

$$|(\sigma_A - \sigma'_A)\prod_{b\in B} f_b| \le 2\prod_{b\in P} f_b \prod_{b\in B\setminus P} f_b \le 2(4\beta)^{|P|}\prod_{b\in P}(1+f_b)\prod_{b\in B\setminus P} f_b$$

using (4.3) in the last inequality (Recall that $f_b \ge 0$!!); and so, picking, as in the proof of Theorem 4.2, for each B, a choice $P(B)$ of P,

$$|\sum_B\sum_{\sigma,\sigma'}(\sigma_A - \sigma'_A)\prod_{b\subset B} f_b| \le 2\sum_P (4\beta)^{|P|}\sum_{\sigma\sigma'}\sum_{B'\in\mathcal{B}(P)}\prod_{b\in P}(1+f_b)\prod_{b\in B'} f_b \tag{4.5}$$

where $\mathcal{B}(P)$ is the set of $B' \subset \bar{\mathcal{B}}_\Lambda \setminus P$ satisfying $P(B' \cup P) = P$. Now,

$$\sum_{B'\in\mathcal{B}(P)}\prod_{b\in B'} f_b \le \sum_{B'\subset\bar{\mathcal{B}}_\Lambda\setminus P}\prod f_b = \prod_{b\in\bar{\mathcal{B}}_\Lambda\setminus P}(1+f_b)$$

so

$$(4.5) \le 2\sum_P (4\beta)^{|P|}\sum_{\sigma\sigma'}\prod_{b\in\bar{\mathcal{B}}_\Lambda}(1+f_b). \tag{4.6}$$

The denominator in (4.2) equals $\sum_{\sigma\sigma'}\prod_{b\in\bar{\mathcal{B}}_\Lambda}(1+f_b)$, so combining (4.2) and (4.6) we obtain

$$|\,\langle\sigma_A\rangle_\Lambda^{\bar\sigma}-\langle\sigma_A\rangle_\Lambda^{\bar\sigma'}|\le 2\sum_P(4\beta)^{|P|}\le 2\sum_{x\in A}\sum_{y\in\partial\Lambda}\sum_{P:x\to y}(4\beta)^{|P|}. \tag{4.7}$$

Taking $8d\beta<1$ the sum over paths is bounded by $C(8d\beta)^{|x-y|}$ and then

$$|\,\langle\sigma_A\rangle_\Lambda^{\bar\sigma}-\langle\sigma_A\rangle_\Lambda^{\bar\sigma'}|\le C|A|\;|\partial\Lambda|(8d\beta)^{\mathrm{dist}(A,\partial\Lambda)}. \tag{4.8}$$

where $\partial\Lambda$ is the boundary of Λ. For $\Lambda=\Lambda_L$ a cube of side L $|\partial\Lambda|\propto L^{d-1}$, $\mathrm{dist}(A,\partial\Lambda)\propto L$ and thus (4.8) is bounded by $\le C|A|L^{d-1}\delta^L$ with $\delta<1$. This tends to zero as $L\to\infty$. . $\square$

Remark Note that the positivity of f_{xy} was crucial in (4.5). This is why we did not use the previous expansion. This method generalizes to a very general class of Hamiltonians, see [7].

Exercises.
1. Estimate $|\langle\sigma_A\rangle_\Lambda^{\bar\sigma}-\langle\sigma_A\rangle_{\Lambda'}^{\bar\sigma}|$ in for $\Lambda\subset\Lambda'$ the same way:

$$\langle\sigma_A\rangle_\Lambda^{\bar\sigma}-\langle\sigma_A\rangle_{\Lambda'}^{\bar\sigma}=\frac{\displaystyle\sum_{\sigma\in\Omega_\Lambda}\sum_{\sigma'\in\Omega_{\Lambda'}}e^{-\beta\mathcal{H}_\Lambda(\sigma)}e^{-\beta\mathcal{H}_{\Lambda'}(\sigma')}(\sigma_A-\sigma'_A)}{\displaystyle\sum_{\sigma\in\Omega_\Lambda}\sum_{\sigma'\in\Omega_{\Lambda'}}e^{-\beta\mathcal{H}_\Lambda(\sigma)-\beta\mathcal{H}_{\Lambda'}(\sigma')}}.$$

Expand $e^{-\beta\mathcal{H}_\Lambda(\sigma)-\beta\mathcal{H}_{\Lambda'}(\sigma')}$ as above and note that only B's connecting A to Λ^c contribute. Use this to show that the limit as $\Lambda\nearrow\mathbb{Z}^d$ exists.

2. Similarily, write

$$|\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma}-\langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma}|=\frac{1}{2}\sum_{\sigma\sigma'}e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')}[\sigma_X-\sigma'_X][\sigma_Y-\sigma'_Y]/\sum_{\sigma\sigma'}e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')},$$

expand $e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')}$ as above and prove:

$$|\langle\sigma_X\sigma_Y\rangle_\Lambda^{\bar\sigma}-\langle\sigma_X\rangle_\Lambda^{\bar\sigma}\langle\sigma_Y\rangle_\Lambda^{\bar\sigma}|\le Ce^{-\alpha\mathrm{dist}(X,Y)}$$

since here only B's connecting X and Y will contribute i.e. all correlations functions decay exponentially.

4.2 Low Temperature Expansions

Consider $\mathcal{H}_\Lambda^+(\sigma)=-\sum_b\sigma_b$. For $\beta\to\infty$, $e^{-\beta\mathcal{H}^+}$ reaches its maximum when all $\sigma_b=\sigma_x\sigma_y=1$ i.e. if all $\sigma_x=+1$ (because of the $+$ bounadry condition). Consider, say in $d = 2$, the configuration $\sigma_x = 1 \;\; \forall x \neq x_0$, $\sigma_{x_0} = -1$:

$$\begin{array}{ccc} + & + & + \\ + & - & + \\ + & + & + \end{array}$$

This has energy $\beta\mathcal{H} = \beta\mathcal{H}_{\text{minimum}} + 8\beta$ ($4d\beta$ in general dimension). Consider a connected region R of $-$:

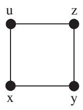
$$\begin{array}{ccccc} + & + & + & + & + \\ + & - & - & - & + \\ + & - & - & - & + \\ + & - & - & - & + \\ + & + & + & + & + \end{array}$$

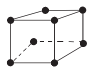
This has $\mathcal{H} = \mathcal{H}_{\min} + 2|\partial R|$ where $|\partial R| = \#$ bonds $\{x, y\}$ with $x \in R, y \in R^c$. We say that there is a *contour* around R. Let us formalize this. Given $\sigma \in \Omega_\Lambda$, let

$$C(\sigma) = \{b \in \bar{\mathcal{B}}_\Lambda |\; \sigma_b = -1\}$$

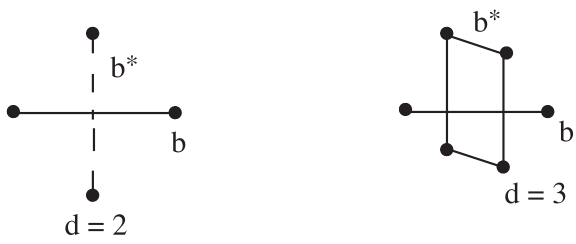
i.e. all bonds in Λ or joining Λ to Λ^c where $\sigma_x \neq \sigma_y$.

The *dual lattice* of $\mathbb{Z}^d$ is the lattice $\mathbb{Z}^d + (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = \{(x_1, \ldots, x_d)|\; x_i - \frac{1}{2} \in \mathbb{Z}, i = 1, \ldots, d\}$ where we consider both imbedded in $\mathbb{R}^d$. Let us visualize bonds $b = \{x, y\}$ as the closed line segments in $\mathbb{R}^d$ from x to y 
.

Similarly 

is a 2-cell $\{x, y, z, u\}$  a 3-cell etc. 2-cells are often called *plaquettes.*

Now a bond b in $\mathbb{Z}^d$ defines a unique $(d-1)$-cell b^* "orthogonal" to it in the dual lattice:



Let $\bar{\mathcal{B}}^*_\Lambda = \{b^* |\; b \in \bar{\mathcal{B}}_\Lambda\}$. Let us denote

$$C^*(\sigma) := \{b^* \in \bar{\mathcal{B}}^*_\Lambda |\; b \in C(\sigma)\}$$

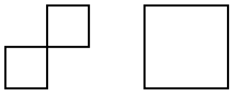
i.e. $C^*(\sigma)$ is a set of bonds (d=2) or plaquettes (d=3) in the dual lattice.

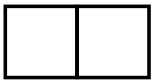
We say two bonds $b \neq b'$ are connected if $|b \cap b'| = 1$ (i.e. they share one site) and two plaquettes $p \neq p'$ are connected if $|p \cap p'| = 2$ (i.e. they share a bond). Given a set B of bonds (plaquettes) consider the graph $G(B)$ with vertex set B and edges $\{b, b'\}$ if b and b' are connected. Let G_α be the connected components of $G(B)$ and B_α the vertices of G_α. We call B_α the connected components of B. We say B is connected if $G(B)$ is connected. By some abuse we say B and B' are disjoint if no $b \in B$, $b' \in B'$ are connected.

We will now characterize the connected components of $C^*(\sigma)$.

Definition 4.3 *A contour* γ is a *connected* set $\gamma \subset \bar{\mathcal{B}}^*_\Lambda$, such that (d=2) each site x of the dual lattice belongs to an even number of bonds b^*, with $b^* \in \gamma$ or (d=3) each bond of the dual lattice belongs to an even number of plaquettes $b^* \in \gamma$. Hence contours are closed paths (d=2) or closed surfaces (d=3). We say a family Γ of contours is *compatible* if all $\gamma \in \Gamma$ are connected and all $\gamma, \gamma' \in \Gamma$ are disjoint.

Example.

Allowed : 

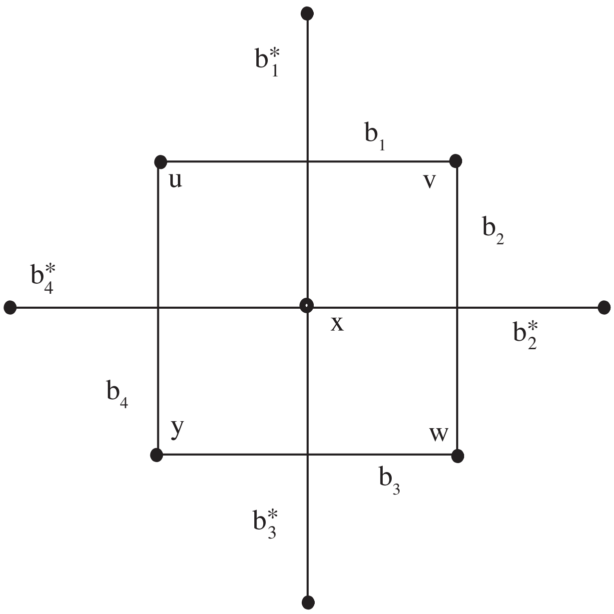
Not allowed : 

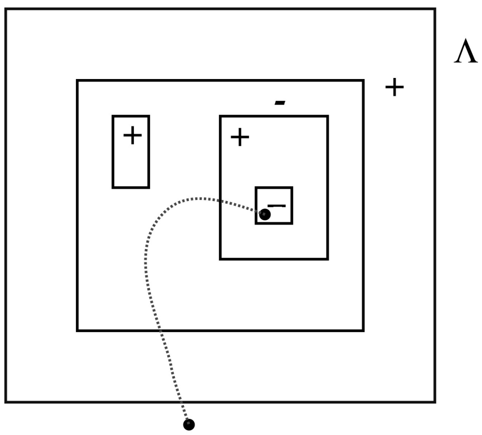
We have since $\sigma^2 = 1, \sigma_u\sigma_v\sigma_v\sigma_w\sigma_w\sigma_y\sigma_y\sigma_u = 1 = \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} \Rightarrow$ even number of $\sigma_{b_i} = -1$

Lemma 4.4. The family $\Gamma(\sigma)$ of connected components of $C^*(\sigma)$ is a compatible family of contours. Conversely, given a compatible family Γ of contours there is a unique σ such that $\Gamma = \Gamma(\sigma)$.

Proof. $C^*(\sigma)$ has the property that x belongs to an even number of bonds (see figure). Hence its connected components have this property.

Conversely, given Γ, let $x_0 \in \Lambda^c$ (so $\sigma_{x_0} = +1$) and let P be any path of bonds from x_0 to x. Let $N(P)$ be the number of $b \in P$ with $b \in C(\sigma)$, i.e. the number of times P crosses contours (see figure). Put $\sigma_x = (-1)^{N(P)}$. This is well defined since if P' is another path then $N(P) - N(P')$ is even (show!). $\square$





Let us denote by $\mathbb{G}_\Lambda$ the set of compatible families Γ of contours $\gamma \in \bar{\mathcal{B}}^*_\Lambda$. The Lemma implies:

$$-\beta\mathcal{H}^+_\Lambda(\sigma) = \beta N_\Lambda - 2\beta \sum_{\gamma\in\Gamma(\sigma)} |\gamma|$$

and

$$Z^+_\Lambda = e^{\beta N_\Lambda} \sum_{\Gamma\in\mathbb{G}} \prod_{\gamma\in\Gamma} e^{-2\beta|\gamma|} \tag{4.9}$$

where we sum over the set of compatible families of contours Γ. Also,

$$\langle\sigma_A\rangle^+_\Lambda = \frac{\sum\limits_{\Gamma\in\mathbb{G}} \prod\limits_{\gamma\in\Gamma} e^{-2\beta|\gamma|} \prod\limits_{x\in A} (-1)^{N_x(\Gamma)}}{\sum\limits_{\Gamma\in\mathbb{G}} \prod\limits_{\gamma\in\Gamma} e^{-2\beta|\gamma|}} \tag{4.10}$$

$N_x(\Gamma) = \#$ of $\gamma \in \Gamma$ surrounding x.

4.3 Magnetization

We prove Theorem 4.1. (b) by proving

Theorem 4.3. (Peierls argument). *There exists $\beta_1 < \infty$, $\delta > 0$ such that for $\beta > \beta_1$*

$$\langle \sigma_0 \rangle_\Lambda^+ = -\langle \sigma_0 \rangle_\Lambda^- \geq \delta$$

for all Λ.

Proof. We have

$$\langle \sigma_0 \rangle_\Lambda^+ = \mathbb{P}(\sigma_0 = 1) - \mathbb{P}(\sigma_0 = -1) = 1 - 2\ \mathbb{P}(\sigma_0 = -1).$$

Now $\mathbb{P}(\sigma_0 = -1)$ equals the probability that there are an odd number of contours surrounding origin which in turn is bounded from above by the probability that there exists a contour surrounding origin. Denote by $\mathbb{G}_\Lambda^0 \subset \mathbb{G}_\Lambda$ those compatible families of contours for which there exists $\gamma_0 \in \mathbb{G}_\Lambda 0$ surrounding origin. Hence

$$\mathbb{P}(\sigma_0 = -1) \leq \sum_{\Gamma \in \mathbb{G}_\Lambda^0} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|} / \sum_{\Gamma \in \mathbb{G}_\Lambda} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}.$$

This we may write as

$$= \sum_{\gamma \textit{ surrounds } 0} e^{-2\beta|\gamma|} \sum_{\Gamma : \{\gamma\} \cup \Gamma \in \mathbb{G}_\Lambda} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|} / \sum_{\Gamma \in \mathbb{G}_\Lambda} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}.$$

Since each term in the sum in the denominator occurs in the numerator we get

$$\mathbb{P}(\sigma_0 = -1) \leq \sum_{\gamma \textit{ surrounds } 0} e^{-2\beta|\gamma|} = \sum_{n=2d}^{\infty} e^{-2\beta n} \#\{\gamma : \gamma \text{ surrounds } 0, |\gamma| = n\}. \quad (4.11)$$

To estimate the number of contours of γ surrounding origin with $|\gamma| = n$ we pick a $b^* \in \bar{\mathcal{B}}_\Lambda^*$ and count the number of connected subsets of $\bar{\mathcal{B}}_\Lambda^*$ of cardinality n containing b^*. By Lemma 4.4. below this number is bounded by c_d^{2n} with $c_2 = 4$ and $c_3 = 12$. Since the distance of b^* to origin is bounded by n the number of choices of b^* is bounded by $(2n)^d$ and so

$$\#\{\gamma : \gamma \text{ surrounds } 0, |\gamma| = n\} \leq (2n)^d c_d^{2n}.$$

Hence, for $\beta > \log c_d$ the series in (4.11) converges, uniformly in Λ and tends to zero as $\beta \to \infty$. The claim follows. Obviously $\langle \sigma_0 \rangle_\Lambda^- = \langle \sigma_0 \rangle_\Lambda^+$. □

Lemma 4.4. . *The number of connected subsets of $\bar{\mathcal{B}}_\Lambda^*$ of cardinality n containing b^* is bounded by c_d^{2n} with $c_2 = 4$ and $c_3 = 12$.*

Proof. A set $B^* \subset \bar{\mathcal{B}}_\Lambda^*$ is connected if and only if the graph $G(B^*)$ is. By Lemma 4.5. any such graph is covered by doing a walk on $\bar{\mathcal{B}}_\Lambda^*$ with starting point b^*, length 2n and jumps between connected vertices. In 2d each b^* has 4 such neighbors and in 3d 12. The number of such walks is c_d^{2n}. The claim follows. $\square$

Lemma 4.5 (Köningsberg bridge Lemma) Let G be a finite connected graph and let α_0 be a vertex of G. Then there is a path starting and ending at α_0 which includes each line of G only twice.

Proof. Induction in the cardinality of the vertex set $v(G)$. Let $v(G) = n$ and $\{\alpha_1, \alpha_2\}$ be an edge of G. Consider the graph $G' = G \backslash \{\alpha_1, \alpha_2\}$. If G' is connected, by induction there is walk $w : \alpha_0 \to \alpha_0$ visiting all its edges twice. The walk $w' : \alpha_0 \to \alpha_1 \to \alpha_2 \to \alpha_1 \to \alpha_0$ where the first α_1 is the first visit of w to α_1 is the desired walk.

If G' is disconnected and α_0 and α_1 are in the same connected component induction provides two walks $\alpha_0 \to \alpha_0$ and $\alpha_2 \to \alpha_2$. Then the desired walk is $\alpha_0 \to \alpha_1 \to \alpha_2 \to \alpha_2 \to \alpha_1 \to \alpha_0$. $\square$

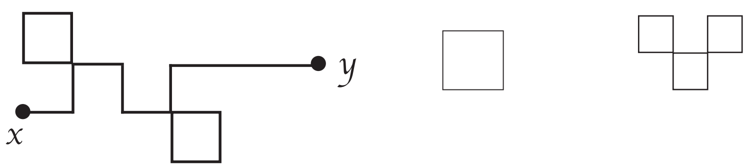
Remark 1 $\langle\sigma_0\rangle \neq 0$ is called *spontaneous symmetry breaking* : $\mathcal{H}_\Lambda(\sigma)$ is invariant (if $h = 0$) under $\sigma \to -\sigma$ except for the *b.c.* ($\mathcal{H}_\Lambda^{free}$ and $\mathcal{H}_\Lambda^{per}$ are). As $\Lambda \nearrow \mathbb{Z}^d$ the *b.c.* have a finite effect ! Another way to state this : Let $h \neq 0$. Construct $\lim_{\Lambda\nearrow\mathbb{Z}^d}\langle - \rangle_{\Lambda,h}^{\bar{\sigma}} \equiv \langle - \rangle_h$. It will be independent on $\bar{\sigma}$ (see below). Then $\lim_{h\downarrow 0}\langle - \rangle_h = \langle - \rangle^+, \lim_{h\uparrow 0}\langle - \rangle^-$. In particular, let $m(h) = \langle\sigma_0\rangle_h$. Then $m(h)$ is *discontinuous* at $h = 0$. This is called a first order phase transition.

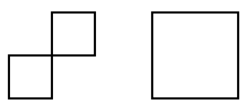
Remark 2 The existence of the limit $\Lambda \nearrow \mathbb{Z}^d$ can be proved along the same lines as for β small above.

4.4 High and Low Temperature Expansions

One can go further and get identities and not only inequalities. Let us return to high temperature expansion

$$\langle\sigma_x\sigma_y\rangle_\Lambda^{free} = \frac{\sum\limits_{\partial B=\{x,y\}} (\tanh\beta)^{|B|}}{\sum\limits_{\partial B=\emptyset} (\tanh\beta)^{|B|}}$$

Numerator : 

Denominator : 

Try to cancel the so-called "vacuum graphs", i.e. the ones not involving x or y. Decompose B into connected components: $B = \bigcup_\alpha B_\alpha, B_\alpha$ with B_α connected sets of bonds (as defined in Section 4.3). Denote by $\mathbf{B}$ a family of connected, mutually disjoint (i.e. disconnected) sets of bonds and by $\mathbb{B}$ the set of such families. In the numerator, we have one component, say B_1, such that $\partial B_1 = \{x, y\}$. So

$$\langle \sigma_x \sigma_y \rangle_\Lambda^{free} = \sum_{B_1 \text{ connected } \partial B_1 = \{x,y\}} \rho(B_1) \sum_{\mathbf{B}:\{B_1\}\cup\mathbf{B}\in\mathbb{B}} \prod_{B\in\mathbf{B}} \rho(B) / \sum_{\mathbf{B}\in\mathbb{B}} \prod_{B\in\mathbf{B}} \rho(B).$$

We have defined

$$\rho(B) = (\tanh\beta)^{|B|}.$$

The algebra we will perform does not depend on the explicit expression of ρ. The cancellation is performed by the following trick. Consider the partition function

$$Z = \sum_{\mathbf{B}\in\mathbb{B}} \prod_{B\in\mathbf{B}} \rho(B) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(B_1, B_2, \ldots, B_n) \\ B_i \cap B_j = \emptyset}} \prod_{\alpha=1}^{n} \rho(B_\alpha) \tag{4.12}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(B_1, \ldots, B_n) \\ \text{(no constraint)}}} \prod_{\alpha=1}^{n} \rho(B_\alpha) \prod_{\alpha<\beta} \chi(B_\alpha, B_\beta) \tag{4.13}$$

where the $n!$ comes from writing the sum over the unordered sets $\mathbf{B} = \{B_1, \ldots, B_n\}$ as one over ordered ones $(B_1, B_2, \ldots, B_n)$ and on the second line we introduced

$$\chi(B_\alpha, B_\beta) = \begin{cases} 0 & B_\alpha \cap B_\beta \neq \emptyset \\ 1 & B_\alpha \cap B_\beta = \emptyset \end{cases}$$

so that the last product imposes the constraint on the first line.

Note that if we didn't have the constraint in (4.13) we could do the sum

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(B_1, \ldots, B_n) \\ \text{(no constraint)}}} \prod_{\alpha=1}^{n} \rho(B_\alpha) = \exp\Big(\sum_{B\subset\mathcal{B}_\Lambda} \rho(B)\Big). \tag{4.14}$$

We will now expand the constraint to derive a generalization of (4.14). To achieve this we write $\chi(B_\alpha, B_\beta) = 1 - \eta(B_\alpha, B_\beta)$ with

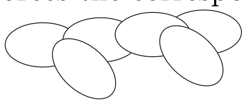
$$\eta(B_\alpha, B_\beta) = \begin{cases} 0 & B_\alpha \cap B_\beta = \emptyset \\ 1 & B_\alpha \cap B_\beta \neq \emptyset \end{cases}.$$

Then

$$\prod_{\alpha<\beta} (1 - \eta(B_\alpha, B_\beta)) = \sum_{\mathcal{G}} \prod_{(\alpha,\beta)\in\mathcal{G}} (-\eta(B_\alpha, B_\beta)) \tag{4.15}$$

where $\mathcal{G}$ is a collection of pairs $\{\alpha, \beta\}$ $\alpha, \beta = 1, \cdots, n, \alpha \neq \beta$. $\mathcal{G}$ can be identified with a *graph* on the vertex set $\{1, \cdots, n\}$ with *edges* $\{\alpha, \beta\}$ that connect vertices. $\mathcal{G}$ is *connected* if any two vertices α, β can be connected by paths in $\mathcal{G}$. Decompose $\mathcal{G}$ into connected components

$$\mathcal{G} = \bigcup_{i=1}^{k} \mathcal{G}_i, \quad \mathcal{G}_i \text{ connected.} \tag{4.16}$$

Note that this forces the corresponding B_α's to form a conneted network since $\eta \neq 0$ only if $B_\alpha \cap B_\beta \neq \emptyset$  . Let $C_i = \bigcup_{\alpha \in v(\mathcal{G}_\alpha)} B_\alpha$.

Reorganize the sum (4.13) with (4.15),(4.18) inserted :

sum over k and a sum over k connected sets $(C_1, \ldots, C_k)$.
sum over $n \geq k$
sum over all families of k subsets of $\{1, \ldots, n\}$ (i.e. $v(\mathcal{G}_i)$ above)
sum over B_α's
sum over connected graphs on those subsets.

So, one gets:

$$Z = \sum_{k=0}^{\infty} \sum_{(C_1, \cdots, C_k)} \sum_{n \geq k} \frac{1}{n!} \sum_{\substack{(n_1, \cdots, n_k) \\ \sum_i n_i = n}} \frac{n!}{n_1! \cdots n_k!} \frac{1}{k!} (*)$$

$$\cdot \prod_{i=1}^{k} \left(\sum_{\substack{(B_1, \cdots, B_{n_i}) \\ \bigcup_{j=1}^{n_i} B_j = C_i}} \sum_{\substack{\mathcal{G} \\ \text{connected graph on} \\ \{1, \cdots, n_i\}}} \prod_{\alpha=1}^{n_i} \rho(B_\alpha) \prod_{(\alpha,\beta) \in \mathcal{G}} (-\eta(B_\alpha, B_\beta)) \right)$$

The combinatorial factor (*) is the number of ways to choose a family of k subsets of $\{1, \ldots, n\}$ of sizes $\{n_1, \ldots, n_k\}$. Indeed, this number equals

$$\begin{pmatrix} n \\ n_1 \end{pmatrix} \begin{pmatrix} n - n_1 \\ n_2 \end{pmatrix} \begin{pmatrix} n - n_1 - n_2 \\ n_3 \end{pmatrix} \cdots \begin{pmatrix} n - \sum_{i=1}^{n-1} n_i \\ n_k \end{pmatrix} \frac{1}{k!}$$

$$= \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{1}{k!} = \frac{n!}{\prod_{i=1}^{k} n_i!} \frac{1}{k!}$$

where $\begin{pmatrix} n \\ n_1 \end{pmatrix}$ comes from choosing the n_1 elements of set 1 etc. and we divide by $k!$ because order of the sets does not matter.

Thus

$$Z = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(C_1,\cdots,C_k)} \sum_{(n_1,\cdots,n_k)} \prod_{i=1}^{k} \frac{1}{n_i!} \sum_{\substack{(B_1,\cdots,B_{n_i}) \\ \bigcup_{j=1}^{n_i} B_j = C_i}} \sum_{\substack{\mathcal{G} \\ \text{connected on} \\ \{1,\cdots,n\}}} \prod_{\alpha=1}^{n_i} \rho(B_\alpha) \prod_{(\alpha,\beta)\in\mathcal{G}} (-\eta(B_\alpha, B_\beta))$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_C f(C) \right)^k = \exp \sum f(C)$$

with

$$f(C) = \sum_n \sum_{\substack{(B_1,\cdots,B_n) \\ \bigcup B_j = C}} \frac{1}{n!} \sum_{\substack{\mathcal{G} \\ \text{connected on} \\ \{1,\cdots,n\}}} \prod_{\alpha=1}^{n} \rho(B_\alpha) \prod_{(\alpha,\beta)\in\mathcal{G}} (-\eta(B_\alpha, B_\beta)) \tag{4.17}$$

We have achieved our goal. With our assumptions, the exponent is $\mathcal{O}(\Lambda)$. The main estimate that guarantees this is

Lemma 4.6 Let $|\rho(B)| \leq \epsilon^{|B|}$. There exists ϵ_0 $s.t$ $\epsilon < \epsilon_0 \Rightarrow |f(C)| \leq (2\epsilon)^{|C|}$.
(2 is arbitrary here). (Recall, B is always connected here).

Proof Omitted, see Simon's book [40] or Seiler [36]. $\square$

Remark 3 Note that this is non-trivial. We can not estimate in (4.17) $|-\eta| \leq 1$. Indeed, the number of graphs on $\{1,\cdots,n\}$ is the subsets of $\{1,\cdots,n\} \times \{1,\cdots,n\}$ i.e. 2^{n^2} and the number of connected graphs is also $\geq e^{an^2}$ $a > 0$ which is much bigger than $n! \leq n^n$ so the sum does not converge absolutely. One needs to account for cancellations. Note that $\sum_{\text{all}\,\mathcal{G}} \prod(-\eta) = \prod \chi(B_\alpha, B_\beta)$ which is ≤ 1. Connectedness of $\mathcal{G}$ makes estimates harder.

As an example consider the trivial case of Ising model in a box of side 2. There is only one nontrivial term in the high temperature expansion for the partition function:

$$Z = 1 + \rho(B) = e^{\log(1+\rho(B)} = e^{-\sum_n \frac{(-1)^n}{n} \rho(B)^n}$$

with $\rho(B) = (\tanh \beta)^4$. On the other hand, in the expression for $f(B)$ we have $B_i = B$ for all i and so

$$f(B) = \sum_n \frac{\rho(B)^n}{n!} \sum_{\mathcal{G}} \prod_{\{\alpha,\beta\}\in\mathcal{G}} (-1)^{\mathcal{G}}$$

where $\mathcal{G}$ runs through connected graphs on $\{1,\ldots,n\}$. Comparing these two expressions we see that this sum equals $\frac{(-1)^n}{n}$.

Given the Lemma, write

$$\sum_{C\subset\mathcal{B}_\Lambda} f(C) = \sum_{b\in\mathcal{B}_\Lambda} \sum_{C\subset\mathcal{B}_\Lambda : b\in C} \frac{1}{|C|} f(C) \equiv \sum_{b\in\mathcal{B}_\Lambda} f_{b,\Lambda}$$

where we noted that in the second sum each $C \subset \mathcal{B}_\Lambda$ is counted $|C|$ times. Then

$$|f_{b,\Lambda}| \leq \sum_{n=1}^{\infty} \frac{1}{n}(2\epsilon)^n S_n$$

where S_n is the number of connected sets $C, |C| = n$, containing a given bond b. By Lemma 4.4. $|S_n| \leq C^n$. Thus, for ϵ small $|f_{b,\Lambda}| < C\epsilon < \infty$ uniformly in Λ and free energy $-\frac{1}{|\beta\Lambda|} \log Z_\Lambda$ is uniformly bounded.

Actually, one easily proves that

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} -\frac{1}{\beta|\Lambda|} \log Z_\Lambda = -d/\beta \lim_{\Lambda \nearrow \mathbb{Z}^d} f_{b,\Lambda} := f$$

independently of b (there are d bonds per site in $\mathbb{Z}^d$).

Also, we get :

$$\begin{aligned} \langle \sigma_x \sigma_y \rangle_\Lambda^{\text{free}} &= \frac{\sum_{\partial B = \{x,y\}} \rho(B) \exp\left(\sum_{C \cap B = \emptyset} f(C) \right)}{\exp(\sum_C f(C))} \\ &= \sum_{\partial B = \{x,y\}} \rho(B) \exp \sum_{C : C \cap B \neq \emptyset} f(C) \equiv \tilde{\rho}(B) \qquad (4.18) \end{aligned}$$

and $|\tilde{\rho}(B)| \leq \epsilon^{|B|} e^{C\epsilon|B|} \leq (2\epsilon)^{|B|}$. (4.18) is a version of the high temperature expansion. It can be further processed but we won't do it here.

4.5 Ising model-summary

One expects the following *phase diagram* for the Ising model.

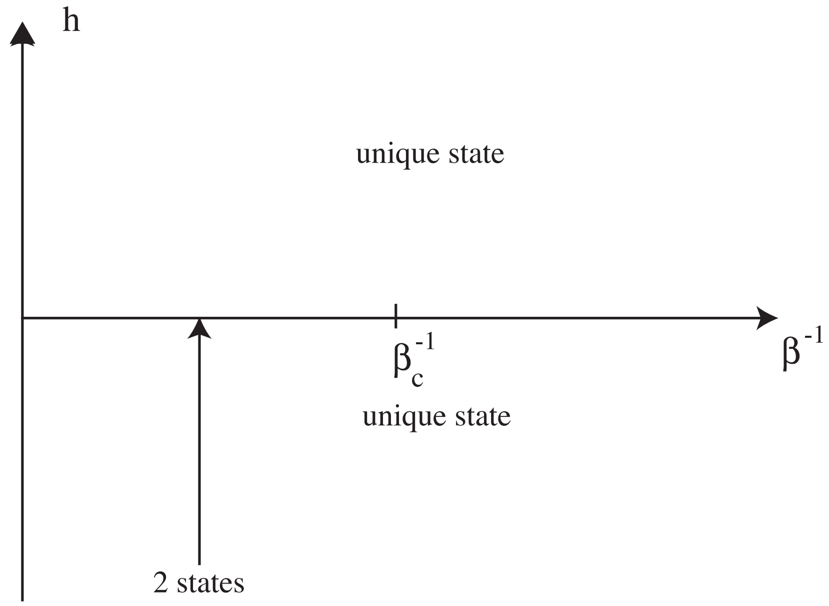
- If $h \neq 0$ the limit of $\langle - \rangle_\Lambda^{\bar{\sigma}}$ is independent of the boundary condition $\bar{\sigma}$.
- If $h = 0$ and $\beta \in (\beta_c, \infty)$ there are two translation invariant states states $\langle - \rangle^\pm$ and if $\beta \leq \beta_c$ the limit of $\langle - \rangle_\Lambda^{\bar{\sigma}}$ is independent of the boundary condition $\bar{\sigma}$.
- The correlation length $\xi(h, \beta) < \infty$ **except** at $h = 0, \beta = \beta_c$. I.e.

$$|\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle| \leq C(A, B) e^{-d(A,B)/\xi}.$$

- At $h = 0, \beta = \beta_c$ one expects

$$\langle \sigma_x \sigma_y \rangle \sim_{|x-y| \to \infty} \frac{c}{|x-y|^a}$$

At	$d = 2$,	$a = 1/4$	(exact)
	$d = 3$	$a = 1.0364$	(numerical, see wikipedia Ising critical exponents)
	$d \geq 4$	$a = d - 2$	(proven for $d > 4$)

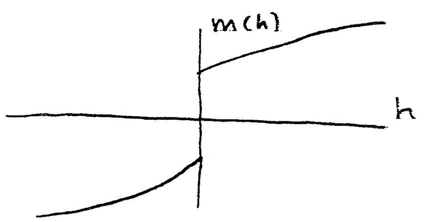


Because of the behaviour for $d \geq 4$, one usually writes $a = d - 2 + \eta$, and one calls η the "anomalous exponent". We will explain the exponent $d - 2$ below.

Rigorous results include : β_c is unique, $\xi < \infty$ up to β_c, $\Lambda \nearrow \mathbb{Z}^d$ limit exists for all β and h, and much else. The deepest is $\eta = 0$ for $d > 4$.

There are two different kinds of phase transitions in the Ising model:

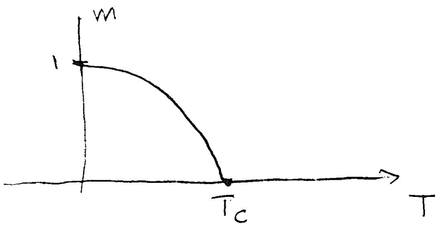
1. First order transition For $h \neq 0$ let $m(h, \beta) = \langle \sigma_x \rangle$ where the expectation is in the unique state (this state is translation invariant so $\langle \sigma_x \rangle$ is independent of x). If $\beta > \beta_c$ (low temperatures) then $\lim_{h \downarrow 0} m(h, \beta) = -\lim_{h \uparrow 0} m(h, \beta) > 0$ i.e. m is *discontinuous* at $h = 0$ (if $\beta \leq \beta_c, m$ is continuous).



Second order transition Let $h = 0$, consider

$$\langle \sigma_0 \rangle_\beta^+ \left(= \lim_{h \downarrow 0} \langle \sigma_0 \rangle_{\beta, h} \right) \equiv m(\beta).$$

Then $m(\beta) > 0$ for $\beta > \beta_c$, $m(\beta) = 0$ for $\beta < \beta_c$. What about $m(\beta_c)$? m is *continuous* at β_c: $\lim_{T \nearrow T_c} m(T) = 0 = m(T_c)$ (we use $T = \beta^{-1}$ as a parameter).



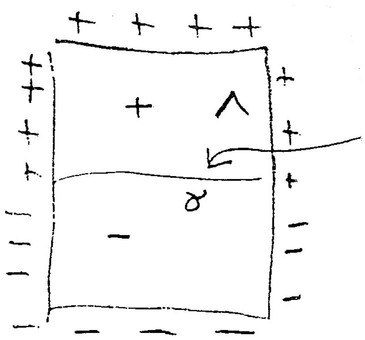
However, $\mid \frac{dm}{dT} \mid \to \infty$ as $T \nearrow T_c$. One has $m(T) \sim |T_c - T|^\beta$ where β (not equal to T^{-1} here!) is a critical exponent for the magnetization. In $d = 2$, $\beta = \frac{1}{8}$ (exact), in $d = 3$, $\beta = 0.3265$ (numerical), in $d \geq 4$ one expects $\beta = \frac{1}{2}$ (this is proven for $d \geq 5$). The continuity or discontinuity of the *order parameter* (m here) is refered to as second order and first order transition. Note also that ξ remains *finite* at a first order transition but becomes infinite at second order transition.

Remark 1. The states above are *translation invariant* i.e.

$$\langle \sigma_A \rangle = \langle \sigma_{A+x} \rangle \quad \forall A \subset \mathbb{Z}^d, \forall x \in \mathbb{Z}^d$$

where $A + x = \{a + x | a \in A\}$. Thus Ising model has two translation invariant states in low temperature and one in high temperature or nonzero field.

Ising model has also *non-translation invariant ones.* One considers so called Dobrushin boundary conditions, see the figure.

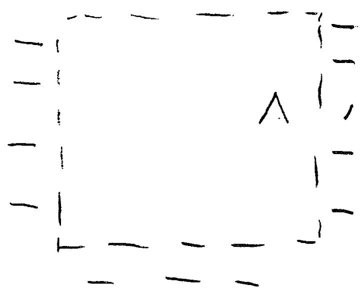


These force a contour (γ in the figure) joining the points where the boundary condition changes. It turns out that in $d = 2$ one does not get a new state this way [1]. In a box of side the contour fluctuates strongly: the expected deviation from the horizontal line is $L^{\frac{1}{2}}$ and an observer sitting at origin will see either a plus state or a minus state: in the $L \to \infty$ limit the state is a convex combination:

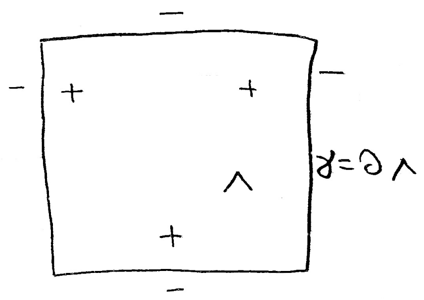
$$\langle - \rangle = p \langle - \rangle^+ + (1 - p) \langle - \rangle^-.$$

with some $p \in (0,1)$. However, in $d = 3$ at low temperatures the fluctuations are $\mathcal{O}(1)$ in L and one does get a new state [12] with an *interface* : $\langle \sigma_x \rangle > 0$ for x far above, $\langle \sigma_x \rangle < 0$ for x far below. However it is believed that there is a $\beta_r > \beta_c$, the *roughening transition* point so that the interface state disappears for $\beta < \beta_r$ due to large fluctuations of the interface.

Remark 2 Why is $h \neq 0$ unique ? Let e.g. $h > 0$ Let us try to construct – a state:



The weight of $\sigma = -1$ i.e. the configuration with no contours is $e^{-\beta(-\sum \sigma_x \sigma_y + h \sum \sigma_x)} = e^{\beta|\bar{\mathcal{B}}_\Lambda|} e^{-\beta h |\Lambda|}$. The weight of the configuration where we have a contour at the boundary i.e. $\gamma = \partial\Lambda$:



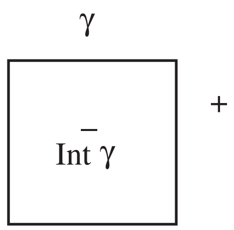
equals $e^{\beta|\bar{\mathcal{B}}_\Lambda|} e^{-2\beta|\partial\Lambda|}$. For Λ a $L-$box $|\partial\Lambda| \sim L^{d-1}, |\Lambda| \sim L^d$ so the contour is more probable! Hence, we expect that the $\Lambda \to \mathbb{Z}^d$ limit is the same as that of the $+$ boundary condition state. One can make a proof along these lines, see [20].

Remark 3 What about h which is not constant :

$$\beta\mathcal{H} = \beta(-\sum \sigma_x \sigma_y + \sum_{x \in \Lambda} h_x \sigma_x).$$

Let us consider h_x's *random* : let each h_x be a random variable with zero mean : $\overline{h_x} = 0$ and different h_x, h_y are independent, identically distributed. Thus we pick a configuration of $h = \{h_x\}$ randomly from such ensemble and consider the Gibbs state with this h. What do we expect ? Consider low temperature and say $+$ boundary conditions. A contour γ costs now an energy and it weight is

$$\exp(-\beta[2|\gamma| + 2 \sum_{x \in \mathrm{Int}\gamma} h_x])$$



Let γ be an L-cube, so $|\gamma| \propto L^{d-1}$, how about $\sum_{x \in \mathrm{Int}\,\gamma} h_x$? This is a sum of L^d independent random variables, therefore of size $\propto L^{d/2}$ since its variance is

$$\mathbb{E}(\sum h_x)^2 = \sum_{x,y} \mathbb{E}h_x h_y = \sum_x \mathbb{E}h_x^2 + \sum_{x \neq y} \mathbb{E}h_x \mathbb{E}h_y = L^d \mathbb{E}h^2$$

Thus, if $d - 1 > \frac{d}{2}$ i.e. if $d > 2$, the magnetic field is unlikely to suppress contours and we expect 2 states. For $d \leq 2$ it seems to have a chance.

Theorem 4.8 *For β large,*
bbEh^2 small, $d \geq 3$ our model has 2 states: one with $\langle \sigma_x \rangle > 0$ and one with < 0. For $d \leq 2$ there is only one Gibbs state (this holds with Probability one in h).

Proof See [5] for $d \geq 3$ and [2] for $d \leq 2$.

4.6 Bounded spin models

Let us say a few words about the general formalism of classical statistical mechanics on a lattice.

1. The "spins" σ_x can take more general values then ± 1. By a *bounded spin model* one means $\sigma_x \in \mathcal{M}$ where $\mathcal{M}$ is "bounded", generally a compact metric space. Let us list a few examples :

a) **q-state Potts model.** $\sigma_x \in \{1, 2, \cdots, q\}$ and the Hamiltonian with free boundary conditions is

$$\mathcal{H}(\sigma) = - \sum_{\{x,y\} \in \mathcal{B}_\Lambda} \delta_{\sigma_x, \sigma_y}$$

(δ is Kronecker delta). Thus to minimize energy nearest neighbours want to be the same. For $q = 2$ this is Ising model: $\delta_{\sigma\sigma'} = \frac{1}{2}(1 + \sigma\sigma')$. The model has S_q symmetry (= group of permutations on q objects): Let $\pi \in S_n$ (so π is a bijection from $\{1, \cdots, q\}$ into itself), and $T_\pi : \Omega_\Lambda \to \Omega_\Lambda : (T_\pi \sigma)_x = \pi \sigma_x$. Then $\mathcal{H}(T_\pi \sigma) = \mathcal{H}(\sigma)$. At low temperatures we have q phases:

Exercise. Devise a Peierls argument to construct a state $\langle - \rangle^p$ with boundary condition $\bar{\sigma}_x = p$ so that as $\beta \to \infty$, $\mathbb{P}(\sigma_0 = p) \to 1$.

At high temperatures there is a unique infinite volume limit which can be constructed with a high temperature expansion. We can copy our arguments for the Ising model by writing

$$e^{-\beta\delta_{\sigma_x,\sigma_y}} = e^{-\beta}(1+f_{xy})$$

where $f_{xy} = e^{\beta(1-\delta_{\sigma_x,\sigma_y})} - 1 = \delta_{\sigma_x,\sigma_y}(e^\beta - 1)$. Then

$$e^{\beta|\mathcal{B}_\Lambda|}e^{-\beta\mathcal{H}_\Lambda(\sigma)} = \prod_{\{x,y\}\in\mathcal{B}_\Lambda}(1+\delta_{\sigma_x,\sigma_y}(e^\beta-1)) = \sum_{B\subset\mathcal{B}_\Lambda}\prod_{\{x,y\}\in B}\delta_{\sigma_x,\sigma_y}(e^\beta-1) \tag{4.19}$$

We have $0 \le f_{xy} \le \beta(1+f_{xy})$ and the thermodynamic limit and decay of correlations can be done as in the Ising case

Exercise. Check this!

The high temperature expansion has actually an interesting structure. Let us consider the partition function i.e. (up to a trivial multiplicative factor) the sum of (4.19) over σ. The set of bonds B defines a graph G with vertex set $v(G) = \Lambda$ and edge set $e(G) = B$. Let G_α be the connected components of G. The Kronecker deltas force all the spins to be equal in each $v(G_\alpha)$ and the spin sum factorizes over the $v(G_\alpha)$. We obtain then

$$\sum_\sigma e^{\beta|\mathcal{B}_\Lambda|}e^{-\beta\mathcal{H}_\Lambda(\sigma)} = \sum_G (e^\beta-1)^{|e(G)|}q^{n(G)} \tag{4.20}$$

where $n(G)$ is the number of connected components of G (note that each $x \in \Lambda$ not belonging to any of the bonds in B is a connected component G_α with $v(G_\alpha) = \{x\}$ and $e(G_\alpha) = \emptyset$). Eq. (4.20) is called the Fortyuin-Kasteleyn representation for the partition function of Potts model. It has an interpretation of a *percolation model*, see [**?**].

As the Ising model Potts model as a critical (inverse) temperature β_c separating the high temperature phase from the low temperature phase. In $d = 2$ and $q \le 4$ the correlation length ξ is infinite at β_c and finite elsewhere and the transition at β_c is second order. For $q > 4$ the transition is *1st order* and $\xi < \infty$. This should be the case in $d > 2$ and $q > 2$ as well. This is proven for q large enough in all dimensions, see [27].

b) $O(N)$**-models**. Here $\sigma_x \in S^{N-1}$ = sphere in $\mathbb{R}^N$ i.e. σ_x is a vector $\vec{\sigma}_x \in \mathbb{R}^N$, $\|\vec{\sigma}_x\|^2 = 1$. We take

$$\mathcal{H} = -\sum_{|x-y|=1}\vec{\sigma}_x.\vec{\sigma}_y.$$

This has $O(N)$ symmetry: Let $R \in O(N)$ i.e. $R = N \times N$ orthogonal matrix, $R^TR = \mathbb{1}$. Then $\mathcal{H}(R\vec{\sigma}\,) = \mathcal{H}(\vec{\sigma}\,)$ where $(R\vec{\sigma}\,)_x = R\vec{\sigma}_x$. The $N = 2$ case is called the XY-model, $N = 3$ the classical Heisenberg model. These are very interesting. First, there is no symmetry breaking when $d = 2$: there are no states with $\langle\vec{\sigma}_x\rangle \neq 0$. This is called the *Mermin-Wagner theorem* [30, 14]. We'll prove it in section 5 below.

In $d \geq 3$ the $O(N)$ models have symmetry breaking :

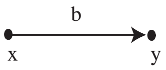
$$\text{for } \beta \leq \beta_c,\ \langle \vec{\sigma}_x \rangle = 0, \quad \text{for } \beta > \beta_c\ \ \langle \vec{\sigma}_x \rangle = m(\beta)\hat{n}, \quad \hat{n} \in S^{N-1}.$$

There are infinitely many low-temperature states, parametrized by unit vectors $\hat{n}$ [18].

For $d = 2$, $N = 2$ and $N \geq 3$ behave qualitatively differently. For $N = 2$ (XY-model or *plane rotator*), there is a critical temperature β_c where ξ becomes ∞: For $\beta < \beta_c$ $\xi(\beta) < \infty$ and for $\beta \geq \beta_c$, $\xi(\beta) = \infty$. The correlation function has a peculiar decay if $\beta > \beta_c$: $\langle \sigma_x \sigma_y \rangle \sim |x - y|^{-a(\beta)}$ where $a(\beta)$ depends on β [26, 19].

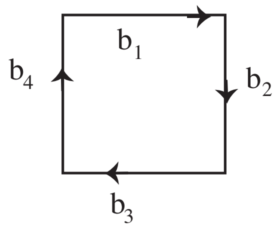
For $d = 2, N \geq 3$, one expects $\xi(\beta)$ diverge as $e^{c\beta}$ when $\beta \to \infty$ (which is proven as a lower bound).

c) **Gauge theories** Consider bonds b with *orientation* $b = (x, y) \neq (y, x)$ and denote $(y, x) = b^{-1}$.



The spins are indexed by oriented bonds b and take values $\sigma_b \in SU(N)$ ($N \times N$ unitary matrices, $\det \sigma_b = 1$). Moreover we let $\sigma_{b^{-1}} = (\sigma_b)^{-1}$. Hence the state space is $\Omega_\Lambda = (SU(N))^{\mathcal{B}_\Lambda}$.

Let p be a plaquette and choose arbitarily an orientation :



Put :

$$\begin{aligned} S_p = \text{Re Tr } \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} &= \frac{1}{2}(\text{Tr } \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} + \overline{\text{Tr } \sigma_{b_1} \cdots \sigma_{b_4}}) \\ &= \frac{1}{2}(\text{Tr } \sigma_{b_1} \cdots \sigma_{b_4} + \text{Tr } (\sigma_{b_1} \cdots \sigma_{b_4})^{-1}) \end{aligned}$$

(since $\overline{\text{Tr } U} = \text{Tr } U^{+} = \text{Tr } U^{-1}$ for $U \in SU(N)$) and

$$\mathcal{H}_\Lambda = \sum_{p \subset \Lambda} S_p.$$

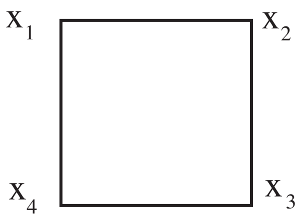
This Hamiltonian has a huge symmetry, the *gauge symmetry* : Let $G_\Lambda = \times_{x \in \Lambda} SU(N)$ i.e. $g \in G_\Lambda$ is a map $\Lambda \ni x \to g_x \in SU(N)$ i.e. at each point $x \in \Lambda$ choose a matrix $g_x \in SU(N)$. G_Λ is the *gauge group* in Λ. It acts on spins as follows. For $\sigma \in \Omega_\Lambda, g \in G_\Lambda$, let

$$(U(g)\sigma)_b = g_x \sigma_b g_y^{-1}$$

where $b = (x, y)$. Then we have

$$S_p(U(g)\sigma) = \text{ReTr } [g_{x_1}\sigma_{b_1} g_{x_2}^{-1} g_{x_2} \sigma_{b_2} g_{x_3}^{-1} \cdots g_{x_4}^{-1} g_{x_4} \sigma_{b_4} g_{x_1}^{-1}] = S_p(\sigma)$$

so $\mathcal{H}_\Lambda(U(g)\sigma) = \mathcal{H}_\Lambda(\sigma)$.



This is a big symmetry and in all dimensions it remains *unbroken*. The reason being that a local symmetry such as $U(g), g \in \sigma_\Lambda$ does not change the Hamiltonian, even with arbitrary boundary conditions, provided those b.c. are on the boundary of Λ', with Λ strictly included in Λ'. Then, the fact that the gauge symmetry remains unbroken is an immediate consequence of the DLR equation (5.4) below.

2. The last example had a 4-spin interaction. Let us now generalize to give the general formalism of bounded spin models. Thus, let $\mathcal{M}$ be a compact metric space, define as before the infinite configuration space $\Omega = \mathcal{M}^{\mathbb{Z}^d}$, and, in a finite volume $\Omega_\Lambda = \mathcal{M}^\Lambda$. Ω is still a compact metric space (by Tychonov), with metric

$$D(\sigma, \sigma') = \sum_{x \in \mathbb{Z}^d} 2^{-|x|} d(\sigma_x, \sigma'_x)$$

with d the metric on $\mathcal{M}$. The measures on Ω we are interested in are characterized by two data, the Hamiltonian and an a priori measure.

Hamiltonian. We want to have general interactions:

Definition 4.9 A potential Φ is a collection of continuous functions

$$\Phi_X : \mathcal{M}^X \to \mathbb{R} \quad \text{for } X \subset \mathbb{Z}^d, \quad |X| < \infty.$$

Example In the Ising model we have $\Phi_X(\sigma) = \begin{cases} \sigma_x \sigma_y & \text{if } X = \{x, y\},\ |x - y| = 1 \\ 0 & \text{otherwise} \end{cases}$

We say the potential Φ is *translation invariant* if $\Phi_X = \Phi_{X+a}$ for all $a \in \mathbb{Z}^d$ where $X + a = \{x + a | x \in X\}$. More explicitly this means $\Phi_X(\tau_a \sigma) = \Phi_{X+a}(\sigma)$ where $(\tau_a \sigma)_x = \sigma_{x+a}$. We will assume our potentials are translation invariant.

Given a potential the *Hamiltonian* in Λ , $|\Lambda| < \infty$ is given by

$$\mathcal{H}_\Lambda(\sigma) = \sum_{X \subset \Lambda} \Phi_X(\sigma).$$

Actually this is the *free boundary condition* Hamiltonian. (Note that e.g. in gauge theories we rather consider spins on bonds, not sites so these definitions should be changed appropriately).

A priori measure : Let ν be a Borel probability measure on $\mathcal{M}$. Thus, in Ising model we had $\nu(-1) = \nu(1) = \frac{1}{2}$, in $\mathcal{O}(N)$-model ν is the uniform measure on S^{N-1} and in $SU(N)$-Gauge theories : ν is the Haar measure on $SU(N)$. The *a priori measure* in $\mathcal{M}_\Lambda$ is the product measure

$$d\nu_\Lambda(\sigma) = \prod_{x\in\Lambda} d\nu(\sigma_x) \tag{4.21}$$

We can the define

Definition 4.10 The Gibbs measure in volume Λ (with free boundary conditions) is

$$d\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda} e^{-\beta\mathcal{H}_\Lambda(\sigma)} d\nu_\Lambda(\sigma) \tag{4.22}$$

For more general boundary conditions we pick as before $\bar\sigma \in \Omega$ and let, for $\sigma \in \Omega_\Lambda$,

$$\mathcal{H}^{\bar\sigma}_\Lambda(\sigma) = \sum_{X:X\cap\Lambda\neq\emptyset} \Phi_X(\sigma \vee \bar\sigma) \tag{4.23}$$

where use the notation : $\sigma \vee \bar\sigma \in \Omega : (\sigma\vee\bar\sigma)_x = \begin{cases} \sigma_x & x \in \Lambda \\ \bar\sigma_x & x \in \Lambda^c \end{cases}$. The corresponding Gibbs measure is then

$$d\mu^{\bar\sigma}_\Lambda(\sigma) = \frac{1}{Z^{\bar\sigma}_\Lambda} e^{-\beta\mathcal{H}^{\bar\sigma}_\Lambda(\sigma)} d\nu_\Lambda(\sigma) \tag{4.24}$$

Note that (4.23) is an infinite sum so we need to address the issue of convergence.

Definition 4.11 Φ is *finite range* if there exists $R < \infty$ such that $\Phi_X = 0$ for all X such that diameter of X is $d(X) > R$. Here $d(X) := \max_{x,y\in X} |x-y|$. For finite range Φ, (4.23) is finite trivially, since continuity of Φ_X in σ implies

$$\|\Phi_X\| = \sup_{\sigma\in\mathcal{M}^X} |\Phi_X(\sigma)| < \infty$$

by compactness of $\mathcal{M}^X$.

Exercise. Let $d = 1$ and Φ finite range R. Show that for all β the correlation functions $\langle F(\sigma)\rangle^{\bar\sigma}_\Lambda$ have thermodynamic limit which is independent on $\bar\sigma$. Here F depends on finitely many spins. Moreover show the correlations decay exponentially. Hint: proceed as in 1d Ising model by introducing a transfer matrix $T_{\sigma,\sigma'}$ where $\sigma, \sigma' \in \mathcal{M}^R$. For simplicity you may take $\mathcal{M}$ a finite set.

Sometimes finite range is not general enough. There are various classes of potentials, that allow for the $\Lambda \nearrow \mathbb{Z}^d$ limit of the free energy, the convergence of high temperature expansions etc. Let us denote

$$\|\Phi\| = \sup_{\sigma\in\Omega_X} |\Phi_X(\sigma)|.$$

Our first class of potentials, $\mathcal{B}_0$, consists of Φ such that

$$\|\Phi\|_0 = \sum_{\substack{X \subset \mathbb{Z}^d \\ 0 \in X}} \|\Phi_X\| < \infty \quad \text{for} \quad \Phi \in \mathcal{B}_0$$

i.e. the interaction energy of σ_0 with all other spins is finite. Note that translation invariance implies $\|\Phi\|_0 = \sum_{\substack{X\subset\mathbb{Z}^d \\ y\in X}} \|\Phi_X\|$ for any $y \in \mathbb{Z}^d$.

$\mathcal{B}_0$ is a vector space, $\| \ \|$ is a norm and $\mathcal{B}_0$ is *complete* in this norm, i.e. if $\|_0\Phi^{(n)} - \Phi^{(m)}\|_0 \underset{n,m\to\infty}{\longrightarrow} 0$ then $\exists\Phi \in \mathcal{B}_0$ such that $\|\Phi - \Phi^{(n)}\|_0 \to 0$ as $n \to \infty$. So $\mathcal{B}_0$ is a Banach space.

The Gibbs measure is well defined:

Lemma 4.12 Let $\Phi \in \mathcal{B}_0$. Then

$$|\mathcal{H}^{\bar{\sigma}}_\Lambda(\sigma)| \leq |\Lambda| \ \|\Phi\|_0.$$

Proof We have

$$\Big|\sum_{X:X\cap\Lambda\neq\emptyset} \Phi_X(\sigma)\Big| = \Big|\sum_{y\in\Lambda} \sum_{\substack{X \subset \mathbb{Z}^d \\ y \in X}} \frac{1}{|X\cap\Lambda|}\Phi_X(\sigma)\Big| \leq \sum_{y\in\Lambda} \sum_{\substack{X \subset \mathbb{Z}^d \\ y \in X}} \|\Phi_X\| = |\Lambda| \ \|\Phi\|_0 \quad \square$$

We have then

Theorem 4.13 *Let* $\Phi \in \mathcal{B}_0$. *Then the free energy*

$$F = -\lim_{L\to\infty} \frac{1}{L^d\beta} \log Z^{\bar{\sigma}}_{\Lambda_L} \qquad (\Lambda_L = L - \text{box})$$

exists and is independent of $\bar{\sigma}$.

Proof. a) Independence of $\bar{\sigma}$. Let $\epsilon > 0$. Consider

$$|\mathcal{H}^{\bar{\sigma}}_\Lambda(\sigma) - \mathcal{H}^{\bar{\sigma}'}_\Lambda(\sigma)| \leq \sum_{X\cap\Lambda\neq\emptyset, X\cap\Lambda^c\neq\emptyset} |\Phi_X(\sigma\vee\bar{\sigma}) - \Phi_X(\sigma\vee\bar{\sigma}')|$$

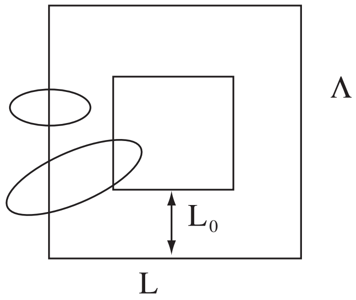
(since terms with $X \subset \Lambda$ cancel)

$$= \sum_{y\in\Lambda} \sum_{\substack{X \text{as above} \\ y \in X}} \frac{1}{|X\cap\Lambda|} |\Phi_X(\sigma\vee\bar{\sigma}) - \Phi_X(\sigma\vee\bar{\sigma}')|$$

$$\leq 2\sum_{y\in\Lambda} \sum_{\substack{X \text{as above} \\ y \in X}} \frac{1}{|X\cap\Lambda|} \|\Phi_X\| \qquad (4.25)$$

Divide the sum over y into two parts:

1° dist $(y, \Lambda^c) \le L_0$
2° dist $(y, \Lambda^c) \ge L_0$



Then the contribution of 1° to (4.25) is bounded by

$$CL_0 L^{d-1} \|\Phi\|_0 \le \tfrac{1}{2}\epsilon L^d \tag{4.26}$$

if we take $CL_0/L \le \frac{1}{2}\epsilon$.

The second sum with the constraint 2° is bounded by

$$2L^d \sum_{X:0\in X,\ d(X)\ge L_0} \|\Phi_X\| \equiv L^d \delta(L_0).$$

Since the sum $\sum_{X:0\in X} \|\Phi_X\|$ converges, $\delta(L_0) \to 0$ as $L_0 \to \infty$. Hence pick L_0 so that $\delta(L_0) \le \frac{1}{2}\epsilon$ and then L so that (4.26) holds. Then we have

$$(4.25) \le \epsilon L^d$$

and thus

$$e^{-\epsilon L^d} Z_\Lambda^{\bar\sigma'} \le Z_\Lambda^{\bar\sigma} \le Z_\Lambda^{\bar\sigma'} e^{\epsilon L^d}.$$

So, for $L > L(\epsilon)$

$$L^{-d} | \log Z_{\Lambda_L}^{\bar\sigma'} - \log Z_{\Lambda_L}^{\bar\sigma} | \le \epsilon.$$

Since this holds for all $\epsilon > 0$

$$\lim_{L\to\infty} L^{-d} | \log Z_{\Lambda_L}^{\bar\sigma'} - \log Z_{\Lambda_L}^{\bar\sigma} | = 0.$$

Obviously we may replace $Z_{\Lambda_L}^{\bar\sigma'}$ by the free boundary condition version above.

b) **Existence.** It suffices to consider the free boundary condition theory. Let $\epsilon > 0$. Then $\exists R$ such that $\|\Phi - \Phi^{(R)}\| \le \epsilon$ where

$$\Phi_X^{(R)} = \begin{cases} \Phi_X & \text{diam } (X) < R \\ 0 & \text{diam } (X) \ge R \end{cases}.$$

So

$$|\mathcal{H}_{\Lambda,\Phi} - \mathcal{H}_{\Lambda,\Phi^{(R)}}| \le \epsilon |\Lambda|$$

and thus

$$\left| \frac{1}{|\Lambda|} \log Z_{\Lambda,\Phi} - \frac{1}{|\Lambda|} \log Z_{\Lambda,\Phi^{(R)}} \right| \le \epsilon.$$

Thus, it suffices to consider Φ of finite range, say R. Let $L_2 > L_1 > R$. We will compare $L_2^{-d} \log Z_{\Lambda_{L_2}}$ to $L_1^{-d} \log Z_{\Lambda_{L_1}}$. Write $L_2 = nL_1 + L$ with $L < L_1$ and $n \in \mathbb{Z}$. Then Λ_{L_2} is a union of n^d disjoint L_1-boxes $\Lambda^{(i)}$ with $i = 1, \ldots, n^d$ and a region Λ of volume $|\Lambda| \le CL_1 L_2^{d-1}$. Let $\sigma|_{\Lambda^{(i)}} \equiv \sigma^{(i)}$. We get

$$\Big| \mathcal{H}_{\Lambda_L}(\sigma) - \sum_{i=1}^{n^d} \mathcal{H}_{\Lambda^{(i)}}(\sigma^{(i)}) \Big| \le CRL_1^{d-1} n^d \|\Phi\|_0 + CL_1 L_2^{d-1} \|\Phi\|_0 \le C' L_2^d (L_1^{-1} + L_1 L_2^{-1})$$

where $\mathcal{H}_{\Lambda^{(i)}}$ is the free boundary condition Hamiltonian. The first term in the middle expression bounds the contribution of Φ_X with $X \cap \Lambda^{(i)} \neq \emptyset$, $X \cap (\Lambda^{(i)})^c \neq 0$ for some i and the second one the Φ_X with $X \cap \Lambda \neq \emptyset$. Thus,

$$e^{-C' L_2^d (L_1^{-1} + L_1 L_2^{-1})} \le \frac{\int e^{-\beta \sum \mathcal{H}_{\Lambda^{(i)}}(\sigma^{(i)})} d\nu_{\Lambda_{L_2}}}{\int e^{-\beta \mathcal{H}_{\Lambda_L}(\sigma)} d\nu_{\Lambda_{L_2}}} \le e^{C' L_2^d (L_1^{-1} + L_1 L_2^{-1})}. \tag{4.27}$$

Now using $d\nu_{\Lambda_{L_2}}(\sigma) = d\nu_\Lambda(\sigma_\Lambda) \prod_{i=1}^{n^d} d\nu_{\Lambda_{L^{(i)}}}(\sigma^{(i)})$ we get

$$\int e^{-\beta \sum \mathcal{H}_{\Lambda^{(i)}}(\sigma^{(i)})} d\nu_{\Lambda_{L_2}} = (Z_{\Lambda_{L_1}})^{n^d}$$

we get

$$\Big| \frac{1}{L_2^d} \log Z_{\Lambda_{L_2}} - \frac{n^d}{L_2^d} \log Z_{\Lambda_{L_1}} \Big| \le C' (L_1^{-1} + L_1 L_2^{-1})$$

Since $|\frac{n}{L_2} - \frac{1}{L_1}| \le \frac{1}{L_2}$ this implies

$$\Big| \frac{1}{L_2^d} \log Z_{\Lambda_{L_2}} - \frac{1}{L_1^d} \log Z_{\Lambda_{L_1}} \Big| \le C'' (L_1^{-1} + L_1 L_2^{-1})$$

and therefore taking $L_2 \to \infty$

$$\Big| \limsup \frac{1}{L_2^d} \log Z_{\Lambda_{L_2}} - \frac{1}{L_1^d} \log Z_{\Lambda_{L_1}} \Big| \le C'' L_1^{-1}$$

and then $L_1 \to \infty$

$$\big| \limsup \frac{1}{L_2^d} \log Z_{\Lambda_{L_2}} - \liminf \frac{1}{L_1^d} \log Z_{\Lambda_{L_1}} \Big| = 0$$

proving the claim. □

Remarks. 1. Idea in both parts was that *boundary energy/volume* $\to 0$ as $L \to \infty$. Note also that we have not proved that $\langle - \rangle^{\bar\sigma}$ is $\bar\sigma$-independent nor that these states converge. We have only shown that the free energy is independent of the boundary conditions.

2. The space $\mathcal{B}_0$ includes Ising type systems

$$\mathcal{H} = \sum_{x,y \in \Lambda} J_{xy} \sigma_x \sigma_y \tag{4.28}$$

provided J_{xy} has enough decay as $|x-y| \to \infty$: e.g. $|J_{xy}| \leq \frac{C}{[|x-y|+1]^{d+\epsilon}}$ will do.
There is a very important system where $\Phi \notin \mathcal{B}_0$: the **Coulomb gas**. It is like (4.28), σ_x = charge at x i.e. + or − charges and

$$\mathcal{H}_\Lambda(\sigma) = \sum_{\substack{x,\, y \,\in\, \Lambda \\ x \,\neq\, y}} \frac{1}{|x-y|}\sigma_x \sigma_y.$$

Hence $\Phi_X = \begin{cases} \frac{1}{|x-y|} \text{ if } X = (x,y) \\ 0 \text{ otherwise} \end{cases}$ and $\sum_{0\in X} \Phi_X = \sum_{y\neq 0} \frac{1}{|y|} = \infty$. Nevertheless, the $\Lambda \nearrow \mathbb{Z}^d$ limit exists. The secret is in signs : note that $e^{-\beta\mathcal{H}}$ gets very small contribution from $\sigma_x = +1 \;\; \forall x$ or $\sigma_x = -1 \;\; \forall x$. The nonzero contribution comes from alternating configurations (neutral ones : σ is charge). See Simon [40], p. 121.

Let us next consider **High-temperature uniqueness**. Let us proceed as in the Ising model case. We first write

$$e^{-\beta\mathcal{H}_\Lambda} = \prod_{X\subset\Lambda} e^{-\beta\Phi_X} = \prod_{X\subset\Lambda} e^{-\beta\|\Phi_X\|} \prod_{X\subset\Lambda} e^{\beta\tilde{\Phi}_X} \tag{4.29}$$

where we note

$$\tilde{\Phi}_X = \|\Phi_X\| - \Phi_X \geq 0.$$

Disregarding the σ-independent constant we then expand

$$\prod_{X\subset\Lambda} e^{\beta\tilde{\Phi}_X} = \prod_{X\subset\Lambda} (1+f_X) = \sum_{\{X_\alpha\}} \prod_\alpha f_{X_\alpha}$$

where the sum is over all families $\{X_\alpha\}$ of subsets of Λ and $f_X = e^{\beta\tilde{\Phi}_X} - 1$ satisfies

$$0 \leq f_X \leq \beta\tilde{\Phi}_X(1+f_X).$$

We can now proceed as in the proof of Theorem 4.1. to study the $\bar{\sigma}$ or Λ dependence of $\langle F_A(\sigma)\rangle_\Lambda^{\bar{\sigma}}$ where F_A is a continuous function on $\mathcal{M}^A$. For instance eq. [?] will be replaced essentially by

$$\sum_{\mathcal{P}} \prod_{i=1}^{n} \beta\|\Phi_{X_i}\|$$

where $\mathcal{P} = \{X_1, \ldots, X_n\}$ is a connected path of sets from A to Λ^c i.e. $A \cap X_1 \neq \emptyset$, $X_i \cap X_{i+1} \neq \emptyset$ and $X_n \cap \Lambda^c \neq \emptyset$. To control this sum we need to assume a bit more from the potential, namely that

$$\|\Phi\|_1 := \sum_{0\in X} |X|\|\Phi_X\| < \infty.$$

Exercise. Show that this condition allows to bound the above sum if β is small enough and show it tends to zero as $d(A, \Lambda^c) \to \infty$.

Indeed one can prove

Proposition 4.14. Let $\|\Phi\|_1 < \infty$. Then, for β small enough the correlation functions $\langle F_A(\sigma) \rangle_\Lambda$ converge as $\Lambda \uparrow \mathbb{Z}^d$ to a $\bar{\sigma}$-independent limit and

$$|\langle F_A G_B \rangle - \langle F_A \rangle \langle B_B \rangle| \to 0$$

as dist $(A, B) \to \infty$.

See [5] for more details. Assuming more on the decay of Φ_X allows one to get information about the decay of correlations. Eg. if we assume that for some $\alpha > 0$

$$\|\Phi\|_2 := \sum_{0 \in X} e^{\alpha |X|} \|\Phi_X\| < \infty$$

then the correlation function deals exponentially with distance

$$|\langle F_A G_B \rangle - \langle F_A \rangle \langle B_B \rangle \leq C e^{-\alpha d(A,B)}$$

with $0 < \alpha' < \alpha$.

5 Gibbs States and DLR Equations

Finally, let us define what one means by Gibbs states in the general framework. Recall the discussion in Section 2 of limits of finite volume correlations. That can be repeated in the general framework. Thus if we knew that for all finite $X \subset \mathbb{Z}^d$ and all $f \in C(\mathcal{M}^X)$

$$\ell(f) = \lim_{n \to \infty} \langle f \rangle^{\bar{\sigma}}_{\Lambda_n}$$

exists where Λ_n is some sequence of boxes converging to $\mathbb{Z}^d$. Then we conclude as before that $\ell(f) = \int f d\mu$ for some Borel measure on $\mathcal{M}^{\mathbb{Z}^d}$. We now want to characterize these limit measures as Gibbs measures.

Consider first the finite volume Gibbs measure $d\mu^{\bar{\sigma}}_{\Lambda_n}(\sigma)$ defined in (4.24) for a potential $\Phi \in \mathcal{B}_0$:

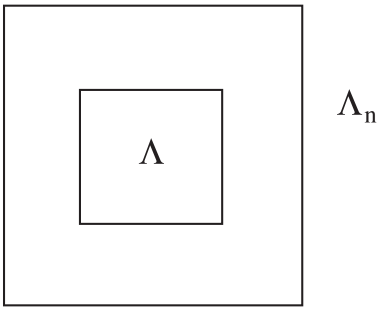
$$d\mu^{\bar{\sigma}}_{\Lambda_n}(\sigma) = \frac{1}{Z^{\bar{\sigma}}_\Lambda} e^{-\beta \mathcal{H}^{\bar{\sigma}}_{\Lambda_n}(\sigma)} d\nu_{\Lambda_n}(\sigma) \tag{5.1}$$

Let $\Lambda \subset \Lambda_n$ and $f \in C(\Omega_\Lambda)$. We use in this section the notation σ_X to denote the configuration in X i.e. $\sigma_X := \{\sigma_x | x \in X\}$. Also, for brevity let $\bar{\sigma}$ denote $\bar{\sigma}_{\Lambda_n^c}$ i.e we have $\bar{\sigma}$ configuration outside Λ_n. Then we have the

Lemma. $\mathbb{E}^{\bar\sigma}_{\Lambda_n}(f) = \mathbb{E}^{\bar\sigma}_{\Lambda_n}(\mathbb{E}^{\sigma_{\Lambda^c}\vee\bar\sigma}_{\Lambda}(f))$.

Proof. We have

$$\mathbb{E}^{\bar\sigma}_{\Lambda_n}(f) = \int f d\mu^{\bar\sigma}_{\Lambda_n} = \frac{1}{Z^{\bar\sigma}_{\Lambda_n}} \int f(\sigma_\Lambda) e^{-\beta\mathcal{H}^{\bar\sigma}_{\Lambda_n}(\sigma)} d\nu_{\Lambda_n}(\sigma) \tag{5.2}$$



Write

$$\mathcal{H}^{\bar\sigma}_{\Lambda_n}(\sigma) = \sum_{X\cap\Lambda_n\neq\emptyset} \Phi_X(\sigma\vee\bar\sigma) = \mathcal{H}^{\sigma\vee\bar\sigma}_{\Lambda}(\sigma) + \sum_{\substack{X\subset\Lambda^c \\ X\cap\Lambda_n\neq\emptyset}} \Phi_X(\sigma\vee\bar\sigma) \equiv \mathcal{H}^{\sigma\vee\bar\sigma}_{\Lambda} + \tilde{\mathcal{H}}$$

where $\tilde{\mathcal{H}}$ is independent of σ_Λ. Thus

$$\begin{aligned}
\mathbb{E}^{\bar\sigma}_{\Lambda_n}(f) &= \int d\nu_\Lambda(\sigma) f(\sigma_\Lambda) \int d\nu_{\Lambda_n\setminus\Lambda}(\sigma) \frac{1}{Z^{\bar\sigma}_{\Lambda_n}} e^{-\beta(\mathcal{H}^{\sigma\vee\bar\sigma}_{\Lambda}+\tilde{\mathcal{H}})} \\
&= \int d\nu_{\Lambda_n\setminus\Lambda}(\sigma) (\int d\mu^{\sigma\vee\bar\sigma}_{\Lambda}(\sigma) f(\sigma_\Lambda)) \frac{1}{Z^{\bar\sigma}_{\Lambda_n}} Z^{\sigma\vee\bar\sigma}_{\Lambda} e^{-\beta\tilde{\mathcal{H}}}.
\end{aligned} \tag{5.3}$$

But

$$Z^{\sigma\vee\bar\sigma}_{\Lambda} e^{-\beta\tilde{\mathcal{H}}} = \int d\nu_\Lambda(\sigma) e^{-\beta\mathcal{H}^{\sigma\vee\bar\sigma}_{\Lambda}} e^{-\beta\tilde{\mathcal{H}}} = \int d\nu_\Lambda(\sigma) e^{-\beta\mathcal{H}^{\bar\sigma}_{\Lambda_n}(\sigma)}$$

so (5.3) becomes

$$\mathbb{E}^{\bar\sigma}_{\Lambda_n}(f) = \int d\mu^{\bar\sigma}_{\Lambda_n}(\sigma) \int d\mu^{\sigma\vee\bar\sigma}_{\Lambda}(\sigma') f(\sigma'_\Lambda)$$

as claimed. $\square$

Another way to say this is the following. The measure $\mu^{\bar\sigma}_{\Lambda_n}$ projects to a measure $(\mu^{\bar\sigma}_{\Lambda_n})_\Lambda$ on $\mathcal{M}^\Lambda$ by the formula

$$\int f d\mu^{\bar\sigma}_{\Lambda_n} = \int f d(\mu^{\bar\sigma}_{\Lambda_n})_\Lambda$$

for all $f \in C(\mathcal{M}^\Lambda)$. Then the Lemma says

$$(\mu^{\bar\sigma}_{\Lambda_n})_\Lambda = \int d\mu^{\bar\sigma}_{\Lambda_n}(\sigma) \mu^{\sigma\vee\bar\sigma}_{\Lambda}$$

i.e. the projection of a Gibbs state in Λ_n to a sub volume Λ is a convex combination of Gibbs states in Λ with different boundary conditions. This simple fact motivates the following definition:

Definition. A Borel measure μ in $\mathcal{M}^{\mathbb{Z}^d}$ is a Gibbs measure with potential $\Phi \in \mathcal{B}_0$ if for all finite $\Lambda \subset \mathbb{Z}^d$ there exists a probability measure $\tilde{\mu}_{\Lambda^c}$ on $\mathcal{M}^{\Lambda^c}$ such that for all $f \in C(\mathcal{M}^\Lambda)$

$$\int f(\sigma)d\mu(\sigma) = \int (\int f(\sigma_\Lambda)d\mu_\Lambda^{\sigma_{\Lambda^c}}(\sigma))\ d\tilde{\mu}_{\Lambda^c}(\sigma)\ \ (DLR) \tag{5.4}$$

We denote the set of Gibbs measures of the potential Φ by $\mathcal{G}_\Phi$.

Remark. The first integral is the integral of f w.r.t. the Gibbs measure in finite volume, with b.c. σ_{Λ^c}, and (5.4) is *some* average over boundary conditions. (DLR) is the *DLR-equation* (Dobrushin [10, 11], Lanford, Ruelle [28]).

What is the connection of this definition to the thermodynamic limits of finite volume Gibbs states? Let $\mu_\Lambda^{\bar{\sigma}}$ be a Gibbs measure on $\mathcal{M}^\Lambda$. We can extend $\mu_\Lambda^{\bar{\sigma}}$ to a measure $\mu^{\bar{\sigma},\Lambda}$ in the infinite volume configuration space $\mathcal{M}^{\mathbb{Z}^d}$ by setting

$$\mu^{\bar{\sigma},\Lambda} = \mu_\Lambda^{\bar{\sigma}} \times \delta_{\Lambda^c}^{\bar{\sigma}}$$

where $\delta_{\Lambda^c}^{\bar{\sigma}}$ is the Dirac measure on $\mathcal{M}^{\Lambda^c}$ i.e.

$$\int f(\tau)d\delta_{\Lambda^c}^{\bar{\sigma}}(\tau) = f(\bar{\sigma}_{\Lambda^c})$$

for all continuous f on $\mathcal{M}^{\Lambda^c}$ (here $\bar{\sigma}_{\Lambda^c}$ means just the restriction of the configuration $\bar{\sigma}$ to Λ^c, where f is defined). Thus, if $g \in C(\mathcal{M}^\Lambda)$, then

$$\int g d\mu^{\bar{\sigma},\Lambda} = \int g d\mu_\Lambda^{\bar{\sigma}} \equiv \langle g \rangle_\Lambda^{\bar{\sigma}}$$

our original state on $C(\mathcal{M}^\Lambda)$.

Consider then a sequence of boxes Λ_n tending to $\mathbb{Z}^d$ and consider the measures $\mu^{\bar{\sigma},\Lambda_n}$. We need the following fact from measure theory:

Let Ω be a compact metric space and let $\mathcal{B}(\Omega) = \{$ Borel probability measures on $\Omega\}$. We say $\mu_n \in \mathcal{B}(\Omega)$ converge if $\int f d\mu_n$ converge for all $f \in C(\Omega)$.Then $\mathcal{B}(\Omega)$ is compact in this (weak*-) topology i.e. if $\mu_n \in \mathcal{B}(\Omega)$, $n \in \mathbb{N}$ then there is a convergent subsequence μ_{n_i} and limit $\mu \in \mathcal{B}(\Omega)$: $\mu_{n_i} \to \mu$.

Hence, there is a sequence of boxes $\Lambda_{n_i} \nearrow \mathbb{Z}^d$ such that $\mu^{\bar{\sigma},\Lambda_{n_i}}$ converges to some measure μ.

Theorem 5.1 $\mu \in \mathcal{G}_\Phi$.

Proof. Let $f \in C(\Lambda)$. By the Lemma

$$\int f d\mu_{\Lambda_n}^{\bar{\sigma}} = \int (\int f d\mu_\Lambda^{\sigma \vee \bar{\sigma}}) d\mu_{\Lambda_n}^{\bar{\sigma}} = \int g(\sigma_{\Lambda^c}) d\mu^{\bar{\sigma},\Lambda_{n_i}}(\sigma)$$

where $g(\sigma_{\Lambda^c}) = \int f d\mu_\Lambda^{\sigma_{\Lambda^c}}$ is easily seen to be a continuous function (check!). Hence the claim follows from the convergence of the sequence $\mu^{\bar{\sigma},\Lambda_{n_i}}$. $\square$

Remarks. In the general theory of Gibbs states one can now show the following.
Let $\Phi \in \mathcal{B}_0$ and $\mathcal{G}_\Phi$ be the set of Gibbs states of Φ. Then :

a) $\mathcal{G}_\Phi$ is a convex set : $\mu_1, \mu_2 \in \mathcal{G}_\Phi \Rightarrow$

$$s\mu_1 + (1-s)\mu_2 \in \mathcal{G}_\Phi, \quad s \in [0,1].$$

b) $\mathcal{G}_\Phi$ is also compact (in the weak* topology).

c) There is a set of extremal elements $\mu_\alpha \in \mathcal{G}_\phi$ $\alpha \in A$ (index set), (μ_α is extremal, if it cannot be written as a convex combination of two distinct μ's) such that every $\mu \in \mathcal{G}_\Phi$ is uniquely a convex combination of μ_α (i.e., if $|A| < \infty, A = \{\alpha_i\}_{i=1}^N$ then $\exists\ s_i \in [0,1]$, $\sum_{i=1}^N s_i = 1$ and $\mu = \sum_{i=1}^N s_i \mu_{\alpha_i}$, in general, there exists a measure ν on A, $\nu(A) = 1$ and $\mu = \int \mu_\alpha d\nu(\alpha)$). The measures μ_α are called *pure phases.*

   Example. Ising model : the translation invariant pure phases are μ^+ and μ^-, general translation invariant Gibbs state $\mu = s\mu^+ + (1-s)\mu^-$. For example $\mu^{free} = \mu^{periodic} = \frac{1}{2}(\mu^+ + \mu^-)$ (Why ?).

d) Pure phases have correlations that tend to 0 at infinity:

$$\langle f(\sigma_A); g(\sigma_B)\rangle^T := \langle f(\sigma_A) g(\sigma_B)\rangle - \langle f(\sigma_A)\rangle\langle g(\sigma_B)\rangle \to 0 \quad \text{as} \quad \text{dist}(A,B) \to \infty.$$

Pure phases are physical, mixtures of them reflect our ignorance.

Let us finish this section by proving the Mermin-Wagner theorem for the XY-model. This generalizes to general systems with compact Lie group symmetries. See Simon's book [40].

Theorem 5.2 Let μ be a Gibbs measure for the XY-model. Then μ is invariant under a global rotation of the spins. I.e. for any $f \in C(\Omega)$ and $R \in SO(2)$,

$$\int f(\sigma) d\mu(\sigma) = \int f(R\sigma) d\mu(\sigma)$$

where $(R\sigma)_x \equiv R\sigma_x$.

Proof 1. Idea is the following (we follow here [25], see [40], p. 296). Consider say $f(\sigma) = F(\sigma_0)$, depends on σ_0 only. Let $R(\phi)$ be rotation by angle $\phi, R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$.

Go to angular variables $\sigma_x = \begin{pmatrix} \cos\theta_x \\ \sin\theta_x \end{pmatrix}$, so $\mathcal{H} = -\sum_{\langle xy\rangle} \cos(\theta_x - \theta_y)$, and the a priori measure is $\dfrac{d\theta_x}{2\pi}$ on $[0, 2\pi]$.

We want to prove :

$$\langle F(\theta_0 + \varphi)\rangle = \langle F(\theta_0)\rangle \quad \forall \phi,$$

it suffices to prove $\Big\langle \dfrac{d}{d\varphi}\Big|_{\varphi=0} F(\theta_0+\varphi)\Big\rangle = 0$, $\forall F$ smooth. Now $\langle - \rangle$ is approximatively $\langle - \rangle_\Lambda^{\bar\theta}$ some large Λ, some b.c. $\bar\theta$ because Gibbs measures are limits of such measures. Thus we could change variables

$$\Big\langle F(\theta_0+\varphi)\Big\rangle_\Lambda^{\bar\theta} = \Big\langle F(\theta_0)\Big\rangle_\Lambda^{\bar\theta+\phi}$$

which follows from $\mathcal{H}_\Lambda^{\bar\theta}(\theta-\varphi) = \mathcal{H}_\Lambda^{\bar\theta+\varphi}(\theta)$

$$\Bigg(\mathcal{H}_\Lambda^{\bar\theta}(\theta) = -\sum_{x,y\in\Lambda}\cos(\theta_x-\theta_y) - \sum_{\substack{x\,\in\,\Lambda\\ y\,\in\,\Lambda^c}}\cos(\theta_x-\bar\theta_y)\Bigg)$$

So the ϕ dependence is now in the b.c. This is tricky to control, so let us try to change variables by slowly rotating the spins : rotate $\theta_0+\varphi \to \theta_0$ and the rest a bit less such that on $\partial\Lambda$ there is no rotation : Let $g : \Lambda \to \mathbb{R}$ $g(0) = 1, g(x) = 0$ for x near $\partial\Lambda$. Then, let $(\tau_\varphi\theta)_x = \theta_x + g(x)\varphi$

$$\begin{aligned}
&\Big\langle F(\theta_0+\varphi)\Big\rangle_\Lambda^{\bar\sigma} = \Big\langle F((\tau_\varphi\theta)_0)\Big\rangle_\Lambda^{\bar\sigma}\\
&= \frac{1}{Z_\Lambda^{\bar\sigma}}\int F(\theta_0)\exp\Big[-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)\Big]\prod\frac{d\theta_x}{2\pi}\\
&\uparrow\\
&\theta_x \to \theta_x - g(x)\phi = (\tau_{-\varphi}\theta)_x\\
&\Rightarrow \langle\frac{d}{d\varphi}\Big|_{\varphi=0}F(\theta_0+\phi)\rangle_\Lambda^{\bar\sigma} = -\beta\Big\langle\Big(\frac{d}{d\varphi}\Big|_{\varphi=0}\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)\Big)F(\theta_0)\Big\rangle_\Lambda^{\bar\sigma}
\end{aligned}$$

Thus, by Schwartz inequality :

$$\Big|\Big\langle\frac{d}{d\varphi}\Big|_{\varphi=0}F\Big\rangle\Big| \;\le \beta\langle F^2\rangle^{1/2}\Big\langle\Big(\frac{d}{d\varphi}\Big|_{\varphi=0}\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta\Big)^2\Big\rangle^{1/2}$$

writing

$$\beta\frac{d}{d\varphi}\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)} = -\frac{d}{d\varphi}e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)}$$

and integrating by parts, we get :

$$\beta\Big\langle\Big(\frac{d}{d\varphi}\Big|_{\varphi=0}\mathcal{H}_\Lambda^{\bar\sigma}\Big)^2\Big\rangle = \Big\langle\frac{d^2}{d\varphi^2}\Big|_{\varphi=0}\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta)\Big\rangle.$$

But

$$\frac{d^2}{d\varphi^2}\mathcal{H}_\Lambda^{\bar\sigma}(\tau_{-\varphi}\theta) = -\frac{d^2}{d\varphi^2}\sum_{\langle xy\rangle x\in\Lambda, y\in\Lambda}\cos\Big(\theta_x-\theta_y-\varphi\Big(g(x)-g(y)\Big)\Big)$$

where we sum only over $x, y \in \Lambda$ since $g(x) = 0$ near $\partial\Lambda$.
So

$$\left|\frac{d^2}{d\varphi^2}\Big|_{\varphi=0} \mathcal{H}_\Lambda^{\bar\sigma}\right| = \Big| \sum_{\langle xy\rangle \text{ in } \Lambda} \Big(g(x) - g(y)\Big)^2 \cos(\theta_x - \theta_y)\Big|$$
$$\leq \sum_{\langle xy\rangle \text{ in } \Lambda} \Big(g(x) - g(y)\Big)^2.$$

Thus,

$$\left|\left\langle \frac{d}{d\varphi}\Big|_{\varphi=0} F\right\rangle\right| \leq C \sum_{\substack{\langle xy\rangle \\ x,\, y\, \in\, \Lambda}} \Big(g(x) - g(y)\Big)^2 \tag{5.5}$$

where C is g-independent. (5.5) holds for all $g : \Lambda \to \mathbb{R}, g(0) = 1, g = 0$ near $\partial\Lambda$. Also (5.5) holds for any Λ (C is Λ-independent).
To make the above argument rigorous, we use the DLR-equation to write

$$\int \frac{dF}{d\varphi} d\mu = \int \left[\int \frac{dF}{d\varphi} d\mu_\Lambda^{\bar\sigma}(\sigma)\right] d\tilde\mu_{\Lambda^c}(\bar\sigma)$$

i.e. an average over $\bar\sigma$ of what we did. Then (5.5) follows as above.

2. Second idea. The point now is that

$$\inf_\Lambda \inf_g \sum_{\langle xy\rangle} \Big(g(x) - g(y)\Big)^2 = 0.$$

Reason : we show that this equals

$$\inf_{h \in C_0^\infty, h(0)=1} \int (\nabla h(u))^2 d^2u \tag{5.6}$$

and this equals zero. Thus, let $h \in C_0^\infty$, $h(0) = 1$ (smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ with compact support). Let the support of h be contained in Λ_R (= cube, center 0, side R). Put $g_L : \mathbb{Z}^2 \to \mathbb{R}$, $g_L(x) = h(\frac{L}{R}x)$ $(x \in \mathbb{Z}^2)$. Then $g_L(0) = 1$, $g_L = 0$ near $\partial\Lambda_L$ and

$$\sum (g_L(x) - g_L(y))^2 = \sum_{x \in \mathbb{Z}^2} \sum_{i=1}^2 \left(h\Big(\frac{L}{R}x\Big) - h\Big(\frac{L}{R}(x - e_i)\Big)\right)^2$$

(where $e_1 = (1, 0), e_2 = (0, 1)$)

$$= \sum_x \sum_i \Big(\frac{R}{L}\Big)^2 \left[\frac{(h(\frac{L}{R}x) - h(\frac{L}{R}(x - e_i))}{R/L}\right]^2$$

$\underset{L\to\infty}{\longrightarrow} \int (\nabla h(u))^2 d^2u$. The infimum of this over $h \in C_0^\infty, h(0) = 1$ equals zero : let $h_\epsilon(x) = \begin{cases} 1 & |x| < 1 \\ |x|^{-\epsilon} & |x| \geq 1 \end{cases}$. This is piecewise C^1 and not 0 on boundary, but one can approximate this function by a function in C_0^∞ (show !). Now,

$$\nabla h_\epsilon(x) = -\epsilon \frac{\vec{x}}{|x|^{2+\epsilon}} \text{ for } |x| \geq 1 \text{ and } = 0, |x| < 1, \Rightarrow$$
$$\int (\nabla h(u))^2 d^2u = 2\pi\epsilon^2 \int_1^\infty r^{-2-2\epsilon} r dr = \pi\epsilon \to 0,$$

as $\epsilon \to 0$ $\square$

6 The Ginzburg-Landau Model

We now consider models where $\sigma_x \in \mathbb{R}$ i.e. is unbounded.

Motivation :
1. Later (see Section 12) we see that our previous models naturally give rise to this by a process called coarse-graining.
2. These are related to quantum fields.

We will denote the spin by $\phi(x)$ i.e. the spin configuration is $\phi : \mathbb{Z}^d \to \mathbb{R}$. Let us consider a simple Hamiltonian

$$\mathcal{H}_\Lambda(\phi) = -\sum_{\{x,y\}\in\mathcal{B}_\Lambda} \phi(x)\phi(y) + a\sum_{x\in\Lambda} \phi^2(x) + \lambda \sum_{x\in\Lambda} \phi^4_(x) \tag{6.1}$$

which again has the $\phi \to -\phi$ symmetry, and consider the probability measure

$$\frac{1}{Z_\Lambda} e^{-\beta\mathcal{H}_\Lambda(\phi)} \prod_{x\in\Lambda} d\phi(x) \tag{6.2}$$

on $\mathbb{R}^{|\Lambda|}$. The normalization (i.e. partition function) is given by $Z_\Lambda = \int_{\mathbb{R}^\Lambda} e^{-\beta\mathcal{H}_\Lambda(\phi)} \prod_{x\in\Lambda} d\phi_x$. This integral exists if $\lambda > 0$. This is easy to see as follows:

$$\begin{aligned} \mathcal{H}_\Lambda(\phi) &= \frac{1}{2} \sum_{\{x,y\}\in\mathcal{B}_\Lambda} (\phi_x - \phi_y)^2 + (-d + a)\sum_{x\in\Lambda} \phi_y^2 + \lambda\sum_{x\in\Lambda} \phi_x^4 + \sum_{x\in\partial\Lambda} n_x \phi_x^2 \\ &\geq (a-d)\sum_{x\in\Lambda} \phi_x^2 + \lambda \sum_{x\in\Lambda} \phi_x^4 \end{aligned}$$

where, for $x \in \partial\Lambda$ n_x is the number of nearest neighbors y of x s.t. $y \notin \Lambda$. Thus

$$Z \leq \left[\int_{-\infty}^\infty e^{-\beta[(a-d)\phi^2 + \lambda\phi^4]} d\phi\right]^{|\Lambda|} < \infty \quad \text{if } \lambda > 0 \text{ or } \ \lambda = 0 \text{ and } a - d > 0.$$

Call $a - d = \frac{r}{2}$ and so

$$\mathcal{H}_\Lambda(\phi) = \frac{1}{2} \sum_{\{x,y\} \in \mathcal{B}_\Lambda} (\phi_x - \phi_y)^2 + \frac{r}{2} \sum \phi_x^2 + \lambda \sum \phi_x^4 + \sum_{x \in \partial\Lambda} n_x \phi_x^2. \tag{6.3}$$

It will be much more convenient to work with the periodic boundary conditions instead of the free ones above. In that case the last term is missing from (6.3) and $\mathcal{B}_\Lambda$ contains also the bonds joining opposite faces of the cube Λ_L.

Remark. The Ising model is a limit of this model : take

$$\frac{r}{2} = -2\lambda \text{ so } \quad \frac{r}{2}\phi_x^2 + \lambda\phi_x^4 = \lambda(\phi_x^2 - 1)^2 - \lambda$$

and $\sqrt{\frac{\lambda}{\pi}} e^{-\lambda(\phi^2-1)^2} \underset{\lambda\to\infty}{\longrightarrow} \delta(\phi^2 - 1)$ (in the sense of distributions).

7 Gaussian Integrals

7.1 Definitions and elementary properties

Consider first the case $\lambda = 0$. Then our measure is *Gaussian.*

Definition 7.1 Let A be a real symmetric $n \times n$ matrix which is (strictly) positive definite (i.e. all eigenvalues > 0 i.e. $(\phi, A\phi) > 0 \ \ \forall \phi \in \mathbb{R}^n, \phi \neq 0$). *The Gaussian measure on* $\mathbb{R}^n$ with covariance A^{-1} and mean 0 is the probability measure

$$d\mu(\phi) = \frac{1}{Z} e^{-\frac{1}{2}(\phi, A\phi)} D\phi$$

where $\phi = (\phi_1, \cdots, \phi_n)$, $D\phi = \prod_i d\phi_i$, $(\phi, A\phi) = \sum_{ij} \phi_i A_{ij} \phi_j$.

Thus, each $\phi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\phi_i(\phi) = \phi_i$ is a random variable, with mean 0 and variance $\langle \phi_i^2 \rangle = (A^{-1})_{ii}$. Also, (ϕ, f) is a random variable, with variance $(f, A^{-1} f)$.

Some calculations

1. Consider the partition function

$$Z = \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\phi, A\phi)} D\phi.$$

A can be diagonalized by an orthogonal matrix $S \in SO(n)$: $S^T A S = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. So $|\det S| = 1$ and changing variables $\phi = S\Psi$ we have $D\phi = |\det S| D\Psi = D\Psi$ and thus

$$Z = \int e^{-\frac{1}{2}\sum_i \lambda_i \Psi_i^2} D\Psi = \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_i \Psi_i^2} d\Psi_i = \prod_{i=1}^n \left(\frac{2\pi}{\lambda_i}\right)^{1/2} = \Big[\det(A/2\pi)\Big]^{-1/2}$$

2. The correlations functions $\langle\prod_{\alpha=1}^{k}\phi_{i_\alpha}\rangle$ where $i_\alpha\in\{i,\dots,n\}$ (note: several i_α may be the same) can be computed using the **generating function** : Let $f\in\mathbb{R}^n$. Define

$$S(f)=\langle e^{(\phi,f)}\rangle=\frac{1}{Z}\int e^{-\frac{1}{2}(\phi,A\phi)+(\phi,f)}D\phi,$$

with $(\phi,f)=\sum_{i=1}^{n}\phi_i f_i$. Then $S(f)$ is smooth in f and

$$\langle\prod_{\alpha=1}^{k}\phi_{i_\alpha}\rangle=\prod_{\alpha=1}^{k}\frac{\partial}{\partial f_{i_\alpha}}\Big|_{f=0}S(f).$$

But

$$\begin{aligned}S(f) &= \frac{1}{Z}\int e^{-\frac{1}{2}(\phi-A^{-1}f,A(\phi-A^{-1}f))+\frac{1}{2}(f,A^{-1}f)}D\phi\\ &= e^{\frac{1}{2}(f,A^{-1}f)}\frac{1}{Z}\int e^{-\frac{1}{2}(\Psi,A\Psi)}D\Psi=e^{\frac{1}{2}(f,A^{-1}f)}\end{aligned}$$

using the change of variables $\Psi=\phi-A^{-1}f$. Hence, say the 2-point function,

$$\langle\phi_i\phi_j\rangle=\frac{\partial^2}{\partial f_i\partial f_j}\Big|_0 e^{\frac{1}{2}(f,A^{-1}f)}=(A^{-1})_{ij}.$$

In general,

$$\begin{aligned}\langle\prod_{\alpha=1}^{k}\phi_{i_\alpha}\rangle &= \begin{cases}0 & k\text{ odd}\\ (\prod\frac{\partial}{\partial f_{i_\alpha}})|_0[\frac{1}{2}(f,A^{-1}f)]^m\frac{1}{m!} & k=2m\end{cases}\\ &= \sum_P\prod_{\{\alpha,\beta\}\in P}(A^{-1})_{i_\alpha i_\beta}\end{aligned}$$

where P is a pairing of the set $\{1,2,\cdots,2m\}$ i.e. a partition into m sets of size 2. The sum runs through all such pairings.

Example For $n=4$ there are three pairings $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$. In general, the number of pairings is $\frac{(2m)!}{2^m m!}$ (show !).

Thus

$$\langle\prod_{\alpha=1}^{2m}\phi_{i_\alpha}\rangle=\sum_P\prod_{\{\alpha,\beta\}\in P}\langle\phi_{i_\alpha}\phi_{i_\beta}\rangle$$

a sum of products of 2-point functions.

7.2 The Gaussian Ginzburg-Landau model

Let us now specialize to the $\lambda=0$ Ginzburg-Landau model with periodic boundary conditions on $\Lambda_L=\mathbb{Z}_L^d$. This gives rise to a gaussian measure on $\mathbb{R}^{|\Lambda_L|}$ with density

$$Z^{-1}e^{-\frac{1}{2}(\sum_{\{x,y\}\in\mathcal{B}_{\Lambda_L}}(\phi_x-\phi_y)^2+r\sum\phi_x^2)}\tag{7.1}$$

The quadratic form $\sum_{\{x,y\}\in\mathcal{B}_{\Lambda_L}}(\phi_x-\phi_y)^2$ defines a matrix Δ_{per} (lattice-Laplacean with periodic b.c.):

$$\sum_{\{x,y\}\in\mathcal{B}_\Lambda}(\phi_x-\phi_y)^2=-(\phi,\Delta_{per}\phi).$$

Concretely

$$-(\Delta_{per}\phi)_x=\sum_{|u|=1}(\phi_x-\phi_{x+u}) \tag{7.2}$$

where the addition is modulo L. For example in $d=1$ we have $(-\Delta\phi)_x=2\phi_x-\phi_{x-1}-\phi_{x+1}$.

(7.2) defines as well an operator Δ in infinite volume i.e. for $\phi\in\mathbb{R}^{\mathbb{Z}^d}$. Then Δ is self adjoint on $\ell^2(\mathbb{Z}^d)\subset\mathbb{R}^{\mathbb{Z}^d}$

$$\ell^2(\mathbb{Z}^d)=\{\phi:\mathbb{Z}^d\to\mathbb{C}\mid\sum_{x\in\mathbb{Z}^d}|\phi_x|^2<\infty\}.$$

Indeed, Δ is bounded: $\|\Delta\|\le 2d$ (show!) and in the scalar product $(\phi,\psi)=\sum_{x\in\mathbb{Z}^d}\bar\phi_x\psi_x$ we have $(\phi,\Delta\psi)=(\Delta\phi,\psi)$.

The periodic Laplacean can be defined in $\mathbb{R}^{\mathbb{Z}^d}$ as well. For this consider the set $F_L^{per}\subset\mathbb{R}^{\mathbb{Z}^d}$ of periodic functions : [we use ϕ_x or $\phi(x)$ below !]

$$\phi\in F_L^{per}:\phi(x+Ln)=\phi(x)\quad\forall n\in\mathbb{Z}^d.$$

Clearly F_L^{per} can be identified with $\mathbb{R}^{\Lambda_L}$, namely ϕ is determined by $\phi|_{\Lambda_L}$, $\Lambda_L=\{x|\;|x_i\in\{0,\dots,L-1\}$. Clearly $\Delta:F_L^{per}\to F_L^{per}$ so the periodic Laplacean Δ_{per} is the corresponding matrix in $\mathbb{R}^{\Lambda_L}$.

We will now diagonalize Δ_{per} and Δ by Fourier series. Consider $p\in[-\pi,\pi]^d$ and $\phi_p\in\mathbb{R}^{\mathbb{Z}^d}$ $\phi_p(x)=e^{ipx}$ (where $px:=\sum_\alpha p_\alpha x_\alpha$) . Then,

$$\Delta\phi_p(x)=\sum_{|u|=1}(e^{ip(x+u)}-e^{ipx})=\sum_{|u|=1}(e^{ipu}-1)e^{ipx}=-\mu(p)\phi_p(x)$$

where

$$\mu(p)=2\sum_{\mu=1}^d(1-\cos p_\mu).$$

Let

$$B_L:=\{p\in\mathbb{R}^d\mid p_i=\frac{2\pi}{L}n_i,\;n_i\in\mathbb{Z},\;|n_i|\le L/2\}. \tag{7.3}$$

Then

$$\{\phi_p|\;p\in B_L\}$$

is a basis for F_L^{per} (they are independent and there are L^d of them). We have

$$\sum_{p\in B_L} e^{ip(x-y)} = L^d\delta_{xy} \quad \text{for} \quad x,y\in\Lambda_L \tag{7.4}$$

and

$$\sum_{x\in\Lambda_L} e^{i(p-q)x} = L^d\delta_{pq} \qquad p,q\in B_L \tag{7.5}$$

so $\{L^{-d/2}\phi_p\}_{p\in B_L}$ is an orthonormal basis of eigenvectors of $-\Delta_{per}$ and from (10.5), (10.6) we conclude

$$(-\Delta_{per}+r)^{-1}_{xy} = L^{-d}\sum_{p\in B_L} e^{ip(x-y)}\frac{1}{\mu(p)+r} \equiv G_L(x-y) \tag{7.6}$$

Thus, the correlations of the $\lambda=0$ GL-model are given in terms of the (periodic b.c.) *Green's function* G_L.

How about $L\to\infty$? In (7.6) we have a Riemann sum over cells of size $(\frac{2\pi}{L})^d$. So, in the limit,

$$\lim_{L\to\infty} G_L(x-y) = \int_{[-\pi,\pi]^d}\frac{d^dp}{(2\pi)^d}\frac{e^{ip(x-y)}}{\mu(p)+r} \equiv G(x-y). \tag{7.7}$$

This equals of course the kernel of the operator $(-\Delta+r)^{-1}$ defined on $\ell^2(\mathbb{Z}^d)$. Indeed, on $\ell^2(\mathbb{Z}^d)$ the operator $-\Delta$ has spectrum $\{\mu(p)|p\in[-\pi,\pi]^d\}$ and we can diagonalize it by Fourier series. If $f\in\ell^2(\mathbb{Z}^d)$, let

$$\hat{f}(p) = \sum_{x\in\mathbb{Z}^d} e^{-ipx}f(x).$$

Then $\hat{f}\in L^2(B), B=[-\pi,\pi]^d$ and

$$(f,g) \equiv \sum_{x\in\mathbb{Z}^d}\overline{f(x)}g(x) = \int_B\frac{d^dp}{(2\pi)^d}\overline{\hat{f}(p)}\hat{g}(p)$$

and we have the inverse formula

$$f(x) = \int_B e^{ipx}\hat{f}(p)\frac{d^dp}{(2\pi)^d}.$$

Thus,

$$((-\Delta+r)^{-1}f)(x) = \int e^{ipx}\frac{1}{\mu(p)+r}\hat{f}(p)\frac{d^dp}{(2\pi)^d} = \sum_y G(x-y)f(y)$$

i.e. $(-\Delta+r)^{-1}_{xy} = G(x-y)$. We have obtained

Proposition 7.5 Let μ_L be the Gaussian measure on $\mathbb{R}^{\Lambda_L}$, with covariance $(-\Delta_{per}+r)^{-1}$. Then

$$\int\prod\phi(x_\alpha)d\mu_L(\phi) \underset{L\to\infty}{\longrightarrow} \sum_P\prod_{\langle\alpha\beta\rangle}G(x_\alpha-x_\beta). \tag{7.8}$$

8 Measures on spaces of distributions

Can we write the RHS of (7.8) in terms of a measure on $\mathbb{R}^{\mathbb{Z}^d}$? Note that this space is not locally compact not to mention compact as we had in the case of bounded spin systems. Hence we can not take the route via Riesz representation theorem to reconstruct a measure out of correlation functions. The way out is to consider instead the *characteristic function* of the measure.

8.1 Bochner's Theorem

Let μ be a probability (Borel) measure on $\mathbb{R}^n$. Recall that we defined the generating function of μ as $S(f) = \int e^{(\phi,f)} d\mu(\phi)$ if $e^{(\phi,f)} \in L^1(\mu)$. It is actually more convenient to consider $S(if), f \in \mathbb{R}^n$ i.e. the characteristic function i.e. the Fourier transform of μ:

$$W(f) = \int e^{i(\phi,f)} d\mu(\phi) \quad f \in \mathbb{R}^n.$$

which always exists by dominated convergence theorem.

$W(f)$ has some obvious and less obvious properties:

a) $|W(f)| \leq \int 1 d\mu = 1.$

b) $W(0) = 1.$

c) W is continuous. Indeed, by the dominated convergence theorem, if $f_n \to f$ then, since $|e^{i(\phi,f_n)}| \leq 1, W(f_n) \to W(f)$.

d) Let $z_\alpha \in \mathbf{C}$, $f_\alpha \in \mathbb{R}^n$, $\alpha = 1, \cdots, N$. Then,
$$\sum_{\alpha,\beta} z_\alpha \bar{z}_\beta W(f_\alpha - f_\beta) = \int \sum z_\alpha \bar{z}_\beta e^{i(\phi,f_\alpha)} e^{-i(\phi,f_\beta)} = \int |\sum z_\alpha e^{i(\phi,f_\alpha)}|^2 d\mu \geq 0.$$

Definition 7.3 $W : \mathbb{R}^n \to \mathbf{C}$ is a function of *of positive type* if a) - d) hold.

Theorem 7.4. (Bochner's theorem) $W(f)$ is of positive type if and only if there is a Borel probability measure on $\mathbb{R}^n$ such that $W(f) = \int e^{i(\phi,f)} d\mu(\phi)$.

Proof. "$\Leftarrow$" is done above.
"$\Rightarrow$". One uses an idea that is useful in other contexts, namely we use W to construct a scalar product. Let

$$\mathcal{H} = \{\psi : \mathbb{R}^n \to \mathbf{C} | \psi(x) \neq 0 \text{ only for finitely many } \ x\}.$$

For $\phi, \psi \in \mathcal{H}$ put
$$(\phi, \psi) = \sum_{x,y \in \mathbb{R}^n} \bar{\phi}(x)\psi(y)W(x-y)$$

(this is a finite sum). $(\ ,\ )$is an inner product except that $(\phi,\phi)=0$ does not imply $\phi=0$. Let $\mathcal{H}_0=\{\phi|(\phi,\phi)=0\}$; $\mathcal{H}_0$ is a subspace of $\mathcal{H}$. Put $\tilde{\mathcal{H}}=\mathcal{H}/\mathcal{H}_0=$ equivalence classes $[\phi]=[\phi+\phi_0],\phi_0\in\mathcal{H}_0$. Then $(\tilde{\mathcal{H}},(\cdot,\cdot))$ is an inner product space which can be completed into a Hilbert space.

Let for $t\in\mathbb{R}^n, U_t:\mathcal{H}\to\mathcal{H}\quad (U_t\psi)(x)=\psi(x+t)$. Clearly, $U_t:\mathcal{H}_0\to\mathcal{H}_0$ so one defines $\tilde{U}_t:\tilde{\mathcal{H}}\to\tilde{\mathcal{H}}$ by $\tilde{U}_t[\phi]=[U_t\phi]$. $(\tilde{U}_t\phi,\tilde{U}_t\psi)=(\phi,\psi)$ so $\tilde{U}_t$ is an isometry. We have $\tilde{U}_{t+s}=\tilde{U}_t\tilde{U}_s$, $\tilde{U}_0=\mathbb{1}$. Also $t\to\tilde{U}_t$ is strongly continuous (i.e. $\forall\psi\in\tilde{\mathcal{H}}\ t\to\tilde{U}_t\psi\in\tilde{\mathcal{H}}$ is continuous from $\mathbb{R}^n\to\tilde{\mathcal{H}}$) because W is continuous:

$$\|\tilde{U}_t\psi-\tilde{U}_s\psi\|^2=2(\psi,\psi)-(\tilde{U}_t\psi,\tilde{U}_s\psi)-(\tilde{U}_s\psi,\tilde{U}_t\psi)=[2(\psi,\psi)-(\psi,\tilde{U}_{t-s}\psi)-(\psi,\tilde{U}_{s-t}\psi)]$$

This equals

$$\sum_{x,y}\bar{\psi}(x)\psi(y)[2W(x-y)-W(x-y+t-s)-W(x-y+s-t]$$

which tends to 0 as $t\to s$. We have now verified the assumptions of

Stone's Theorem. Let $\mathbb{R}^n\ni t\to U_t$ unitary in Hilbert space be strongly continuous and $U_tU_s=U_{t+s}$, $\forall s,t$, $U_0=\mathbb{1}$. Then for each $\phi\in\mathcal{H}, \|\phi\|=1\ \exists$ a probability Borel measure μ_ϕ on $\mathbb{R}^n$ such that $(\phi,U_t\phi)=\int e^{i(\lambda,t)}d\mu_\phi(\lambda)$.

Proof See Reed-Simon [33], vol. 1, Th. VIII.12. [The heuristics behind the proof is that, if $\mathcal{H}$ were finite dimensional, we could easily prove that U_t is differentiable in t, $\frac{\partial U_t}{\partial t_j}|_{t=0}=iA_j$, A_j Hermitean, $[A_i,A_j]=0$ and $U_t=e^{i\sum t_jA_j}$. Diagonalize all A_j simultaneously $\Rightarrow A_j=\begin{pmatrix}\lambda_1^{(j)} & & 0\\ & \ddots & \\ 0 & & \lambda_n^{(j)}\end{pmatrix}\Rightarrow d\mu_\phi(\vec{\lambda})=\sum_{i=1}^n\delta(\vec{\lambda}^{(i)}-\vec{\lambda})|\phi_i|^2$.

To finish the proof of the Theorem, using Stone's Theorem, take $\tilde{\phi}\in\tilde{\mathcal{H}}$

$$\tilde{\phi}=[\phi],\qquad \phi(x)=\begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases}$$

$$W(t)=\sum_{x,y\in\mathbb{R}^n}\delta(x)\delta(y+t)W(x-y)=(\tilde{\phi},\tilde{U}_t\tilde{\phi})=\int e^{i(\lambda,t)}d\mu_{\tilde{\phi}}(\lambda)\qquad\square$$

Suppose now we have measures μ_k on $\mathbb{R}^n$ and want to prove μ_k converge as $k\to\infty$. The properties 1,2 and 4 of W_k usually carry to the limit so the main issue to check is 2 i.e. continuity. In our case W_L however live in spaces whose dimensions tend to infinity as $L\to$ infinity. We aim at conditions 1-4 in such a setup and a generalization of Bochner's theorem there.

8.2 Cylinder measures

Suppose we had a measure μ on some σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}^{\mathbb{N}}$ such that sets of the form $A \times \mathbb{R}^{\mathbb{N}\backslash n} \in \mathcal{A}$ where $A \in \mathcal{A}_n$ = some σ-algebra of subsets of $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}\backslash n}$ denotes $\times_{i=n+1}^{\infty}\mathbb{R}$ (and $\mathbb{R}^{\mathbb{N}} \equiv \times_{i=1}^{\infty}\mathbb{R}$). Then we get a measure μ_n on $\mathcal{A}_n$ by "integrating out" the other variables :

$$\mu_n(A) = \mu(A \times \mathbb{R}^{\mathbb{N}\backslash n}). \tag{8.9}$$

This resulting set of measures is *consistent* i.e.

$$\mu_n(A) = \mu_{n+m}(A \times \mathbb{R}^m) \tag{8.10}$$

provided $A \times \mathbb{R}^m \in \mathcal{A}_{n+m}$ if $A \in \mathcal{A}_n$. Thus, conversely, let $\{\mu_n\}_{n=1}^{\infty}$ be a family of measures, μ_n a Borel probability measure on $\mathbb{R}^n$. We say μ_n *are consistent if* (8.10) holds for all Borel sets $A \subset \mathbb{R}^n$. A natural σ-algebra on $\mathbb{R}^{\mathbb{N}}$ is the σ-algebra generated by the *cylinder sets*. Denote by $\mathcal{B}$ Borel sigma-algebra on $\mathbb{R}$ and by $\mathcal{B}^n$ the one on $\mathbb{R}^n$.

Definition 8.1 A *cylinder set* on $\mathbb{R}^{\mathbb{N}}$ is a set of the form $S_{B_1,\cdots,B_n} = \{x \in \mathbb{R}^{\mathbb{N}} | x_i \in B_i, i = 1,\cdots,n, B_i \in \mathcal{B}, n < \infty\}$. The σ-algebra of subsets of $\mathbb{R}^{\mathbb{N}}$ generated by cylinder sets is denoded by $\mathcal{B}^{\mathbb{N}}$. A measure μ defined on $\mathcal{B}^{\mathbb{N}}$ is called a *cylinder measure*. We have then the

Kolmogorov's extension theorem. Let $\{\mu_n\}_{n=1}^{\infty}$ be a consistent family of measures on $\mathcal{B}^n$. Then there is a unique cylinder measure μ on $\mathcal{B}^{\mathbb{N}}$ such that

$$\mu(S_{B_1,\cdots,B_n}) = \mu_n(B_1 \times B_2 \times \cdots \times B_n). \tag{8.11}$$

Proof. Main problem is countable additivity. We can do this with Rietz theorem by the following trick.

Let $\dot{\mathbb{R}}$ be the one-point compactification of $\mathbb{R}$ (i.e. $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and open sets are open sets in $\mathbb{R}$ and sets of the form $A \cup \{\infty\}$ where $A \subset \mathbb{R}$ is open and $A^c \subset [-n, n]$ some $n < \infty$, $\dot{\mathbb{R}}$ is easily seen to be compact). Let

$$\mathcal{M} = \times_{i=1}^{\infty}\dot{\mathbb{R}} = \dot{\mathbb{R}}^{\mathbb{N}}.$$

Then $\mathcal{M}$ is compact in the product topology. Let $C(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{R} | f \text{ continuous }\}$ and $C_0(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{R}|,\ f$ is continuous and depends on finitely many $x_i\}$. Define, for $f \in C_0(\mathcal{M})$,

$$\ell(f) = \int f d\mu_n$$

where n is large enough, and we extend μ_n to $\dot{\mathbb{R}}^n$ by putting $\mu_n(B_1 \times \cdots \times \{\infty\} \times \cdots \times B_n) = 0$. By consistency, ℓ is n independent if n is large enough. So ℓ defines a positive linear functional $\ell : C_0(\mathcal{M}) \to \mathbb{R}$, $\ell(1) = 1$, $|\ell(f)| \le \|f\|_\infty$. The last one implies that ℓ extends to $C(\mathcal{M})$ since $C_0(\mathcal{M})$ is dense in $C(\mathcal{M})$. Hence by Riesz theorem there exists unique Borel-measure μ on $\mathcal{M}$ such that $\ell(f) = \int f d\mu$. Recall that $\mathcal{M} = \dot{\mathbb{R}}^{\mathbb{N}}$. But

$$\mu(\{x \in X | x_n = \infty \text{ some } n) \le \sum_{n=1}^{\infty} \mu(\{x | x_n = \infty\}) = \sum_{n=1}^{\infty} \mu_n(\{x | x_n = \infty\}) = 0.$$

Hence μ is supported on $\mathbb{R}^{\mathbb{N}}$. By construction (8.11) holds ($\mathcal{B}^{\mathbb{N}} \subset$ Borel sets of $\dot{\mathbb{R}}^{\mathbb{N}}$) [2]). $\square$

As it turns out, we can embed all interesting function (and distribution) spaces into $\mathbb{R}^{\mathbb{N}}$. Let us start with two subsets :

8.3 Minlos Theorem : Measures on $\mathbf{s}'$

Let

$$\mathbf{s}_m = \{x \in \mathbb{R}^{\mathbb{N}} | \sum_{n=1}^{\infty} n^{2m} |x_n|^2 \equiv \|x\|_m^2 < \infty\}$$

for $m \in \mathbb{Z}$. Let

$$\mathbf{s} = \bigcap_{m \in \mathbb{Z}} \mathbf{s}_m \qquad \mathbf{s}' = \bigcup_{m \in \mathbb{Z}} \mathbf{s}_m$$

(note : $\mathbf{s}_m \supset \mathbf{s}_{m+1} \supset \cdots$). $\mathbf{s}$ is the set of sequences which decay as $n \to \infty$ faster than any power, s' the ones with at most polynomial growth. Give $\mathbf{s}$ the topology generated by the neighbourhoods of 0 of the form

$$N(m, \epsilon) = \{x | \ \|x\|_m < \epsilon\}$$

(so neighbourhoods of $y \in \mathbf{s}$ are $y + N(m, \epsilon)$).

Lemma 8.2 The space $\mathbf{s}$ with the above topology is a complete metrizable space, i.e. $\mathbf{s}$ has a complete metric d such that d gives the same topology as above.

Proof. Define the metric

$$d(x, y) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|x - y\|_m}{1 + \|x - y\|_m}.$$

Then (**Exercise**) : Show that this distance defines the same topology as above and that $\mathbf{s}$ is complete (Cauchy sequences converge). $\square$

Remark. $\mathbf{s}$ is an example of a *Frechet space* i.e. a complete locally convex metric space, and which is not a Banach space.

[2]Borel sets of $\dot{\mathbb{R}}^{\mathbb{N}}$ = smallest σ-algebra containing open sets and cylinder sets are generated by open sets.

Lemma 8.3 $\mathbf{s}'$ is the dual space of $\mathbf{s}$ i.e. the space of continuous linear functionals on $\mathbf{s}$.

Proof Let $y \in \mathbf{s}'$. y defines a continuous linear functional ℓ_y on s :

$$\ell_y(x) \equiv (y, x) \equiv \sum_{n=1}^{\infty} y_n x_n.$$

Indeed, this converges since $|y_n| \leq Cn^m$ some $m < \infty$ and $|x_n| \leq C_m n^m$ for all m. To prove continuity, let $d(x^{(k)}, x) \to 0$ as $k \to \infty$. Then, by Schwartz' inequality,

$$\begin{aligned} |\ell_y(x^{(k)}) - \ell_y(x)| &\leq \sum_{n=1}^{\infty} Cn^m |x_n^{(k)} - x_n| = C \sum n^{m+1} |x_n^{(k)} - x_n| \frac{1}{n} \\ &\leq C\Big(\sum \frac{1}{n^2}\Big)^{1/2} \|x^{(k)} - x\|_{m+1} \\ &\leq C' 2^{m+1} d(x^{(k)}, x) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Conversely, let $\ell : \mathbf{s} \to \mathbb{R}$ be a continuous linear map. Then there exists $C < \infty$, $m < \infty$ such that

$$|\ell(z)| \leq C\|z\|_m \tag{8.12}$$

for all z. Indeed, by continuity there exists a neighbourhood U of zero in $\mathbf{s}$ such that $|\ell(x| < 1$, for $x \in U$. On the other hand U contains a set $\{z| \ \|z\|_m < \delta\}$ some m, δ. Hence, given a $z \in \mathbf{s}$ we have $\delta \frac{z}{\|z\|_m} \in U$ and so $|\ell(\delta \frac{z}{\|z\|_m})| < 1$ i.e. $|\ell(z)| \leq \delta^{-1}\|z\|_m)$.

Let now $e^{(k)} \in \mathbf{s}$ be given by $e_n^{(k)} = \delta_{nk}$. Put $y_n = \ell(e^{(n)})$. By (8.12)

$$|y_n| = |\ell(e^{(n)})| \leq C\|e^{(n)}\|_m = Cn^m$$

so $y \in \mathbf{s}'$. Given $x \in \mathbf{s}$, let

$$x^{(n)} = \sum_{k=1}^{n} e^{(k)} x_k.$$

Then $x^{(n)} \to x$ in $\mathbf{s}$ (prove !) and so

$$\ell(x^{(n)}) \to \ell(x).$$

But $\ell(x^{(n)}) = \sum_{k=1}^n y_k x_k$. Hence $\ell(x) = (y, x)$ □

The reason we introduced $\mathbf{s}$ and $\mathbf{s}'$ is the following. Let μ be a cylinder (probability) measure on $\mathcal{B}^{\mathbb{N}}$. Note that $\mathbf{s}'$ is a μ-measurable set (prove !). Suppose $\mu(\mathbf{s}') = 1$ i.e. $\mu(\mathcal{B}^{\mathbb{N}} \backslash \mathbf{s}') = 0$ i.e. μ is supported in $\mathbf{s}'$. Then,

Exercise. Let $x \in \mathbf{s}$. Then the function $y \to (y, x)$ from $\mathbf{s}'$ to $\mathbb{R}$ is μ-measurable.

Definition 8.4 Let μ be a cylinder probability measure on $\mathbf{s}'$. The characteristic function $W : \mathbf{s} \to \mathbf{C}$ of μ is

$$W(x) = \int_{\mathbf{s}'} e^{i(y,x)} d\mu(y).$$

Lemma 8.5 W satisfies

a) $W(0) = 1$, $|W(f)| \leq 1$

b) W is *positive definite* : given $z_i \in \mathbf{C}$, $x^{(i)} \in \mathbf{s}$ $\; i = 1, \cdots, n$ we have

$$\sum_{i,j=1}^{n} \overline{z_i} z_j W(x_i - x_j) \geq 0$$

c) $W : \mathbf{s} \to \mathbb{R}$ is continuous.

Proof As before, for c) use again the dominated convergence theorem. $\square$

Finally, our goal:

Minlos' Theorem A necessary and sufficient condition for a function W on $\mathbf{s}$ to be a characteristic function of a probability measure on $\mathbf{s}$' is that it obeys a)-c) above.

Proof. Let W be given. Let $u \in \mathbb{R}^n$, put $x = (u_1, \cdots, u_n, 0, 0, \cdots) \in \mathbf{s}$. Then $W(x) \equiv \widetilde{W}_n(u)$ is a function of positive type in $\mathbb{R}^n \Rightarrow \exists$ probability measure μ_n on $\mathbb{R}^n$ such that

$$\widetilde{W}_n(u) = \int_{\mathbb{R}^n} e^{i(y,u)} d\mu_n(u)$$

$\{\mu_n\}_{n=1}^{\infty}$ are consistent since $\widetilde{W}_{n+m}((u_1 \cdots u_n, 0, \cdots, 0)) = \widetilde{W}_n(u)$ and $\widetilde{W}_n$ uniquely determines μ_n (Why ?). Hence $\exists \mu$ on $\mathbb{R}^{\mathbb{N}}$ such that

$$W(x) = \int_{\mathbb{R}^{\mathbb{N}}} e^{i(y,x)} d\mu(y) \tag{8.13}$$

for all $x \in \mathbf{s}$ with only finitely many non zero x_i. We need to show : $\mu(\mathbf{s}') = 1$. In that case the integral can be restricted to $\mathbf{s}'$ and (8.13) holds for all $x \in \mathbf{s}$ (because both sides are continuous in x and the set of x with a finite number of nonzero x_i is dense in $\mathbf{s}$).

$\underline{\mu(\mathbf{s}') = 1}$: Recall that $\mathbf{s}' = \bigcup_{m=-\infty}^{\infty} \mathbf{s}_m$ with

$$\mathbf{s}_m = \{y | \sum_{n=1}^{\infty} n^{2m} y_n^2 \equiv \|y\|_m^2 < \infty\}.$$

We show: given $\epsilon > 0$, there exists an m such that

$$\mu(\mathbf{s}_m) \geq 1 - \epsilon. \tag{8.14}$$

Since $\mu(\mathbf{s}') \geq \mu(\mathbf{s}_m)$ we get $\mu(\mathbf{s}') \geq 1 - \epsilon$ $\forall \epsilon$ i.e. the claim.

To prove (8.14), we use first the monotone convergence theorem to get a more manageable expression :

$$\lim_{\alpha \downarrow 0} \lim_{N \to \infty} \int \exp\left(-\frac{\alpha}{2} \sum_{n=1}^{N} n^{2m} y_n^2\right) d\mu = \lim_{\alpha \downarrow 0} \int e^{-\frac{\alpha}{2}\|y\|_m^2} d\mu = \mu(\mathbf{s}_m)$$

since $e^{-\frac{\alpha}{2}\|y\|_m^2} \nearrow \chi(y \in \mathbf{s}_m) \equiv \begin{cases} 1 & y \in \mathbf{s}_m \\ 0 & y \notin \mathbf{s}_m, \text{i.e. } \|y\|_m = \infty \end{cases}$

We consider $\displaystyle\int \exp\left(-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2\right) d\mu$. Choose m:

Continuity of $W(x) \Rightarrow \exists\ m, \delta$ such that

$$|W(x) - 1| \leq \epsilon \ \text{ if } \ \|x\|_{-m-1}^2 \leq \delta \tag{8.15}$$

(recall that $W(0) = 1$). (Note that m here tends to be < 0 !).

Then, for all $x \in \mathbf{s}$,

$$\text{Re } W(x) \geq 1 - \epsilon - \frac{2}{\delta}\|x\|_{-m-1}^2 \tag{8.16}$$

since if $\|x\|_{-m-1}^2 < \delta$, (8.16) holds by (8.15) and if $\|x\|_{-m-1}^2 > \delta$, (8.16) holds because $|W(x)| \leq W(0) = 1$ implies Re $W(x) \geq -1$.

Now write

$$\exp\left(-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2\right) = \int_{\mathbb{R}^N} e^{i(y,x)} d\nu(x)$$

$$d\nu(x) = \prod_{n=1}^{N} (2\pi\alpha n^{2m})^{-1/2} \exp\Big(-\frac{x_n^2}{2\alpha n^{2m}}\Big) dx_n$$

to get

$$\begin{aligned}\int \exp\left(-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2\right) d\mu &= \int_{\mathbb{R}^N} \left(\int \exp\left(i\sum_{i=1}^{N} y_i x_i\right) d\mu(y)\right) d\nu(x) \\ &= \int_{\mathbb{R}^N} W(x) d\nu(x) = \int \text{ Re } W(x) d\nu(x)\end{aligned}$$

since the left hand side is real.

Hence by (8.16),

$$\int \exp\left(-\frac{\alpha}{2}\sum^{N} n^{2m} y_n^2\right) d\mu \geq 1 - \epsilon - \frac{2}{\delta}\int_{\mathbb{R}^N} \|x\|_{-m-1}^2 d\nu(x).$$

But

$$\int_{\mathbb{R}^N} \|x\|_{-m-1}^2 d\nu = \sum_{n-1}^{N} n^{-2m-2} \int x_n^2 d\nu(x) = \sum_{n=1}^{N} n^{-2m-2} \alpha n^{2m} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \alpha.$$

Thus

$$\lim_{N\to\infty} \int \exp\left(-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2\right) d\mu \geq 1 - \epsilon - \frac{c\alpha}{\delta}$$

and our claim follows $\square$

Example. Gaussian measures. Let C be a matrix C_{ij} $i,j=1,2,\cdots$ with $|C_{ij}| \leq ai^m.j^m$ some $a<\infty, m<\infty$ and $(x,Cx)=\sum x_i x_j C_{ij} > 0$ $\forall x \neq 0$ $x \in \mathbf{s}$ (note (x,Cx) makes sense since $|x_i| < b_n i^{-n}$ all n). Then the function

$$W_C(x) = e^{-\frac{1}{2}(x,Cx)}$$

is of positive type.

Proof W is continuous since $|(x,Cx)| \leq A\|x\|_n^2$ for some n. Hence $W_C(x) = \lim_{k\to\infty} W_C(x^{(k)})$, $x_i^{(k)} = \begin{cases} x_i & i \leq k \\ 0 & i > k \end{cases}$. But

$$W_C(x^{(k)}) = \exp\left(-\frac{1}{2}\sum_{ij=1}^{k} x_i x_j C_{ij}\right) = \int_{\mathbb{R}^k} \exp\left(i\sum_{i=1}^{k} x_i y_i\right) d\mu_k(y)$$

where μ_k is the Gaussian measure on $\mathbb{R}^k$ with covariance matrix C_{ij}, $i,j=1,\cdots,k$. By assumption, this is a positive $k\times k$ matrix. Hence, let $W_C^k(x) = W_C(x^{(k)})$ $x \in \mathbf{s}$ so W_C^k is of positive type and $W_C^k(x) \to W_C(x)$ $\forall x \in \mathbf{s} \to W_C(x)$ is of positive type. $\square$

We call the measure μ_C corresponding to W_C the *Gaussian measure in* $\mathbf{s}'$ *with covariance* C.

Example. Order $\mathbb{Z}^d$ arbitarily such that $\mathbb{Z}^d = \{x_k | k=1,2,\cdots\}$ and $|x_{k+1}| \geq |x_k|$. Then $\phi : \mathbb{Z}^d \to \mathbb{R}$ becomes $\tilde{\phi} : \mathbb{N} \to \mathbb{R}$ $\tilde{\phi}_i = \phi_{x_i}$. Thus, if C_{xy} $x,y \in \mathbb{Z}^d$ is a positive matrix with $|C_{xy}| \leq a(|x|+1)^m(|y|+1)^m$ we can define the Gaussian measure with covariance C on $\mathbf{s}'(\mathbb{Z}^d) = \{\phi : \mathbb{Z}^d \to \mathbb{R} | \sum_{x\in\mathbb{Z}^d} (|x|+1)^{2m} |\phi_x|^2 < \infty \text{ some } m \in \mathbb{Z}\}$. Thus $C = (-\Delta + r)^{-1}$ is an example (it is positive and $|C_{xy}| \leq$ constant; this holds : $r \geq 0, d \geq 3$ and $r > 0, d = 2$).

8.4 Measures on Spaces of Distributions

Let

$$S(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is } C^\infty \text{ and } \sup_{x\in\mathbb{R}} (1+|x|^m) \frac{d^n f}{dx^n}| < \infty \text{ for all } n, m \geq 0\} \tag{8.17}$$

i.e. f and its derivatives decay faster than any power. Let us define ("creation and annihilation operators")

$$a = \frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big),\ a^+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big)$$

and $N = a^+a = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}$. For $f \in S(\mathbb{R})$, let

$$\|f\|_m = \left(\int_{\mathbb{R}} |(N+1)^m f|^2 dx\right)^{1/2}$$

(this is finite due to (8.17)[3].

We give S the topology where neighbourhoods of 0 are

$$N(m,\epsilon) = \{f \in S| \ \ \|f\|_m < \epsilon\}$$

Proposition 8.6 S is homeomorphic to **s**.

Proof Consider the functions $\phi_n \in S$:

$$\phi_0 = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}x^2}, \phi_n = \frac{1}{\sqrt{n!}}(a^+)^n \phi_0$$

These satisfy :

$$(\phi_n, \phi_m) \equiv \int_{-\infty}^{\infty} \phi_n \phi_m dx = \delta_{nm}$$

Proof $a\phi_0 = 0$, $aa^+ = a^+a + 1$ imply

$$\begin{aligned} a\phi_n &= \frac{1}{\sqrt{n!}} a(a^+)^n \phi_0 = \frac{1}{\sqrt{n!}}\Big[(a^+)^{n-1}\phi_0 + a^+a(a^+)^{n-1}\phi_0\Big] \\ &= \frac{1}{\sqrt{n!}}\Big[2(a^+)^{n-1}\phi_2 + (a^+)^2 a a^{+n-2}\phi_0\Big] = \ldots = \frac{n}{\sqrt{n!}}(a^+)^{n-1}\phi_0 \Rightarrow a^n\phi_n = \frac{n!}{\sqrt{n!}}\phi_0 \end{aligned}$$

But also, for $\phi, \psi \in S, (a^+\phi, \psi) = (\phi, a\psi)$ so $(\phi_n, \phi_n) = \frac{1}{\sqrt{n!}}(\phi_0, a^n\phi_n) = (\phi_0, \phi_0) = 1$. $n \neq m$ gives 0 since $a\phi_0 = 0$.

The functions ϕ_n are thus orthonormal in $L^2(\mathbb{R})$. They are a basis (they are the Hermite functions). Now our map $S \to \mathbf{s}$ is $f \to x = (x_1, x_2, \cdots)$, with $x_n = (f, \phi_{n-1})$. Note that

$$N\phi_n = a^+a\phi_n = n\phi_n$$

so $(N+1)^m\phi \to x'$ with $x'_n = n^m x_n$.

Thus

$$\|f\|_m^2 = \int |(N+1)^m f|^2 dx = \sum_{k=0}^{\infty} |((N+1)^m f, \phi_k)|^2 = \sum_{n=1}^{\infty} n^{2m}|x_n|^2 = \|x\|_m^2$$

i.e. the map $f \to x$ maps the neighbourhoods to each other. To conclude, we need to show that this map is a bijection.

[3]See Remark 1 below.

A) Injection : $\{\phi_n\}$ is a basis of L^2.

B) Surjection, or the map is onto : Let $x \in \mathbf{s}$. Then we need to show that $\sum_{n=1}^{N} x_n \phi_{n-1}$ converges as $N \to \infty$ to an element f of S and $x_n = (f, \phi_{n-1})$. $\sum_{n=1}^{N} x_n \varphi_{n-1}$ converges in L^2 to $f \in L^2$. $(N+1)^m \sum^{N} x_n \varphi_{n-1} = \sum^{N} x_n n^m \varphi_{n-1}$ converges in $L^2 \Rightarrow f \in C^\infty$. And $\|f\|_m = \|x\|_m$ so $f \in S$. $\square$

Definition 8.7 The space of *tempered distributions* on $\mathbb{R}$ is the set $S'(\mathbb{R})$ of continuous linear functionals on $S(\mathbb{R})$.

Let $\varphi \in S'(\mathbb{R})$. Put $y_n = \varphi(\phi_{n-1})$. Then φ continuous means[4] : $\exists C, m : |\varphi(f)| \leq C\|f\|_m$ $\forall f \in S \Rightarrow |y_n| = |\varphi(\phi_{n-1})| \leq C\|\phi_{n-1}\|_m = Cn^m$. Hence $y \in \mathbf{s}'$. Conversely, $y \in \mathbf{s}'$ gives rise to $\varphi_y \in S'(\mathbb{R})$ by $\varphi_y(f) = \sum_{n=1}^{\infty} y_n(f, \phi_n)$. Indeed, from above, $x_n = (f, \varphi_n)$ is in $\mathbf{s}$ so $|\varphi_y(f)| \leq C\|x\|_m$ for some m and this $= C\|f\|_m$.

Remark 1 We see from above that $\| \ \|_m$ is a *norm* on S : $\|f\|_m^2 = 0 \Rightarrow (f, \varphi_n) = 0 \ \forall n \Rightarrow f = 0$. Moreover, since $\|f\|_m^2 = \Big((a^+a+1)^m f, (a^+a+1)^m f\Big) = \Big(f, (a^+a+1)^{2m} f\Big) = \sum$ of $\Big(f, (a^+a)^k f\Big)$ and $(a^+a)^k f =$ (Polynomial in x and $\frac{d}{dx}$) f, we get

$$\|f\|_m \leq C_m \sum_{\alpha,\beta \leq 2m} \sup_x |x^\alpha \frac{d^\beta}{dx^\beta} f(x)| \equiv C_m \|f\|^{(m)}$$

and conversely (show !). So, $\varphi \in S' \iff \exists C, N : |\varphi(f)| \leq C\|f\|^{(N)}$. This is in practice useful, see Examples 1,2 below.

Example 1 Let φ be a polynomially bounded function : $|\varphi(x)| \leq C(1+|x|)^m$. Then it defines a distribution $\varphi \in S'$ by

$$\varphi(f) = \int \varphi(x) f(x) dx$$

since

$$\begin{aligned} &|\int \varphi(x) f(x) dx| \leq C \int (1+|x|)^m |f(x)| dx \leq \\ &C \int (1+|x|)^{m+2} |f(x)| \frac{1}{[1+|x|]^2} dx \leq C' \sup_x (1+|x|)^{m+2} |f(x)| \leq C'' \|f\|^{(m+2)} \end{aligned}$$

[4] φ continuous : $\forall \epsilon \ \exists \ N(m,\delta) : f \in N(m,\delta) \Rightarrow |\varphi(f)| < \epsilon$ i.e. $\|f\|_m < \delta \Rightarrow |\varphi(f)| < \epsilon$. Thus for any $f \in S$, $|\varphi(f)| = |\varphi\Big(\frac{f}{\|f\|_m}\delta\Big)| \frac{\|f\|_m}{\delta} \leq \frac{\epsilon}{\delta}\|f\|_m = C\|f\|_m$.

so $\varphi \in S'$.

Example 2 φ can be a "delta function" or its derivatives : let

$$\varphi(f) = \frac{d^k f}{dx^k}(x_0).$$

Then,

$$|\varphi(f)| \leq \|D^k f\|_\infty$$

so $\varphi \in S'$. We denote $\varphi = \frac{d^k}{dx^k}\delta(x = x_0)$.

Remark This works in $\mathbb{R}^n$ too :

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \ \|x^\alpha D^\beta f\|_\infty \leq C_{\alpha\beta} \ \ \forall \alpha, \beta\}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n), x^\alpha = \displaystyle\prod_{i=1}^n x_i^{\alpha_i}, D^\beta = \prod_{i=1}^n \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}$. Basis $\{\varphi_k\}_{k \in \mathbb{N}^n}$ $\varphi_k = \displaystyle\prod_{i=1}^n \varphi_{k_i}(x_i)$ and we define $\mathbf{s}^{(n)} = \{x \in \mathbb{R}^{\mathbb{N}^n} | x = (x_k)_{k \in \mathbb{N}^n} \displaystyle\sum_k (\prod k_i)^{2m} |x_k|^2 < \infty \ \forall m\}$. Do the rest !

Definition 8.8 *1. The cylinder set σ-algebra in $S'(\mathbb{R})$* is the smallest σ-algebra containing all sets

$$A(\varphi, B) = \{\varphi \in S'(\mathbb{R}) | \varphi(f) \in B\}$$

for $f \in S, B$ Borel in $\mathbb{R}$. It is just the image of our old cylinder algebra on $\mathbf{s}'$ to S' by the map constructed above. Indeed, the cylinder σ-algebra C in $\mathbf{s}'$ is the smallest σ-algebra of subsets of $\mathbf{s}'$ that contains the sets $\sigma_n(B) = \{y \in \mathbf{s}' | y_n \in B\}$ where B is Borel in $\mathbb{R}$. I.e. it is the smallest σ-algebra such that functions

$$p_n : \mathbf{s}' \to \mathbb{R} \quad , \quad p_n(y) = y_n$$

are measurable (i.e. $p_n^{-1}(B) \in C \ \forall B$ Borel i.e. $\sigma_n(B) \in C \ \forall B$). Let $\tilde{C}$ be the smallest σ-algebra in $\mathbf{s}'$ such that the functions $\pi_x : \mathbf{s}' \to \mathbb{R} \ \ \pi_x(y) = (y, x)$ are measurable for all $x \in \mathbf{s}$. Since $p_n = \pi_{e_n^{(n)}}$, $e_m^{(n)} = \delta_{nm}$ we see that $C \subset \tilde{C}$. But, also, $\pi_x = \displaystyle\sum_{n=1}^\infty x_n p_n$ so π_x is C-measurable. Hence $\tilde{C} \subset C$. Thus $C = \tilde{C}$. Clearly the σ-algebra defined above in $S'(\mathbb{R})$ is the image of $\tilde{C}$ under the map $y \in \mathbf{s}' \to \varphi_y \in S'$. This proves the claim.

Definition 8.9 *2. A cylinder measure on* $S'(\mathbb{R})$ is a measure on this σ-algebra.

Translating from $\mathbf{s}, \mathbf{s}'$ to S, S' we get :

Theorem (Minlos) A necessary and sufficient condition for a function W on $S(\mathbb{R}^n)$ to be the generating function of a cylinder probability measure μ on $S'(\mathbb{R}^n)$,

$$W(f) = \int_{S'} e^{i\varphi(f)} d\mu(\varphi)$$

is that $W(0) = 1$, W be of positive type and continuous.

Example Gaussian measures. Here $W(f) = \exp\left[-\frac{1}{2}C(f,f)\right]$ where $C(f,g)$ is a continuous bilinear function on $S(\mathbb{R}^n) \times S(\mathbb{R}^n)$, with $C(f,f) > 0$ for $f \neq 0$. Important examples are given by integral kernels

$$C(f,g) = \int d^n x \int d^n y C(x,y) f(x) g(y)$$

here $\int |C(x,y)|[(1+|x|)(1+|y|)]^m dxdy < \infty$ for some m suffices. We are interested in *translation-invariant* C's i.e. $C(x,y) = G(x-y)$. These are most conveniently expressed in terms of Fourier transform : If $f \in S(\mathbb{R}^n)$ we put

$$\hat{f}(k) = \int f(x) e^{-ikx} d^n x$$

and one gets $\hat{f} \in S(\mathbb{R}^n)$. [This is because $(x^\alpha D^\beta f)^\wedge(k) = i^{\alpha+\beta} D^\alpha k^\beta \hat{f}(k)$, do the details !]
Inverse :

$$f(x) = \int e^{ikx} \hat{f}(k) \frac{d^n k}{(2\pi)^n}.$$

Then we have

$$\begin{aligned} \int |f(x)|^2 d^n x &= \int |\hat{f}(k)|^2 \frac{d^n k}{(2\pi)^n} \\ (f*g)(x) &\equiv \int f(x-y) g(y) d^n y \\ (f*g)^\wedge(k) &= \hat{f}(k)\hat{g}(k) \end{aligned}$$

and

$$\int f(x) G(x-y) f(y) d^n x d^n y = \int |\hat{f}(k)|^2 \hat{G}(k) \frac{d^n k}{(2\pi)^n}$$

which holds for $f, G \in S$ to start with, but extends to much more general G's.

Example The *free Euclidean field with mass* m is the Gaussian measure μ_G where

$$\widehat{G}(k) = \frac{1}{k^2+m^2} \qquad (k \in \mathbb{R}^n).$$

Clearly $S \ni f \to \int |\hat{f}(k)|^2 \dfrac{d^n k}{(k^2+m^2)(2\pi)^n}$ is continuous, so

$$W(f) = \exp\left[-\frac{1}{2}\int \frac{|\hat{f}(k)|^2}{k^2+m^2} \frac{d^n k}{(2\pi)^n}\right]$$

determines a measure on $S'(\mathbb{R}^n)$, by Minlos' theorem. Strictly speaking, $G(x)$ is defined as a distribution[5], by the Fourier transform $\widehat{G}$ but actually is a locally integrable function :

$$|G(x)| \leq C\frac{e^{-m|x|}}{|x|^{n-2}}. \tag{8.18}$$

To see this, let us consider a *ultraviolet cutoff* : let $\Lambda > 0$ and define

$$\widehat{G}_\Lambda(k) = \frac{1}{k^2+m^2}\frac{\Lambda^2}{k^2+\Lambda^2}.$$

This is called Pauli-Villars cutoff. Note that $\widehat{G}_\Lambda(k) \underset{\Lambda\to\infty}{\longrightarrow} \widehat{G}(k)$ pointwise, and as elements of S'.

<u>Note</u> S' is given a topology such that $\varphi_i \to \varphi$ as $i \to \infty$ if $\varphi_i(f) \to \varphi(f)$ $\forall f \in S$.
For $n < 4$, $\widehat{G}_\Lambda \in L^1(\mathbb{R}^n)$ and so $G_\Lambda(x)$ is a continuous function (for $n \geq 4$: replace $\dfrac{\Lambda^2}{p^2+\Lambda^2}$ by $\left(\dfrac{\Lambda^2}{p^2+\Lambda^2}\right)^k$).
So, by rotational symmetry,

$$G_\Lambda(x) = \int \frac{d^nk}{(2\pi)^n}\frac{e^{ik_1|x|}}{k^2+m^2}\frac{\Lambda^2}{k^2+\Lambda^2}$$

So the k_1 integral :

$$\begin{aligned}
&k^2+m^2 = (k_1 + i\sqrt{\vec{k}^2+m^2})(k_1 - i\sqrt{\vec{k}^2+m^2}),\\
&k^2+\Lambda^2 = (k_1 + i\sqrt{\vec{k}^2+\Lambda^2})(k_1 - i\sqrt{\vec{k}^2+\Lambda^2}),\\
&\vec{k} = (k_2, k_3, \cdots).
\end{aligned}$$

By Cauchy:

$$G_\Lambda(x) = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}}\left[\frac{e^{-r\sqrt{\vec{k}^2+m^2}}}{2\sqrt{\vec{k}^2+m^2}}\frac{\Lambda^2}{m^2+\Lambda^2} + \frac{e^{-r\sqrt{\vec{k}^2+\Lambda^2}}}{2\sqrt{\vec{k}^2+\Lambda^2}}\frac{1}{m^2+\Lambda^2}\right]$$

which for $r = |x| \neq 0$ is *smooth* and converges as $\Lambda \to \infty$ to

$$G(x) = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}}\frac{e^{-r\sqrt{\vec{k}^2+m^2}}}{2\sqrt{\vec{k}^2+m^2}};$$

this is a L^1 function satisfying (8.18).

[5] Fourier transform $\hat{\varphi}$ of a distribution $\varphi \in S'$ is a distribution, $\hat{\varphi} \in S'$, defined by

$$\hat{\varphi}(\hat{f}) = (2\pi)^n\varphi(\tilde{f}) \quad \tilde{f}(x) = f(-x)$$

which is the usual (see example 1 above) if $\varphi \in S$: $\int \hat{\varphi}(k)\hat{f}(k)\frac{d^nk}{(2\pi)^n} = \int \varphi(x)f(-x)d^nx$.

Note that, a priori, for $\varphi \in S', \varphi(x)$ is not defined, only $\varphi(f),\ f \in S$. These are measurable functions in S' and actually Gaussian random variables on the probability space $(S'(\mathbb{R}^n), \mu)$:

$$\langle e^{it(\varphi,f)} \rangle = e^{-\frac{1}{2}t^2(f,Gf)}$$

so the variance of (φ, f) is (f, Gf). Let $f_i \in S, i = 1, \cdots, N$. Then $x_i \equiv \varphi(f_i)$ are jointly Gaussian :

$$\langle e^{i\sum t_i x_i} \rangle = \langle e^{i\varphi(\sum t_i f_i)} \rangle = e^{-\frac{1}{2}\sum_{ij} t_i t_j (f_i, G f_j)}$$

i.e. $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ is Gaussian with covariance matrix $A^{-1}_{ij} = (f_i, Gf_j)$. Hence e.g.

$$\int \varphi(f)\varphi(g) d\mu_G(\varphi) = -\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \langle e^{-i[t\varphi(f)+s\varphi(g)]} \rangle = (f, Gg) = \int d^n x d^n y f(x) G(x-y) g(y)$$

and, more generally,

$$\int \prod_{i=1}^{2N} \varphi(f_i) d\mu_G(\varphi) = \sum_P \prod_{\langle ij \rangle \in P} (f_i, G f_j)$$
$$= \int d^n x_1 \cdots d^n x_{2N} f_1(x_1) \cdots f_N(x_N) \sum_P \prod_{\langle ij \rangle \in P} G(x_i - x_j)$$

Hence we may use the notation

$$\int \varphi(x_1) \cdots \varphi(x_{2N}) d\mu_G(\varphi) = \sum_P \prod_{\langle ij \rangle \in P} G(x_i - x_j)$$

even if strictly speaking $\varphi(x)$ is not a well defined random variable.

Remark Indeed, let us try to see what $\varphi(x)^2$ would be. Formally

$$\int \varphi(x)^2 d\mu_G(\varphi) = \lim_{x \to y} \int \varphi(x)\varphi(y) d\mu_G = \lim_{x \to y} G(x-y) = G(0) = \infty.$$

This is a first instance of an *ultraviolet divergence* in quantum field theory. Our G is such that $\varphi(x)$ is not a nice random variable (it is a "distribution valued random variable"). We'll return to this.

9 The Gaussian Ginzburg-Landau Model : Critical Point and Continuum Limit

Let us return to the lattice and the Ginzburg-Landau-model. The $\lambda = 0$ case is the Gaussian measure μ_C on $\mathbb{R}^{\mathbb{Z}^d}$ (or $\mathrm{s}'(\mathbb{Z}^d)$) with covariance

$$C(x-y) = \int \phi(x)\phi(y) d\mu_C(\phi) = \int_{[-\pi,\pi]^d} \frac{e^{ip(x-y)}}{\mu(p) + r} \frac{d^d p}{(2\pi)^d}$$

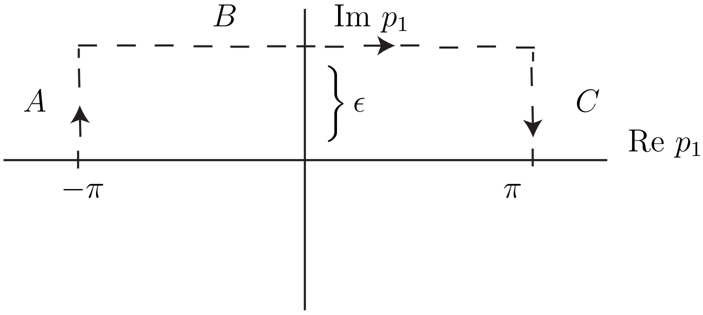
where $x, y \in \mathbb{Z}^d, \phi \in \mathbb{R}^{\mathbb{Z}^d}$.
We have

$$0 \le C(x-y) \le Ce^{-|x-y|/\xi} \qquad \xi > 0 \ \ , \ \ \xi \to \infty \text{ as } r \to 0.$$

Proof Let $R = \max_{i=1,\cdots,d} |x_i - y_i|$. We may assume $R = |x_1 - y_1| = x_1 - y_1$.

$$C(x-y) = \int \left(\int_{-\pi}^{\pi} e^{ip_1 R} \frac{1}{2(1-\cos p_1) + \mu(\vec{p}) + r} \frac{dp_1}{2\pi} \right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}.$$

Use Cauchy for the p_1 integral



A and C cancel since $\cos p_1, e^{ip_1 R}$ are periodic

$$= e^{-\epsilon R} \int \left(\int_{-\pi}^{\pi} e^{ip_1 R} \frac{1}{2(1-\cos(p_1 + i\epsilon)) + \mu(\vec{p}) + r} \frac{dp_1}{2\pi} \right) e^{i\vec{p}(\vec{x}-\vec{y})} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}$$

Choose ϵ so that : $\mathrm{Re} 2(1-\cos(p_1 + i\epsilon)) + r > 0$ i.e. $2(1 - \cos p_1 \cosh \epsilon) + r > 0$ i.e. $2\cosh\epsilon < 2 + r$ or $\epsilon < \cosh^{-1}(1 + r/2)$. [So, for r small, one can choose $\epsilon \sim \sqrt{r}$]. So, since $\mu(p) \ge 0$,

$$C(x-y) \le Ce^{-\epsilon R} \qquad \square$$

Thus $r > 0$ is non critical. For $r = 0$, we have

$$C(x) = |x|^{2-d} \int_{[-\pi|x|,\pi|x|]^d} \frac{e^{ip\frac{x}{|x|}}}{x^2 \mu(p/|x|)} \frac{d^d p}{(2\pi)^d} \equiv |x|^{2-d} h(\hat{x}, |x|)$$

$\hat{x} = x/|x|$,
provided $\boxed{d > 2}$.

<u>Homework</u> Prove that

$$h(\hat{x}, |x|) \underset{|x|\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \frac{e^{ip_1}}{p^2} \frac{d^d p}{(2\pi)^d}$$

$$= \int_{\mathbb{R}^d} \frac{e^{-\sqrt{\vec{p}^2}}}{2\sqrt{\vec{p}^2}} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} = \frac{A_{d-1}}{2(2\pi)^{d-1}} \int_0^\infty e^{-r} r^{d-3} dr = \frac{\Gamma(d-2) A_{d-1}}{2(2\pi)^{d-1}} \equiv C_d < \infty$$

for $d > 2$, where A_{d-1} = volume of S^{d-2}.

We can state the previous calculation for $r = 0$ in two ways :

a) We showed $C(x) \sim G(x)$ as $|x| \to \infty$. Here $G(x)$ = 2-point function of the massless free field, $\widehat{G}(p) = \frac{1}{p^2}$. Note that $\widehat{G}$, for $d > 2$, defines a continuous function $e^{-\frac{1}{2}(f,Gf)}$ on S : We have

$$(f, Gg) = \int_{\mathbb{R}^d \times \mathbb{R}^d} d^dx d^dy \frac{C_d}{|x-y|^{d-2}} f(x)g(y) \leq C\|f\|_m \|g\|_m \text{ for } m \text{ large enough.}$$

For $d = 2$, see below.

b) Consider the function

$$\begin{aligned} &L^{d-2}C(Lx - Ly) = L^{d-2}\langle\phi(Lx)\phi(Ly)\rangle \\ &= L^{d-2}\int_{[-\pi,\pi]^d} \frac{e^{ip(Lx-Ly)}}{\mu(p)} \frac{d^dp}{(2\pi)^d} = \int_{[-L\pi,L\pi]^d} \frac{e^{ip(x-y)}}{L^2\mu(p/L)} \frac{d^dp}{(2\pi)^d}. \end{aligned}$$

Formally, this converges

$$\underset{L\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \frac{e^{ip(x-y)}}{p^2} \frac{d^dp}{(2\pi)^d} = G(x-y),$$

since $L^2\mu(p/L) \to p^2$ as $L \to \infty$.

We call this the *scaling limit* : take the statistical mechanics at the critical point, look at long distances, scale ϕ (by $L^{\frac{d-2}{2}}$ above) $\Rightarrow$ get quantum field theory. In this Gaussian model we get : let $x_i \neq x_j$ $i \neq j$, and $\varphi_L(x) = L^{\frac{d-2}{2}}\phi(Lx)$ (for $x \in (L^{-1}\mathbb{Z})^d$).
Then

$$\int_{s'} \prod_{i=1}^{2N} \varphi_L(x_i) d\mu_C(\phi) \underset{L\to\infty}{\longrightarrow} \int_{S'} \prod_{i=1}^{2N} \varphi(x_i) d\mu_G(\varphi).$$

Here the LHS is with a measure defined on variables $\phi(x), x \in \mathbb{Z}^d$, a lattice model and the RHS with a measure defined on variables $\varphi(x)$ $x \in \mathbb{R}^d$, a continuum model (to make this precise, observe that $(L^{-1}\mathbb{Z})^d)$ becomes "dense" in $\mathbb{R}^d$ as $L \to \infty$).

The massive case

How about $r > 0$? Consider again the Ginzburg-Landau model, $C_r = (-\Delta + r)^{-1}$ on $\ell^2(\mathbb{Z}^d)$. So,

$$C_r(x) = |x|^{2-d} \int_{[-|x|\pi,|x|\pi]^d} \frac{e^{ip\frac{x}{|x|}}}{|x|^2\mu(p/|x|) + |x|^2 r} \frac{d^dp}{(2\pi)^d}.$$

For $|x| < r^{-1/2}$ (let r be small now)

$$C_r(x) \sim |x|^{2-d}$$

and for $|x| > r^{-1/2}$ scale differently, $p \to r^{1/2}p$,

$$\begin{aligned} C_r(x) &= r^{\frac{d-2}{2}} \int_{[-\frac{\pi}{r},\frac{\pi}{r}]^d} \frac{e^{ipr^{1/2}x}}{r^{-1}\mu(r^{1/2}p)+1} \frac{d^dp}{(2\pi)^d} \\ &\sim r^{\frac{d-2}{2}} e^{-r^{1/2}|x|}. \end{aligned}$$

Stated differently, let us put $r = L^{-2}m^2$ and consider, after the change of variable $pm \to p$,

$$\begin{aligned} L^{d-2}C_{m^2/L^2}(Lx) &= \int_{[-L\pi,L\pi]^d} \frac{e^{ipx}}{L^2\mu(p/L)+m^2} \frac{d^dp}{(2\pi)^d} \\ &\underset{L\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \frac{e^{ipx}}{p^2+m^2} \frac{d^dp}{(2\pi)^d} \equiv G_{m^2}(x). \end{aligned}$$

Hence

$$\int_{\mathbf{s}'} \prod_{i=1}^{2N} L^{\frac{d-2}{2}} \phi(Lx_i) d\mu_{C_{m^2/L^2}}(\phi) \to \int_{S'} \prod \varphi(x_i) d\mu_{G_{m^2}}(\varphi)$$

i.e. if we approach the critical point $r \to 0$ in a suitable way and scale distances and ϕ we get the continuum theory at nonzero m i.e. correlation length $< \infty$.

Remark All the scalings are completely natural :
Formally

$$d\mu_{C_r}(\phi) = \frac{1}{Z} e^{-\frac{1}{2}(\phi,(-\Delta+r)\phi)} \prod_{x\in\mathbb{Z}^d} d\phi(x)$$

and $\phi(x) = L^{\frac{2-d}{2}}\varphi_L(x/L)$ gives

$$\begin{aligned} (\phi,(-\Delta+r)\phi)) &= \sum_{x\in\mathbb{Z}^d} \phi(x)((-\Delta+r)\phi)(x)) \\ &= \sum_{x\in\mathbb{Z}^d} \phi(x) \left[\sum_{|u|=1} (\phi(x)-\phi(x+u)) + r\phi(x) \right] \\ &= \sum_{x\in\mathbb{Z}^d} L^{2-d}\varphi_L(\frac{x}{L}) \left[\sum_{|u|=1} (\varphi_L(\frac{x}{L}) - \varphi_L(\frac{x}{L}+\frac{u}{L})) + r\varphi_L(\frac{x}{L}) \right] \\ &= \sum_{y\in(\frac{1}{L}\mathbb{Z})^d} L^{-d}\varphi_L(y) \left[\sum_{|u|=1} \frac{\varphi_L(y)-\varphi_L(y+\frac{u}{L})}{L^{-2}} + L^2r\varphi_L(y) \right] \end{aligned}$$

which, if φ_L went to a smooth function as $L \to \infty$,

$$\underset{\substack{L\to\infty \\ r=m^2/L^2}}{\longrightarrow} \int_{\mathbb{R}^d} \varphi(y)(-\Delta+m^2)\varphi(y)d^dy = (\varphi, G_{m^2}^{-1}\varphi),$$

10 Non-Gaussian Theory on the lattice

10.1 Measures

Let us now consider the $\lambda \neq 0$ Ginzburg-Landau model.
There are various finite volume theories that we can consider : Let $\Lambda \subset \mathbb{Z}^d$ be an L-cube.

A) The measures

$$\frac{1}{\widetilde{Z}_\Lambda} \exp\left(-\lambda \sum_{x\in\Lambda} \phi(x)^4\right) d\mu_{C_\Lambda}(\phi)$$

on $\mathbb{R}^{|\Lambda|}$ where μ_{C_Λ} is Gaussian with covariance $C_\Lambda = (-\Delta_\Lambda + r)^{-1}$ and Δ_Λ is the Laplacean in $\mathbb{Z}^d$ with some boundary conditions on $\partial\Lambda$ (we have considered periodic and free above; one can consider Dirichlet or Neumann on the lattice too). $\widetilde{Z}_\Lambda$ normalizes the total measure to 1.

B) The measure

$$\frac{1}{Z} \exp\left(-\lambda \sum_{x\in\Lambda} \phi(x)^4\right) d\mu_C(\phi)$$

on $\mathbf{s}'(\mathbb{Z}^d)$ when μ_C is Gaussian with covariance $C = (-\Delta + r)^{-1}$. Thus μ_C is in infinite volume and the ϕ^4 perturbation is in finite volume. Note that, whereas in A, $\widetilde{Z}_\Lambda = \int \exp\left(-\lambda \sum_{x\in\Lambda} \phi_x^4\right) d\mu_{C_\Lambda}$ is strictly > 0, this is not immediately obvious in B). However, recall Jensen's inequality :

If ν is a probability measure in $\mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is convex: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$, $t \in (0,1)$, then

$$\int f(x) d\nu(x) \geq f\left(\int x d\nu(x)\right).$$

Apply it to the random variable $x = \lambda \sum_{y\in\Lambda} \phi_y^4$ to get

$$\int \exp\left(-\lambda \sum_{y\in\Lambda} \phi_y^4\right) d\mu_C \geq \exp[-\lambda \sum_{y\in\Lambda} \langle \phi_y^4 \rangle]$$

where

$$\langle \phi_y^4 \rangle \equiv \int d\mu_C(\phi) \phi_y^4 = 3C(0)^2 = 3\left[\int \frac{1}{\mu(p) + m^2} \frac{d^d p}{(2\pi)^d}\right]^2$$

from our rules for Gaussian integrals. Also, since $\langle \phi_y^4 \rangle > 0$, we get

$$\exp[-3\lambda|\Lambda|C(0)^2] \leq Z_\Lambda \leq 1. \tag{10.1}$$

Note how the extensivity of $Z (= e^{\mathcal{O}(\Lambda)})$ is visible here. Also, note that the free energy : $-3\lambda C(0)^2 \leq \frac{1}{|\Lambda|} \log Z_\Lambda \leq 0$, uniformly in Λ. Similar inequalities hold in the case A) above.

10.2 Perturbation Theory

It is possible to prove quite generally, using Ising-model approximations and various correlation inequalities that the $\Lambda \to \mathbb{Z}^d$ limit of the above measures exist. These are, however, special tricks and we want to understand these issues eventually using the renormalization group. But let us for now proceed more heuristically and study the correlation functions perturbatively in λ. Consider e.g. the pair correlation

$$\langle \phi(x)\phi(y)\rangle_\Lambda \equiv \frac{\int \phi(x)\phi(y) \exp\left(-\lambda \sum_{x\in\Lambda} \phi(x)^4\right) d\mu_C(\phi)}{\int \exp\left(-\lambda \sum_{x\in\Lambda} \phi(x)^4\right) d\mu_C(\phi)} \equiv G_2(x,y). \tag{10.2}$$

It is not hard to see that, for $|\Lambda| < \infty$, this is C^∞ in λ. Indeed consider e.g. the denominator and let $V = \sum_{x\in\Lambda} \phi(x)^4$ and set $F_k(\lambda) := \int V^k e^{-\lambda V} d\mu_C(\phi)$. Since

$$|F_{n-1}(\lambda+\epsilon) - F_{n-1}(\lambda)| = V^{n-1}|e^{-\epsilon V} - 1|e^{-\lambda V} \le |\epsilon| V^n e^{-(\lambda-|\epsilon|)V} \le |\epsilon| n! \delta^{-n} e^{-(\lambda-|\epsilon|-\delta)V}$$

and $\int e^{-(\lambda-|\epsilon|-\delta)V} d\mu_C(\phi) < \infty$ for $\lambda - |\epsilon| - \delta \ge 0$ we see that $F_{n-1}(\lambda)$ is differentiable with $F'_{n-1}(\lambda) = F_n(\lambda)$ and the latter equals the n:th derivative of the denominator. Proceeding similarly with the numerator one shows that both numerator and denominator are C^∞ and from (10.1) above, $Z_\Lambda > 0$. Hence G_2 is smooth (of course this way we get terrible bounds for the derivatives as Λ gets large).

Thus, we have Taylor's expansion for G_2:

$$G_2(x,y) = \sum_{n=0}^{N} G_{2,n}(x,y)\lambda^n + R_N$$

where $\lim_{\lambda\to 0} \lambda^{-N} R_N(\lambda) = 0$. The Taylor coefficients $G_{2,n}$ have a nice graphical representation which we now derive.

Let us start with small n. Denote (10.2) by $\frac{\mathcal{N}}{\mathcal{D}}$.

$n = 0$. $\mathcal{N} = \int \phi(x)\phi(y) d\mu_C + \mathcal{O}(\lambda)$, $\mathcal{D} = 1 + \mathcal{O}(\lambda)$ so $G_{2,0} = C(x-y)$ our Gaussian covariance.

$n = 1$.

$$\mathcal{N} = C(x-y) - \lambda \sum_{z\in\Lambda} \int \phi(x)\phi(y)\phi(z)^4 d\mu_C$$

and

$$\mathcal{D} = 1 - \lambda \sum_{z\in\Lambda} \int \phi(z)^4 d\mu_C.$$

From our rules of Gaussian integrals we get:

$$\int \phi(z)^4 d\mu_C = 3C(0)^2$$

from the three pairings. The integral

$$\int \phi(x)\phi(y)\phi(z)^4 d\mu_C$$

has $\frac{6!}{2^3 \cdot 3!} = 15$ pairings and it equals

$$3C(x-y)C(0)^2 + 12C(x-z)C(y-z)C(z-z).$$

So,

$$\mathcal{N} = C(x-y)(1 - 3\lambda|\Lambda|C(0)^2) - 12\lambda \sum_{z\in\Lambda} C(x-z)C(y-z)C(0)$$

and

$$\mathcal{D} = 1 - 3\lambda|\Lambda|C(0)^2$$

Hence altogether

$$G_{2,1} = -12\lambda C(0) \sum_{z\in\Lambda} C(x-z)C(z-y).$$

To proceed further, we need to introduce some notation. Clearly, we need to calculate expressions like

$$\int \phi(x)\phi(y) \prod_{i=1}^{N} \phi(z_i)^4 d\mu_C \text{ and } \int \prod_{i=1}^{N} \phi(z_i)^4 d\mu_C.$$

Both are given as sums of products of pairings $\sum_P \prod_{\langle\alpha\beta\rangle} C(u_\alpha - u_\beta)$ where $\{u_\alpha\}$ are the points x, y, z_i, however such that each z_i *occurs 4 times*. More precisely, let $u_{4(i-1)+j} = z_i$ for $i = 1, \ldots, N$, $j = 1, 2, 3, 4$ and $u_{4N+1} = x$, $u_{4N+2} = y$. Then the sum is over pairings of the set $1, \ldots, 4N+2$.

To each pairing P we associate a *graph* $\mathcal{G}(P)$ as follows : $\mathcal{G} = (V, \mathcal{L})$ consists of a set V of vertices and a set $\mathcal{L}$ of lines. Vertices are the set $\{x, y, z_1, \cdots, z_N\} = V$ (for the 1st case). Lines ℓ join vertices : $\ell = \{u_{\ell_1}, u_{\ell_2}\}$ where $u_{\ell_1}, u_{\ell_2} \in V$ and $u_{\ell_1} = u_{\ell_2}$ is allowed. Let $\Gamma_2(V)$be the set of such graphs $\mathcal{G}$ such that each vertex z_i belongs to four lines (where we count lines $\{z_i, z_i\}$ twice) and each vertex x, y belongs to one line.

Given a $\mathcal{G}$ satisfying these conditions, there may be several P's such that $\mathcal{G}(P) = \mathcal{G}$. Call the number $|\{P|\mathcal{G}(P) = \mathcal{G}\}| := n(\mathcal{G})$.
Then

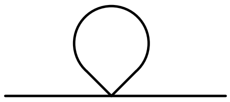
$$\int \prod_{i=1}^{N} \phi(z_i)^4 \phi(x)\phi(y) d\mu_C = \sum_{\mathcal{G}\in\Gamma_2(V)} n(\mathcal{G}) \prod_{\ell\in\mathcal{G}} C(u_{\ell_1} - u_{\ell_2}). \tag{10.3}$$

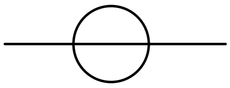
Example. In $\langle\phi(z)^4\phi(x)\phi(y)\rangle$ we had 2 graphs, $\mathcal{G}_1$ has lines $\{z,z\}, \{z,z\}, \{x,y\}$ and $\mathcal{G}_2$ lines $\{x,z\}, \{y,z\}, \{z,z\}$. $n(\mathcal{G}_1) = 3, n(\mathcal{G}_2) = 12$.

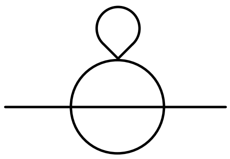
Now, once we sum (10.3) over the points $z_1, \ldots, z_N$ the order of the points z_i will not anymore matter. We get finally the following graphical rules which we state for arbitary correlations :

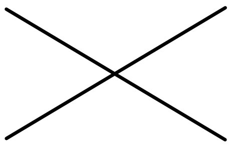
Definition 10.1 A ϕ^4- graph with $2m$ *external legs*, labelled $1, \cdots, 2m$, and N unlabelled *vertices* is a graph on N points (vertices) such that each vertex has 4 lines attached (one line can start and end at a vertex and is counted as two here), $2m$ lines have one end with no other lines attached.

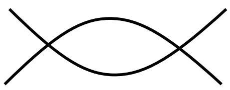
Examples.

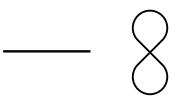
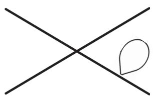
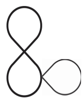

$2m = 2,\ N = 1$


$2m = 2,\ N = 2$


$2m = 2,\ N = 3$


$2m = 4,\ N = 1$


$2m = 4,\ N = 2$

More :  ,  ,  etc.

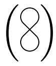
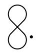
With this notation, we have

$$\frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \int \prod_{i=1}^{2m} \phi(x_i)\ \exp\left(-\lambda \sum_{x\in\Lambda} \phi(x)^4\right) d\mu_C(\phi) = (-1)^n \sum_{\mathcal{G}} n(\mathcal{G}) A(\mathcal{G})$$

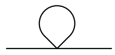
where $\mathcal{G}$ run through $(2m, n)$-graphs, $n(\mathcal{G})$ is the number of pairings giving that $\mathcal{G}$ and the *amplitude* $A(\mathcal{G})$ corresponding to the graph $\mathcal{G}$ is the following expression:

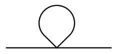
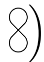
1. Label the n vertices of $\mathcal{G}$ as $z_1, \cdots, z_n$
2. To each line $\ell = \{u, v\}$ of $\mathcal{G}$ put $C_\ell = C(u - v)$.

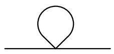
$$A(\mathcal{G}) = \sum_{z_1,\cdots,z_n\in\Lambda} \prod_{\ell} C_\ell. \tag{10.4}$$

We denote for short A  by .

Example. We get to $\mathcal{O}(\lambda)$:

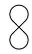
$G_2 =$ [— $-\lambda\cdot 4\cdot 3$  $-\lambda\cdot 3(-$  $)+\mathcal{O}(\lambda^2)$] / [$1-\lambda\cdot 3$  $+(\lambda^2)$]

$=$ — -12λ  $-3\lambda\Big(-$  $\Big)+3\lambda\Big(-$  $\Big)+\mathcal{O}(\lambda^2)$

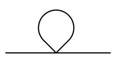
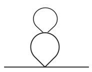
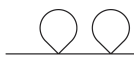
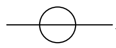
$= C - 12\lambda$  $+\mathcal{O}(\lambda^2)$

$$= C(x-y) - 12\lambda\sum_{z_1} C(x-z_1)C(z_1-z_1)C(z_1-y) + \mathcal{O}(\lambda^2)$$

$$= C(x-y) - 12\lambda C(0)\sum_{z_1\in\Lambda} C(x-z_1)C(z_1-y) + \mathcal{O}(\lambda^2)$$

Remark. Note the presence of *disconnected* graphs like —  that are cancelled upon normalization.

Homework : Prove :

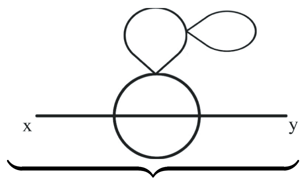
$G_2 =$ — -12λ  $+\lambda^2(\alpha$  $+\beta$  $+\gamma$  $)+\mathcal{O}(\lambda^3)$

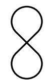
and find α, β, γ

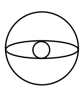
Theorem 10.2

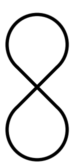
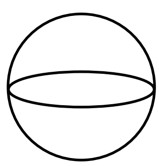
$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} \frac{\displaystyle\int \phi(x)\phi(y)e^{-\lambda\sum\phi_x^4(x)}d\mu_C}{\displaystyle\int e^{-\lambda\sum\phi^4(x)}d\mu_C} = (-)^n \sum_{\mathcal{G}}{}^{c}\, n(\mathcal{G})A(\mathcal{G})$$

where the sum $\sum{}^{c}$ runs over *connected* graphs.

Remark. In the numerator the graphs are of the type 

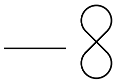
$\cdot$  $\cdot$



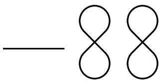
In the denominator  $\cdot$  $\cdot$ 

I.e. connected components have either both x and y or neither.

Note that disconnected graphs are here proportional to a power of the volume :



$$= C(x-y)\sum_{z\in\Lambda} C(0)^2 = |\Lambda| C(0)^2 C(x-y)$$



$$= C(x-y)C(0)^4|\Lambda|^2 \text{ etc}$$

Proof of the Theorem. Notation :

$$\langle F\rangle_\lambda = \frac{\int F(\phi)e^{-\lambda\sum\phi^4}d\mu_C}{\int e^{-\lambda\sum\phi^4}d\mu_C}$$

so $\langle F\rangle_0 = \int F d\mu_C$.
Then,

$$\frac{d}{d\lambda}\langle F\rangle_\lambda = -\lambda\sum_x\langle\phi(x)^4 F\rangle_\lambda + \lambda\sum\langle\phi(x)^4\rangle_\lambda\langle F\rangle_\lambda$$

i.e. $\frac{d}{d\lambda}\langle F\rangle_\lambda = -\lambda\sum_x\langle\phi(x)^4; F\rangle_\lambda$ where we denote by

$$\langle F; G\rangle_\lambda = \langle FG\rangle_\lambda - \langle F\rangle_\lambda\langle G\rangle_\lambda$$

the so called *truncated* (or connected) correlation functions.

Lemma 10.3 Let $F = \prod_{\alpha\in A}\phi(x_\alpha), G = \prod_{\beta\in B}\phi(x_\beta)$.
Then $\langle F; G\rangle_0 = \sum_P\prod_{\{\gamma,\delta\}\in P} C(x_\gamma - x_\delta)$ where each pairing P has at least one pair $\{\gamma,\delta\}$ such that $\gamma\in A, \delta\in B$.

Proof Obvious, since $\langle F\rangle\langle G\rangle$ has those pairings where no such pairs occur and $\langle FG\rangle$ has all pairings. □
Now (Prove !)

$$\frac{d^n}{d\lambda^n}\Big|_{\lambda=0}\langle F\rangle_\lambda = (-1)^n\sum_{x_1\cdots x_n}\langle F;\phi(x_1)^4;\phi(x_2)^4;\cdots;\phi(x_n)^4\rangle_0$$

where

$$\langle F_1; F_2;\cdots;F_N\rangle = \sum_\pi(-1)^{|\pi|-1}(|\pi|-1)!\prod_\alpha\langle\prod_{i\in\pi_\alpha}F\rangle_0$$

where $\pi = \{\pi_\alpha\}_{\alpha=1}^{|\pi|}$ is a partition of $\{1,\cdots,N\}$ into $|\pi|$ subsets.

Example. $\langle F_1; F_2; F_3\rangle = \langle F_1F_2F_3\rangle - \langle F_1F_2\rangle\langle F_3\rangle - \langle F_1F_3\rangle\langle F_2\rangle - \langle F_2F_3\rangle\langle F_1\rangle + 2\langle F_1\rangle\langle F_2\rangle\langle F_3\rangle$

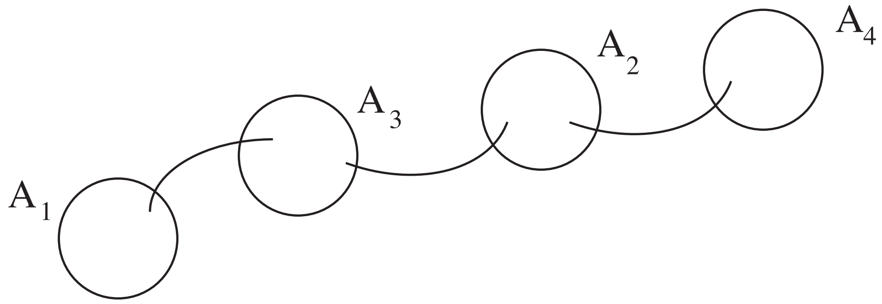
Now the thing to check is

Lemma 10.4 Let $F_i = \prod_{\alpha\in A_i} \phi(x_{i_\alpha})$. Then

$$\langle F_1; \cdots; F_N\rangle_0 = \sum_P \prod_{\{\gamma,\delta\}\in P} C(x_\gamma - x_\delta)$$

where for each partition into pairs P, there is a $\{\{\gamma_i, \delta_i\}\}_{i=1}^{N-1} \subset P$ that forms a connected path, connecting the A_i's, i.e. there exists a permutation ρ of the A_i's such that

$$\gamma_1 \in A_{\rho(1)}, \delta_1 \in A_{\rho(2)}, \gamma_2 \in A_{\rho(2)}, \delta_2 \in A_{\rho(3)} \cdots \delta_{N-1} \in A_{\rho(N)}.$$



Ex

The Lemma yields our theorem since there cannot be a connected graph with only one leg. $\square$

Problems. 1. Let $G_4(x_1 \cdots x_4) = \langle\prod_{i=1}^4 \phi(x_i)\rangle$ and define the *connected four point function* $G_4^c(x_1 \cdots x_4) = \langle\phi(x_1); \ldots; \phi(x_4)\rangle$. Prove that

$$\begin{aligned} G_4(x_1 \cdots x_4) &= G_2(x_1 - x_2)G_2(x_3 - x_4) + G_2(x_1 - x_3)G_2(x_2 - x_4) + G_2(x_1 - x_4)G_2(x_2 - x_3) \\ &\quad + G_4^c(x_1, \cdots, x_4) \end{aligned}$$

Show that G_4^c has an expansion $\sum_{n=1}^N \lambda^n G_{4,n}^c + \mathcal{O}(\lambda^{N+1})$ with $G_{4,n}^c$ consisting of *connected* graphs with 4 legs.

2. Generalize 1. to the *connected N-point function*

$$G_N^c(x_1 \cdots x_N) = \langle\phi(x_1); \ldots; \phi(x_N)\rangle$$

by showing that

$$G_N(x_1 \cdots x_N) = \sum_\pi \prod_{I\in\pi} G_{|I|}^c(x_I)$$

and that the perturbation expansion of G_N^c consists of connected graphs with N legs.

3. Let

$$Z(J) := \langle e^{(\phi,J)} \rangle.$$

Hence

$$G_N(x_1,\ldots,x_N) := \langle \prod_{i=1}^N \phi(x_i) \rangle = (\prod_{i=1}^N \frac{\partial}{\partial J(x_i)})|_{J=0} Z(J).$$

Define $F(J) := \log Z(J)$. Show

$$G_N^c(x_1,\ldots,x_N) = (\prod_{i=1}^N \frac{\partial}{\partial J(x_i)})|_{J=0} W(J).$$

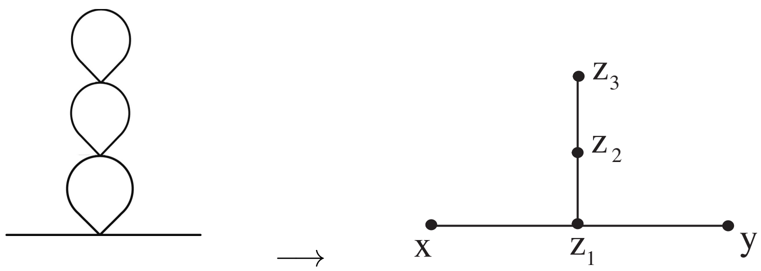
4. Prove that $\frac{d^n}{d\lambda^n}\Big|_0 \log Z$ is sum of connected *vacuum graphs* with n vertices. (a vacuum graph is a graph with no legs).

How to calculate the amplitude $A(\mathcal{G})$ of a given graph $\mathcal{G}$? First, let us observe that in our model $G_{2m,n}$ has a $\Lambda \to \mathbb{Z}^d$ limit : Clearly it suffices to consider a connected graph which is an expression (let say $m = 1$)

$$\sum_{z_1 \cdots z_n \in \Lambda} A(x_1, x_2, z_1 \cdots z_n)$$

and A is a product of $C(y_i - y_j)$ where the y's are x's or z's. Use $|C(x)| \leq$ const. $e^{-|x|/\xi}$, and note that, since $\mathcal{G}$ is connected, there exists a connected tree graph in $\mathcal{G}$ containing all the vertices and all the legs (a tree graph has no loops).

Example.



Now do the z_i sums using $|\sum_{z_j \in \Lambda} C(z_i - z_j)| \leq$ const (independent on Λ) starting at ends of branches (e.g. z_3 above).

10.3 Momentum space representation

Actual calculations are easier in Fourier transform. In order not to worry about analysis let us first work in finite volume $\Lambda = \mathbb{Z}_L^d$. Recalling from Section 7.2. we have for $p \in B_L$ (defined in (7.3))

$$\hat{f}(p) = \sum_{x \in \Lambda_L} e^{-ipx} f(x).$$

and the inverse formula

$$f(x) = L^{-d} \sum_{p \in B_L} e^{ipx} \hat{f}(p) \equiv \int_L dp e^{ipx} \hat{f}(p)$$

where $\int_L dp$ is a convenient shorthand for the Riemann sum, converging as $L \to \infty$ to $\int_{[-\pi,\pi]^d} dp$ where dp is the normalized Lebesgue measure $\prod_i \frac{dp_i}{2\pi}$. We have

$$\int_L dp e^{ip(x-y)} = \delta_{xy} \quad \text{for} \quad x, y \in \Lambda_L \tag{10.5}$$

and

$$\sum_{x \in \Lambda_L} e^{i(p-q)x} = L^d \delta_{pq} \equiv (2\pi)^d \delta_L(p-q) \qquad p, q \in B_L. \tag{10.6}$$

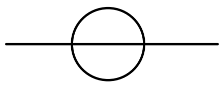
where we defined the discrete delta function $\delta_L(p-q)$. Products work out as

$$(fg)\hat{}(p) = \hat{f} * \hat{g} := \int_L dq \hat{f}(p-q) \hat{g}(q)$$

and

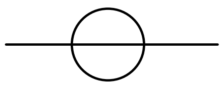
$$(f * g)\hat{} = \hat{f}\hat{g}.$$

Example.



$$= \sum_{z_1 z_2} C(x-z_1) C(z_1-z_2)^3 C(z_2-y) = (C * C^3 * C)(x-y)$$

Hence using above rules

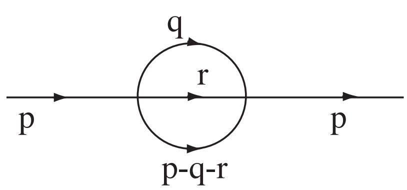


$$= G(x-y) = \int_L dp \hat{G}(p) e^{ip(x-y)}$$

with

$$\hat{G}(p) = \left(\int_L dq \int_L dr \hat{C}(q) \hat{C}(r) \hat{C}(p-q-r) \right) \hat{C}(p)^2$$

or pictorially

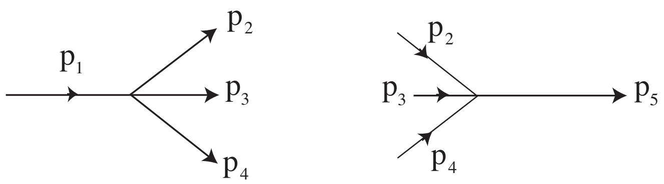


A more useful way to derive the same result is to insert for each C its Fourier representation:

$$= \sum_{z_1 z_2} \int_L e^{i[p_1(x-z_1)+p_2(z_1-z_2)+p_3(z_1-z_2)+p_4(z_1-z_2)+p_5(z_2-y)]} \prod_{i=1}^{5} \hat{C}(p_i) dp_i$$

$$= \int (2\pi)^d \delta(-p_1+p_2+p_3+p_4)(2\pi)^d \delta(-p_2-p_3-p_4+p_5) e^{ip_1 x - ip_5 y} \prod_{i=1}^{5} \hat{C}(p_i) dp_i$$

The first delta-function is due to the sum over z_1 and the second to the sum over z_2. Note "momentum conservation" at vertices.



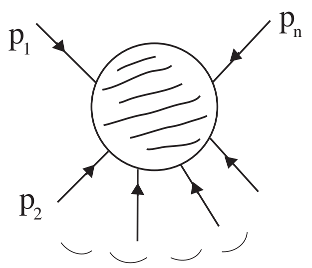
$$p_1 = \sum_{i=2}^{4} p_i \quad , \quad p_5 = \sum_{i=2}^{4} p_i$$

Solving for the constraints we get again

$$= \int_L dp_1 dp_2 dp_3 \hat{C}(p_1)\hat{C}(p_2)\hat{C}(p_3)\hat{C}(p_1 - p_2 - p_3)\hat{C}(p_1)e^{ip_1(x-y)}$$

using $p_5 = p_1$, $p_4 = p_1 - p_2 - p_3$. These expressions now have the obvious limits as $L \to \infty$ with $\int_L$ replaced by $\int$.

The general graph is now obvious :



1) But momentum p_i to legs, such that $\sum p_i = 0$

Reason : $G(x_1, \cdots, x_n)$ is translation invariant :

$$G(x_1, \cdots, x_n) = G(x_1 + y, x_2 + y, \cdots, x_n + y)$$

for all y so

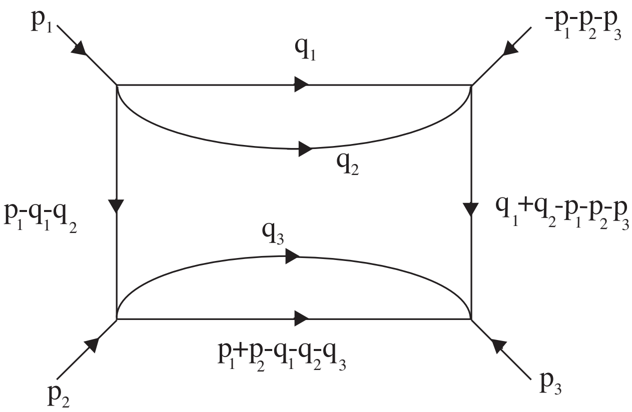
$$\begin{aligned}
\hat{G}(p_1, \cdots, p_n) &= \sum_{x_1,\ldots,x_n} G(x_1, x_2, \cdots, x_n)e^{i\sum_i p_i x_i} \\
&= \sum_{x_1,\ldots,x_n} G(0, x_2 - x_1, \cdots, x_n - x_1)e^{i\sum_i p_i x_i} \\
&= \sum_{x_1,\ldots,x_n} G(0, x_2, \cdots, x_n)e^{i\sum_{i=2}^{n} p_i x_i}e^{ix_1\sum_i p_i} \\
&= \hat{g}(p_2, \ldots, p_n)(2\pi)^d \delta_L(\sum p_i)
\end{aligned}$$

where $g(x_2, \ldots, x_n) = G(0, x_2 \cdots, x_n)$.

2) Satisfy momentum conservation at each vertex (coming from the delta functions).

3) Integrate over each independent internal momentum.

Example.



$$\hat{C}(p_1)\hat{C}(-p_1-p_2-p_3)\hat{C}(p_2)\hat{C}(p_3)$$
$$\cdot \int \prod_{i=1}^{3} dq_i \hat{C}(q_1)\hat{C}(q_2)\hat{C}(q_3)$$
$$\cdot \hat{C}(p_1-q_1-q_2)\hat{C}(q_1+q_2-p_1-p_2-p_3)$$
$$\cdot \hat{C}(p_1+p_2-q_1-q_2-q_3) \equiv \Gamma(p_1,p_2,p_3)$$

and in x space the graph is

$$G(x_1,\cdots,x_4) = \int e^{i(p_1(x_1-x_4)+p_2(x_2-x_4)+p_3(x_3-x_4))}\Gamma(p_1,p_2,p_3)\prod_{i=1}^{3}\frac{dp_i}{(2\pi)^d}$$

Remark 1. We have seen so far that :
a) If $|\Lambda| < \infty$, the functions G_{2m} are C^∞ in λ
b) The Taylor coefficients $G^{(\Lambda)}_{2m,n}$ have a $\Lambda \to \mathbb{Z}^d$ limit.

It can be shown that if $r > 0$ and λ is small enough the functions G_{2m} have a $\Lambda \to \mathbb{Z}^d$ limit, they are C^∞ in λ and their Taylor coefficients are the $\lim\limits_{\Lambda\to\mathbb{Z}^d} G^{(\Lambda)}_{2m,n}$.

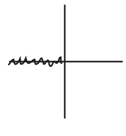
Remark 2 The expansion is *not* convergent. Consider e.g. a lattice consisting of one point i.e. the integral

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\varphi^4 - \frac{r}{2}\varphi^2} d\varphi$$

Then

$$\alpha_n = \frac{d^n F}{d\lambda^n}\Big|_{\lambda=0} = (-)^n \int \varphi^{4n} e^{-\frac{r}{2}\varphi^2} d\varphi = r^{-2n}(-1)^n\sqrt{2\pi}\frac{(4n)!}{2^{2n}(2n)!} \sim C^n(2n)!$$

as $n \to \infty$. Thus the Taylor series $\sum \frac{1}{n!}\alpha_n\lambda^n$ has $\frac{\alpha_n}{n!} \sim C^n n!$ i.e. is very badly divergent.

Homework. Prove that $F(\lambda)$ is analytic in the region $\mathbf{C}\backslash\{\lambda \in \mathbb{R}, \lambda \leq 0\}$ 

and has an essential singularity at $\lambda = 0$.

11 Infrared and Ultraviolet Divergencies

11.1 Infrared

Recall that for $\lambda = 0$, the critical point of our model is at $r = 0$. What happens for $\lambda > 0$? The following is expected to be true :

For $\lambda > 0$, there exists $r_c(\lambda)$ such that for $r > r_c(\lambda)$

$$0 \le \langle \phi(x)\phi(y) \rangle \le A e^{-|x|/\xi} \quad , \xi < \infty \quad , \quad \xi(r) \xrightarrow[r \downarrow r_c(\lambda)]{} \infty .$$

For $r < r_c(\lambda)$ there are two phases, where $\langle \phi(x) \rangle = \pm\, m \neq 0$ and

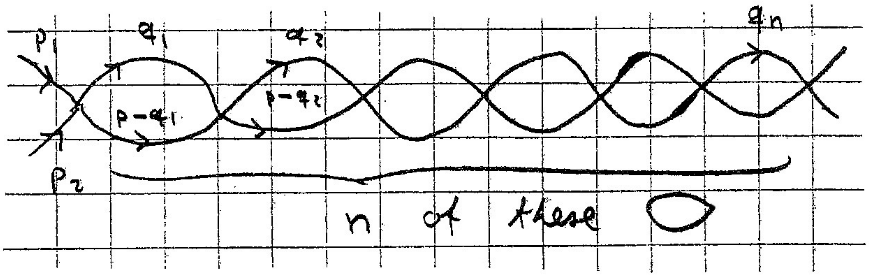
$$|\langle \phi(x); \phi(y) \rangle| \le A e^{-|x|/\xi} \quad \xi < \infty,$$

with $\xi(r) \to \infty$ as $r \uparrow r_c(\lambda)$.
For $r = r_c(\lambda)$,

$$\lim_{|x-y| \to \infty} \langle \phi(x)\phi(y) \rangle |x-y|^{d-2+\eta} \neq 0$$

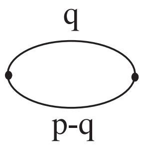
where η is independent of $\lambda > 0$, it depends on the dimension d, $\eta = 0$ if $d \ge 4$, and it equals the η of the Ising model. These claims are proven, for small λ and $d \ge 4$. The rest remains a conjecture, although with plenty of theoretical and numerical evidence.

Let us see how these facts are reflected in the behavior of the perturbation theory when $r = 0$. Let us consider the momentum space expressions for the graphs at $r = 0$. Since all integrals are over a bounded region $[-\pi, \pi]^d$ the only problem could come from $p = 0$ where $\mu(p) = 2\sum_{i=1}^{d}(1 - \cos p_i) = p^2 + \mathcal{O}(p^4)$ vanishes. Consider the following graph



$$p = p_1 + p_2$$

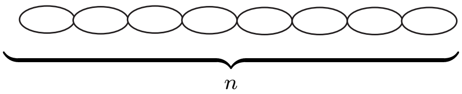
Now

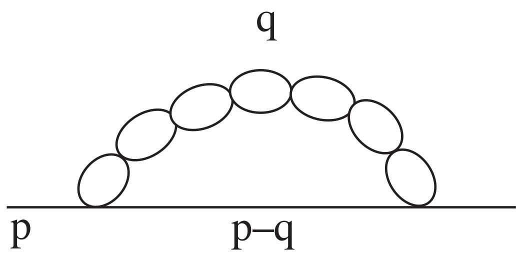


$$= \int_{[-\pi,\pi]^d} \frac{d^d q}{(2\pi)^d} \frac{1}{\mu(q)} \frac{1}{\mu(p-q)} \equiv I(p)$$

This is integrable in $d > 2$ if $p \neq 0$ and behaves as

$$I(p) \sim_{p \to 0} \begin{cases} |p|^{d-4} & d < 4 \\ \log |p| & d = 4 \\ \text{constant} & d > 4 \end{cases}$$

Thus  $= I(p)^n \sim |p|^{n(d-4)}$ which is not integrable if $n(4-d) \geq d$ (i.e. for $d = 3$ if $n \geq 3$). Hence e.g. the graph



has amplitude

$$\int \frac{I(q)^n}{\mu(p-q)} \frac{d^d q}{(2\pi)^d}$$

which is ill-defined for $d < 4$ and n large.

Conclusion. The individual terms of our perturbation expansion

$$G_m = \sum \lambda^n G_{m,n}$$

$G_{m,n} = G_{m,n}(x_1 \cdots x_m, r)$ have no limit as $r \to 0$ if $d < 4$ (at least some of them). For $d \geq 4$ it may be shown that this limit exists. Hence we expect to need radically new ideas for $d < 4$. The $p \sim 0$ divergencies are called **IR** (infrared) divergencies.

11.2 Ultraviolet

Consider now field theory. Let $d\mu_G(\varphi)$ be the Gaussian measure on $S'(\mathbb{R}^d)$ with covariance $G = (-\Delta + m^2)^{-1}$. We would like to consider the continuum limit of the Ginzburg-Landau model i.e. to start with finite volume $\Lambda \subset \mathbb{Z}^d$ we would consider the measure

$$e^{-\lambda \int_\Lambda \varphi(x)^4 d^d x} d\mu_G(\varphi).$$

We immediately run into trouble: $\varphi(x)^4$, as we saw, is not a well defined random variable. Indeed, $\int \varphi(x)^4 d\mu_G = \infty$. Actually, we only know $\varphi(f) \equiv \int \varphi(x) f(x) dx$ for $f \in S(\mathbb{R}^d)$ is a measurable function on S' and has finite moments $\int \varphi(f)^n d\mu < \infty$, $\forall n$.

One way to proceed is to *regularize* the theory. Recall that $\int \varphi(x)^4 d\mu_G = 3G(0)^2$ and $G(0) = \int_{\mathbb{R}^d} \hat{G}(p) \frac{d^d p}{(2\pi)^d}$. For us, $\hat{G}(p) = \frac{1}{p^2+m^2}$ so the divergence is due to insufficient decay of $\hat{G}(p)$ as $|p| \to \infty$ i.e. **UV** (ultraviolet) divergence. Thus, let us replace $\hat{G}(p)$ by

$$\hat{G}_\epsilon(p) = \hat{G}(p)\chi(\epsilon p) \equiv \hat{G}(p)\chi_\epsilon(p)$$

where $\chi \in S(\mathbb{R}^d)$ cuts of large $|p|$. We demand $\chi(0) = 1$ and $\chi \geq 0$. E.g. $\chi(p) = e^{-p^2}$ is a good choise. As $\epsilon \to 0, \hat{G}_\epsilon(p) \to \hat{G}(p)$ pointwise. Now $\varphi(x)$ is a well defined random variable :

$$\int \varphi(x)^{2n} d\mu_{G_\epsilon} = \frac{(2n)!}{2^n n!} G_\epsilon(0)^n \quad , \quad G_\epsilon(0) = \int \hat{G}(p)\chi(\epsilon p)\frac{d^d p}{(2\pi)^d} < \infty$$

[To be pedantic : a priori we know only that $\varphi(f)$ is measurable, for $f \in S$, so consider $\varphi(f_{\epsilon,x}), f_{\epsilon,x}(y) = \dfrac{e^{-\frac{(x-y)^2}{2\epsilon}}}{(2\pi\epsilon)^{d/2}}$. This is a measurable function $\ell_\epsilon : S' \to \mathbb{R} : \varphi \to \varphi(f_{\epsilon,x})$. Thus $\lim_{\epsilon\to 0} \ell_\epsilon$ also is measurable and this is what we call $\varphi(x)$ (formally : $\varphi(f_{\epsilon,x}) = \int \varphi(y) f_{\epsilon,x}(y)dy \xrightarrow[\epsilon\to 0]{} \int \varphi(y)\delta(x-y)dy = \varphi(x)$)].
Hence, consider the measure

$$\frac{1}{Z_{\Lambda,\epsilon}} e^{-\int_\Lambda [a\varphi(x)^2 + \lambda\varphi(x)^4] d^d x} d\mu_{G_\epsilon}(\varphi) \tag{11.1}$$

where we added for later purpose also a quadratic term to the Hamiltonian. This measure is well defined provided $a \in \mathbb{R}, \lambda > 0$. Indeed, as before by Jensen's inequality we get

$$Z \geq \exp\left[-|\Lambda|(aG_\epsilon(0) + 3\lambda G_\epsilon(0)^2)\right]$$

and since $a\varphi^2 + \lambda\varphi^4 \geq -\frac{a^2}{4\lambda}$, $Z \leq e^{+|\Lambda|\frac{a^2}{4\lambda}}$.

The correlation functions are again C^∞ in λ and a, and we have the expansion

$$G_{2k} = \sum_{n,p=0}^{N} \lambda^n a^p G_{2k,n,p} + R_N \qquad R_N = \mathcal{O}(\lambda^{N+1}, a^{N+1})$$

given in terms of graphs. The coefficients, $G_{2k,n,p}$ for $m^2 > 0$, have a $|\Lambda| \to \mathbb{R}^d$ limit but *not* an $\epsilon \to 0$ limit, as we saw above.

The question of continuum limit will not arise in statistical mechanics where the lattice spacing ϵ is a fixed nonzero quantity. However, in quantum field theory this question arises. In this case there is no fundamental length and one would like to have a quantum theory of continuum fields. Moreover these fields depend also on time and are defined on the four dimensional space time $\mathbb{R}^4$. Brushing aside for the moment the fact that this space time carries the Minkowski metric instead of the euclidean one we have been considering and that in quantum mechanics the correlation functions of fields are replaced by expectation values field operators in Hilbert space states it turns out that in case of a scalar field theory one needs to address precisely the question of $\epsilon \to 0$ limit as above.

The problem of divergences of the coefficients of perturbation theory arose in the 1930's when physicists were studying Quantum Electrodynamics (QED), the quantum theory of electromagnetic field interacting with electrically charged matter. In this theory there are two sorts of fields, the electromagnetic field $A(x)$ which is resembles our $\phi(x)$ but instead takes values in $A(x) \in \mathbb{R}^4$. $A(x)$ is the four dimensional electromagnetic vector potential. It is a gaussian field with zero mass, thus resembling our $r = 0$ free field. The second field describes electrons and has non zero mass. The interaction term in the Hamiltonian makes the theory non gaussian; the analogue of our parameter λ is played by the electron charge e. One can then proceed to derive a perturbation series as above in powers of e and as above the individual terms are ill defined, diverging due to small scale (large momentum) behavior of the integrands.

One would like to think about the electron charge (and its mass) as given physical constants that enter the Hamiltonian describing the dynamics. The divergence of the perturbation theory indicates that this point of view is incorrect. Rather one should think about these parameters *depending on scale*. Thus physicists introduced in the 50's the idea of *renormalization*. We should think about the parameters entering ϵ-cutoff theory describing the charge and mass of the electron in that scale. These can very well be different from the ones we measure that are interpreted as the charge and mass of the electron in the measurement scale. Once we have introduced the renormalization group we will make this picture more precise. For the time being we just remark that the parameters a and λ entering the measure (11.1) should be allowed to depend on the scale ϵ. Then the $\epsilon \to 0$ limit question can be posed as follows.

Question. Find three functions $Z(\epsilon), a(\epsilon), \lambda(\epsilon)$ such that

$$Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(x_i) \rangle_\epsilon$$

converge, as $\epsilon \to 0$, to elements of $S'(\mathbb{R}^{2md})$ (or $Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(f_i) \rangle_\epsilon$, converge for all $f_i \in S(\mathbb{R}^d)$). Here $\langle\ \rangle_\epsilon$ is the expectation w.r.t. the limit of the measure (11.1), as $\Lambda \to \mathbb{R}^d$, $a = a(\epsilon)$ and $\lambda = \lambda(\epsilon)$.

Remarks. 1. We allowed (it turns out this often is necessary) for a third parameter that renormalizes the value of the field: $\varphi(x_i)$ is replaced by $Z(\epsilon)^{\frac{1}{2}} \varphi(x_i)$.

Obviously $\lambda(\epsilon) = 0, a(\epsilon) = 0, Z = 1$ is a solution. We ask if there are any limits that are *non-Gaussian*. The answer is the following:

For $d = 2, Z = 1, a = -6\lambda G_\epsilon(0), \lambda(\epsilon) = \lambda$ works.

For $d = 3, Z = 1, a = -6\lambda G_\epsilon(0) + \alpha\lambda^2 \log \epsilon$ (α explicit), $\lambda(\epsilon) = \lambda$ works.

For $d \geq 4$ there is no non-Gaussian limit.

Our objective is to explain these claims.

Remark. χ_ϵ is a regulator. It cuts off momentum $|p| \gtrsim \frac{1}{\epsilon}$ very effectively. We could also use a lattice-cutoff. Thus consider the following Hamiltonian on fields defined on $(\epsilon\mathbb{Z})^d$:

Let $\varphi : (\epsilon\mathbb{Z})^d \to \mathbb{R}$ and set

$$\mathcal{H} = \tfrac{1}{2} \sum_{|x-y|=\epsilon} \epsilon^d \frac{(\varphi(x)-\varphi(y))^2}{\epsilon^2} + \frac{r}{2}\sum_x \epsilon^d \varphi(x)^2 + \sum_x \epsilon^d (a(\epsilon)\varphi(x)^2 + \lambda(\epsilon)\varphi(x)^4). \quad (11.2)$$

Consider as above the renormalized correlations

$$G_\epsilon(x_1,\dots,x_n) := Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(x_i)\rangle_\epsilon .$$

As we saw before, setting

$$\phi(x) = \epsilon^{\frac{d-2}{2}} \varphi(\epsilon x) \quad , \quad x \in \mathbb{Z}^d \quad (11.3)$$

we have

$$G_\epsilon(x_1,\dots,x_n) = Z(\epsilon)^m \epsilon^{(2-d)m} G^{(\epsilon)}(x_1/\epsilon,\dots,x_n/\epsilon) \quad (11.4)$$

where

$$G^{(\epsilon)}(y_1,\dots,y_n) = \langle \prod_{i=1}^{2m} \phi(y_i)\rangle^{(\epsilon)}$$

and $\langle - \rangle^{(\epsilon)}$ is the expectation w.r.t. the measure

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{Z_{\Lambda,\epsilon}} \exp\left[-\sum_{x\in\Lambda} \epsilon^2 a(\epsilon)\phi(x)^2 + \epsilon^{4-d}\lambda(\epsilon)\phi(x)^4\right] d\mu_{C_\epsilon}(\phi)$$

where $d\mu_{C_\epsilon}$ is the Gaussian measure on unit lattice fields $\phi \in s'(\mathbb{Z}^d)$ with covariance $C_\epsilon = (-\Delta + \epsilon^2 r)^{-1}$.

Thus our **UV** problem $\epsilon \to 0$ is the same thing as staying on a fixed lattice, taking distances to ∞ (x_i/ϵ), and going to critical point $(\epsilon^2 r \to 0)$. It should now be obvious that both **IR** and **UV** problem have something to do with Statistical Mechanics at the critical point.

Remark. The ϵ-lattice theory can be interpreted as a particular cutoff of the continuum theory. Namely, the measure with $\mathcal{H}$ as in (11.2) with $a = \lambda = 0$ is the Gaussian measure on $\mathbf{s}'((\epsilon\mathbb{Z})^d)$ with covariance $C_\epsilon = (-\Delta_\epsilon + r)^{-1}$ where the latter is an operator on $\ell^2(\epsilon\mathbb{Z}^d)$ i.e. $(-\Delta_\epsilon\varphi)(x) = \sum_{|i|=1}[\varphi(x) - \varphi(x+\epsilon i)]\epsilon^{-2}$ for $\varphi : (\epsilon\mathbb{Z})^d \to \mathbb{R}$. Concretely

$$\frac{1}{-\Delta_\epsilon + r}(x,y) = \int_{[-\frac{\pi}{\epsilon},\frac{\pi}{\epsilon}]^d} \frac{e^{ip(x-y)}}{\epsilon^{-2}\mu(\epsilon p) + r} \frac{d^d p}{(2\pi)^d}$$

which shows that p is cut-off to $|p_i| \le \pi/\epsilon$. Similarily, instead of the lattice model we may consider the Statistical Mechanics where φ is on $\mathbb{R}^d$ but has the cutoff χ, i.e. the measure

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{Z_\Lambda} e^{-\lambda\int_\Lambda \varphi(x)^4 dx} d\mu_{G_1}(\varphi)$$

with $G_1(p) = \frac{1}{p^2+r}\chi(p)$.

12 The Renormalization Group

12.1 The Block-spin Transformation

Suppose that we have a Statistical Mechanics model at its critical point such that

$$| \langle \phi(x)\phi(y) \rangle - \frac{A}{|x-y|^a} |\leq \frac{B}{|x-y|^{a+\epsilon}} \tag{12.5}$$

(as $|x-y| > 0$) where $\epsilon > 0$. We are after a theory that explains the leading term in this asymptotics and the fact that this term (or the exponent a) is *universal* i.e. stays the same when the Hamiltonian is changed (at least under some changes). Thus suppose there is a class of Hamiltonians that satisfy (12.5). Then the details of these Hamiltonians affect the sub leading asymptotics i.e. the RHS of (12.5). We can get a special theory where the scale invariance satisfied by the leading term is exact by taking the *scaling limit*. Recall the relations (11.3) and (11.4) between cutoff quantum field theory and statistical mechanics. Let, for $L > 0$

$$\varphi_L(x) = L^{\frac{a}{2}}\phi(x/L). \tag{12.6}$$

Then if (12.5) is satisfied we get

$$\lim_{L\to\infty} \langle \varphi_L(x)\varphi_L(y)\rangle = \frac{A}{|x-y|^a}.$$

On the other hand φ_L can be seen as a cutoff $1/L$ field as it depends on $x \in (L^{-1}\mathbb{Z})^d$. Hence if the LHS is given by an expectation $\langle \varphi(x)\varphi(y)\rangle$ the field φ lives on $\mathbb{R}^d$. Thus we expect the limits, if they exist,

$$\lim_{L\to\infty} \langle \prod_i \varphi_L(x_i)\rangle$$

to be the correlation functions of a field theory. This field theory is called the scaling limit of our model. By definition it is scale invariant, e.g.

$$L^a\langle \varphi(Lx)\varphi(Ly)\rangle = \langle \varphi(x)\varphi(y)\rangle.$$

Universality then would mean that several Hamiltonians give rise to the same scaling limit.

The drawback of this formulation is that, first, the scaling limit is a different object than the one we started with. The latter one is a fixed (unit) lattice model whereas the former one is a continuum object. More importantly, we have not given any constructive approach to the study of the scaling limit: we still need to understand the statistical mechanics at the critical point, in particular we need to show (12.5). The Renormalization group addresses these two problems. First, it supplements scaling by another operation, *coarse graining*, that allows one to stay in the category of fixed lattice spacing theories. Second, it provides actually a tool to study the critical theory.

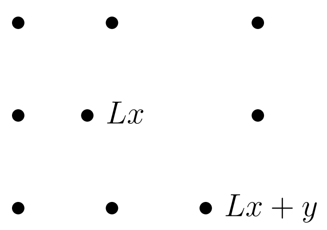
We replace the scale transformation (12.6) by

$$\phi_L(x) = L^{\frac{a}{2}} \phi^{average}(Lx) \tag{12.7}$$

where $\phi^{average}(Lx)$ is the average of ϕ in a L-sided cube centered at Lx:

$$\phi^{average}(Lx) = L^{-d} \sum_{y:|y_i|<\frac{L}{2}} \phi(Lx+y) \tag{12.8}$$

(12.7) and (12.8) define a map $\phi \in \mathbb{R}^{\mathbb{Z}^d} \to \phi_L \in \mathbb{R}^{\mathbb{Z}^d}$, i.e. the scaled field is also defined on $\mathbb{Z}^d$. We take for convenience $L > 1$ odd integer.



Here $L = 3$

We see that $\phi \to \phi_L$ involves :

a) "Coarse graining" : average over details of ϕ for scales $\leq L$.

b) Scaling : $\phi_L(x)$ depends on ϕ near Lx and we multiply ϕ by $L^{a/2}$

$\phi \to \phi_L$ is called the *Block-spin transformation.* (12.7) and (12.8) define a linear map $C_L : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$, concretely,

$$(C_L)_{xy} = L^{\frac{a}{2}-d} \begin{cases} 1 & |(Lx-y)_i| < L/2 \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$C_L^2 = C_{L^2}.$$

Thus, the iteration of (12.8) n times is the same as doing it once with L^n :

$$\begin{aligned} \phi_{L^n}(x) &= (C_{L^n}\phi)(x) = (C_L^n \phi)(x) \\ &= L^{n(\frac{a}{2}-d)} \sum_{|y_i|<L^n/2} \phi(L^n x + y) \end{aligned}$$

Now observe :

Proposition 12.1 Suppose that (12.5) holds. Then, for $\max_i |x_i - y_j| > 1$,

$$\langle (C_L^n\phi)(x)(C_L^n\phi)(y)\rangle \underset{n\to\infty}{\longrightarrow} AG^*(x,y)$$

with

$$G^*(x,y) = \int_\square du \int_\square dv |x-y+u-v|^{-a}$$

and this holds also for all x, y, if $a < d$. Here $\square = [-\frac{1}{2}, \frac{1}{2}]^d$.

Remark. G^* decays as $\frac{1}{|x-y|^a}$ as $|x-y| \to \infty$.

Proof. We get, for $x \neq y$,

$$\begin{aligned}
&| \langle C_L^n \phi(x) C_L^n \phi(y) \rangle - AL^{(a-2d)n} \sum_{\substack{u,v \\ |v_i|,|u_i|<L^n/2}} |L^n(x-y)+u-v|^{-a} \, | \\
&\leq BL^{(a-2d)n} \sum_{u,v} |L^n(x-y)+u-v|^{-a-\epsilon} \\
&= BL^{-n\epsilon} \sum_{u,v} L^{-2dn} |x-y+L^{-n}(u-v)|^{-a-\epsilon} \\
&\underset{n\to\infty}{\longrightarrow} 0 \cdot \int_\square du \int_\square dv |x-y+u-v|^{-a-\epsilon} = 0
\end{aligned}$$

and the A-term $\to A \int_\square du \int_\square dv |x-y+u-v|^{-a}$. We used $\max |x_i - y_i| > 1, |(u-v)_i| < 1 \Rightarrow |x-y+u-v| > 0$, so integrals converge. The rest is similar. $\square$

Example For the Gaussian Ginzburg-Landau model with covariance $\frac{1}{-\Delta}$ we take $a = d-2$ and get

$$\begin{aligned}
\langle C_L^n \phi(x) C_L^n \phi(y) \rangle &= L^{n(d-2)} L^{-2nd} \sum_{u,v} \int_{[-\pi,\pi]^d} \frac{d^d p}{(2\pi)^d} \frac{e^{ip(L^n x - L^n y + u - v)}}{\mu(p)} \\
&= L^{-2nd} \sum_{u,v} L^{-2n} \int_{[-L^n\pi, L^n\pi]^d} \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y+L^{-n}(u-v))}}{\mu(L^{-n}p)} \\
&\underset{n\to\infty}{\longrightarrow} \int_\square du \int_\square dv \left(\frac{1}{-\Delta_{\mathbb{R}^d}} \right) (x-y+u-v),
\end{aligned}$$

using $L^{2n}\mu(L^{-n}p) \to p^2$, as $n \to \infty$, and where $-\Delta_{\mathbb{R}^d}$ is the usual $-\Delta$ on $L^2(\mathbb{R}^d)$. Now,

$$\frac{1}{-\Delta_{\mathbb{R}^d}}(x-y) = \frac{\text{const}}{|x-y|^{d-2}}.$$

Remark. In this example, $C_L^n \phi(x)$ are Gaussian, with covariance $C_L^n \frac{1}{-\Delta} (C_L^n)^T$. T denotes the transpose, this is just what we write above, i.e.

$$C_L^n \frac{1}{-\Delta} (C_L^n)^T (x,y) = \sum_{z,w} (C_L^n)_{xz} (C_L^n)_{yw} \left(\frac{1}{-\Delta} \right)_{zw}.$$

How about generally ?

12.2 Transformations on measures

Suppose μ is a cylinder measure in $\mathbb{R}^{\mathbb{Z}^d}$ and consider the random variables $C_L\phi$. Their generating function is

$$\begin{aligned} W_L(f) &= \int e^{i(C_L\phi,f)}d\mu(\phi) = \int e^{i(\phi,C_L^Tf)}d\mu(\phi) \\ &= W(C_L^Tf) \end{aligned}$$

which is obviously of positive type, so there is a measure μ_L such that

$$W_L(f) = \int e^{i(\Psi,f)}d\mu_L(\Psi).$$

μ_L is the probability distribution of the block spins $C_L\phi$. Thus $C_L : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ induces a map

$$C_L^* : \mathcal{B} \to \mathcal{B}$$

where $\mathcal{B}$ are the cylinder measures on $\mathbf{s}'(\mathbb{Z}^d)$.
C_L^* is called the *Renormalization Group Transformation* on measures.

12.3 Transformations on Hamiltonians

Concretely, let us work in finite volume Λ_N and take $\Lambda_N = L^N$-box centered at the origin. Thus $\phi \in \mathbb{R}^{\Lambda_N}, C_L\phi \in \mathbb{R}^{\Lambda_{N-1}}$ and $C_L^n\phi \in \mathbb{R}^{\Lambda_{N-n}}$. Let μ be a probability measure

$$d\mu(\phi) = \frac{1}{Z}e^{-\mathcal{H}(\phi)}d^{\Lambda_N}\phi$$

when $\mathcal{H}$ is some function $\mathcal{H} : \mathbb{R}^{\Lambda_N} \to \mathbb{R}$ such that $e^{-\mathcal{H}}$ is integrable. Then,

$$d\mu_L(\Psi) = F(\Psi)d^{\Lambda_{N-1}}\Psi$$

with

$$F(\Psi) = \frac{1}{Z}\int e^{-\mathcal{H}(\phi)} \prod_{x\in\Lambda_{N-1}} \delta(\Psi(x) - (C_L\phi)(x))d^{\Lambda_N}\phi$$

(Z normalizes $\int F d^{\Lambda_{N-1}}\Psi = 1$).
Let us write this more explicitely : $\Psi(x)$ equals $L^{\frac{a}{2}}$ times the average of ϕ in the L-cube centered at Lx. Thus, if we fix $\Psi(x), \forall x \in \Lambda_{N-1}$, we need to integrate out all fluctuations around this average. In other words, let ϕ_0 be a configuration that is constant in each L-cube, and equals the average of ϕ there. I.e. $\phi_0(Lx + y) = L^{-\frac{a}{2}}\Psi(x)$ for all $x \in \Lambda_{N-1}$, all y with $|y_i| < L/2$. Using our previous notation this equals

$$\phi_0 = L^{-\frac{a}{2}}L^{-\frac{a}{2}+d}C^T\Psi = L^{d-a}C^T\Psi$$

(recall, $C^T_{xy} = L^{\frac{a}{2}-d}$ if $x \in L$-cube at Ly, and equals 0 otherwise). Thus $\phi = \phi_0 + Z$ with Z having 0 average over L-cubes i.e. $C_L Z = 0$. Thus

$$F(\Psi) = \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a} C^T \Psi + Z)} \prod_{x \in \Lambda_{N-1}} \delta\Big((C_L Z)(x) \Big) d^{\Lambda_N} Z$$

Even more concretely, take $z \in \mathbb{R}^{\Lambda_N \setminus L\Lambda_{N-1}}$ i.e. $z(x) \in \mathbb{R}$, $\forall x \in \Lambda_N$ where $x \neq Lw$, $w \in \mathbb{Z}^d$. Put

$$Z(x) = \begin{cases} z(x) & x \in \Lambda_N \setminus L\Lambda_{N-1} \\ -\sum\limits_{|y_i|<L/2} z(Lw+y) & x = Lw \end{cases}$$

Then $C_L Z = 0$ and so

$$F(\Psi) = \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a} C^T \Psi + Z)} d^{\Lambda_N \setminus L\Lambda_{N-1}} z$$

a concrete integral over "fluctuation variables" $z(x)$.
Let $F = e^{-\mathcal{H}'}$. The map $R_L : \mathcal{H} \to \mathcal{H}'$ is called the *RG Transformation on Hamiltonians*, i.e.

$$R_L \mathcal{H} = -\log \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a} C^T \Psi + Z)} dz \tag{12.9}$$

Here $\mathcal{H} : \mathbb{R}^{\Lambda_N} \to \mathbb{R}$ and $R_L \mathcal{H} : \mathbb{R}^{\Lambda_{N-1}} \to \mathbb{R}$. Note how the volume contracted from an L^N-box to an L^{N-1}-box.

Remarks. 1. It was trivial to define the RG on measures and in infinite volume : $C^*_L \mu$ is automatically a measure. On Hamiltonians, we need to
a) work in finite volume,
b) make some assumptions on $\mathcal{H}$ so that $R_L \mathcal{H}$ in (12.9) is well defined.

2. RG is called a "group" because $C^n_L = C_{L^n}$ i.e. $(C^*_L)^n = C^*_{L^n}$ and $(R_L)^n = R_{L^n}$. Actually, it is a semigroup : $R_1 = \mathrm{id}$ but R_L^{-1} is not defined.
Our Proposition 12.1 says that

$$\int (C^n_L \phi)(x)(C^n_L \phi)(y) d\mu \underset{n\to\infty}{\longrightarrow} G^*(x,y)$$

(absorbing the constant A in the definition of G^*), i.e.

$$\int \Psi(x)\Psi(y) d\mu_{L^n} \to G^*(x,y)$$

Thus, we might hope that the measures μ_{L^n} converge to some measure μ^* such that

$$\int \Psi(x)\Psi(y) d\mu^*(\Psi) = G^*(x,y)$$

and μ^* is a *fixed point* of C_L^* :

$$C_L^* \mu^* = \mu^*$$

Equivalently, provided we make sense of $\mathcal{H}$ in infinite volume, we might expect

$$(R_L)^n \mathcal{H} \underset{n\to\infty}{\longrightarrow} \mathcal{H}^*, \qquad R_L \mathcal{H}^* = \mathcal{H}^*$$

in some topology in a space of $\mathcal{H}$'s.

Remarks. 1. It is actually quite difficult to set up nice spaces of Hamiltonians for unbounded spins : things are very sensitive to the dependence of $\mathcal{H}$ on large values of $\phi(x)$'s : we need the $\phi(x)$ integral to converge at $\pm\infty$. This problem did not occur for bounded spins where we could define $\mathcal{H}$ via potentials Φ_X and talk about the Banach space $\mathcal{B}$ of Hamiltonians (i.e. of Φ_X's) : Given $\Phi \in \mathcal{B}$, we could define the set of Gibbs measures μ in infinite volume corresponding to Φ. Moreover, above the critical temperature, we expected Φ to uniquely determine a μ. For unbounded spins no similar formalism exists.

2. For bounded spins σ the RG, as defined above, will in general change the range of values that σ takes : e.g. for Ising spins $\sigma(x) \in \{\pm 1\}$ $C_L\sigma$ takes values in the set $\{\sigma \in L^{\frac{a}{2}-d}\mathbb{Z} \| \sigma| \leq L^{\frac{a}{2}}\}$ i.e. for $a > 0$, larger values. Upon iteration, $C_L^n \sigma(x) \in L^{(\frac{a}{2}-d)n}\mathbb{Z}$, $|C_L^n \sigma(x)| \leq L^{\frac{an}{2}}$. Thus, e.g. at $d = 3$, where we expect $a = d - 2 + \eta$ with $\eta \approx 0.1$, as $n \to \infty$ the block spins become *unbounded* and continuous! Anyway, for $n < \infty$, we have a bounded spin model and can ask whether R_L actually maps Hamiltonians to Hamiltonians. I.e. suppose μ is a Φ-Gibbs measure, $\Phi \in \mathcal{B}$. Is $C_L^* \mu$ a Gibbs measure for some Φ' ? If so, define $R_L \Phi = \Phi'$. This is expected to be true in Ising model, but proven only for small β or large β (where one needs to enlarge $\mathcal{B}$, surprisingly !).

3. For unbounded spins, R_L and its iterations have been rigorously studied for the Ginzburg-Landau model. It turns out that one can, to a certain extent set up a space of Hamiltonians where R_L acts; however, one has to supplement this with a different representation of μ for large values of ϕ's.

For the rest of the time, we will not discuss these problems. Rather we

a) Pretend that R_L is defined in some space of $\mathcal{H}$'s and see what implications this could have (Section 13).

b) Carry out a perturbative analysis of $R_L^n \mathcal{H}$ for $\mathcal{H}$ = Ginzburg-Landau model and understand the **IR** and **UV** problem this way (Sections 14-17).

13 R_L near a Fixed Point : Critical Exponents

13.1 General Framework

We assume the following setup

1. We have some space $\mathcal{K}$ of "Hamiltonians" $\mathcal{H}$ such that $\mathcal{H} \in \mathcal{K}$ determines a unique measure $\mu_{\mathcal{H}}$ on $\mathbb{R}^{\mathbb{Z}^d}$. We denote $\langle F\rangle_{\mathcal{H}} = \int F d\mu_{\mathcal{H}}$. Here $\mu_{\mathcal{H}}$ is the limit of $\frac{1}{Z_\Lambda} e^{-\mathcal{H}_\Lambda}\phi$ as $\Lambda \nearrow \mathbb{Z}^d$ and $\mathcal{H}_\Lambda$ is some finite volume version of $\mathcal{H}$ (or more properly, $\mathcal{H} = \{\mathcal{H}_\Lambda\}_{\Lambda\subset\mathbb{Z}^d}$).

2. The RG is defined in $\mathcal{K}$: $R_L : \mathcal{K} \to \mathcal{K}$ i.e.

$$C_L^* \mu_{\mathcal{H}} = \mu_{R_L\mathcal{H}}$$

We also suppose that there is a metric in $\mathcal{K}$ and if $\mathcal{H}_n \to \mathcal{H}$, then $\mu_{\mathcal{H}_n} \to \mu_{\mathcal{H}}$.
Let us see what kind of picture of critical phenomena emerges from such assumptions.
Let $\mathcal{H}^*$ be a fixed point of R_L : $R_L\mathcal{H}^* = \mathcal{H}^*$.

Definition 13.1 The stable manifold of $\mathcal{H}^*$ in $\mathcal{K}$ is

$$\mathcal{M}_s = \{\mathcal{H} \in \mathcal{K} \mid R_L^n\mathcal{H} \to \mathcal{H}^*\}$$

What can we say about the decay of correlations of $\mathcal{H} \in \mathcal{M}_s$? Let us define

Definition 13.2 $\mathcal{H} \in \mathcal{M}_s$ is *critical* if

$$\sup_{x,y} |\langle\phi(x)\phi(y)\rangle_{\mathcal{H}}| e^{|x-y|/\xi} = \infty$$

for all $\xi < \infty$.

Hence for critical $\mathcal{H}$ there are no $A > 0$, $\xi < \infty$ such that $|\langle\phi(x)\phi(y)\rangle_{\mathcal{H}} \leq Ae^{-|x-y|/\xi}$ for all x, y.

Recall a in the definition of C_L : $(C_L\phi)(x) = L^{\frac{a}{2}} L^{-d} \sum_{|y_i|<\frac{L}{2}} \phi(Lx + y)$. We have

Proposition 13.2 a) Suppose $a < d$. Then all $\mathcal{H} \in \mathcal{M}_s$ are critical.
b) If $a = d$ and $\mathcal{H} \in \mathcal{M}_s$ is not critical, then $\langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*} = 0$ for $x \neq y$.

Proof a) Suppose $\exists A, \xi$. Then,

$$
\begin{aligned}
0 \neq |\langle\phi(x_0)\phi(y_0)\rangle_{\mathcal{H}^*}| &= |\lim_{n\to\infty}\langle\phi(x_0)\phi(y_0)\rangle_{R^n\mathcal{H}}| \\
&= |\lim_{n\to\infty}\langle C_L^n\phi(x_0)C_L^n\phi(y_0)\rangle_{\mathcal{H}}| \\
&= |\lim_{n\to\infty} L^{n(a-2d)} \sum_{|u_i|<L^n/2}\ \sum_{|v_i|<L^n/2} \langle\phi(L^n x_0+u)\phi(L^n y_0+v)\rangle_{\mathcal{H}}| \\
&\leq A \lim_{n\to\infty} L^{n(a-2d)} \sum_{|u_i|<L^n/2}\ \sum_{|v_i|<L^n/2} \exp[-L^n|x_0-y_0+L^{-n}|u-v|]/\xi \\
&= A \lim_{n\to\infty} L^{na} \int_\square du \int_\square dv e^{-L^n|x_0-y_0+u-v]/\xi} \\
&\leq AC \lim_{n\to\infty} L^{n(a-d)} = 0 \Rightarrow \text{ contradiction}
\end{aligned}
$$

b) Now, if $x_0 \neq y_0$ this is $\leq AC \lim L^{n(a-d-1)} = 0$. $\square$

Remark. Also, if $\langle\phi(x_0)\phi(y_0)\rangle_{\mathcal{H}^*} \neq 0$ for $|x_0 - y_0| > 1$ get above $\leq \lim L^{na}e^{-L^n} = 0$. Thus, if $a < d$, both $\mathcal{H}$ and $\mathcal{H}^*$ are critical.
If $a = d$ our block spins are

$$\phi_L(x) = L^{-\frac{d}{2}} \sum \phi(Lx + y)$$

i.e. we normalize like independent random variables. The calculation above just showed that if $\phi(x)$'s are (exponentially) weakly dependent then $\phi_{L^n}(x)$ become *independent* as $n \to \infty$ (central limit theorem).
Thus $a < d$ is the interesting case.
[By the way, it can be proven quite generally that if $\langle\phi(x)\phi(y)\rangle$ decays as $|x-y|^{-a}, a > d$ then it decays exponentially].
Identical argument shows that it can not be the case that $\mathcal{H} \in \mathcal{M}_s$ $\langle\phi(x)\phi(y)\rangle_{\mathcal{H}} \leq A|x-y|^{-a'}$ with $a' > a$. Thus it is reasonable to expect:

Summary. $\mathcal{M}_s$ is a *critical surface* consisting of $\mathcal{H} \in \mathcal{K}$, all having the same critical exponent for the 2-point function.

Next, consider

$$\mathcal{M}_\xi = \{\mathcal{H}|\mathcal{H} \text{ has correlation length } \xi\}$$

For this we assume that all $\mathcal{H} \in K$ have a well defined ξ, i.e. the limit

$$\lim_{|x|\to\infty} -\frac{1}{|x|}\log\langle\phi(0)\phi(x)\rangle = \xi^{-1} \text{ exists}$$

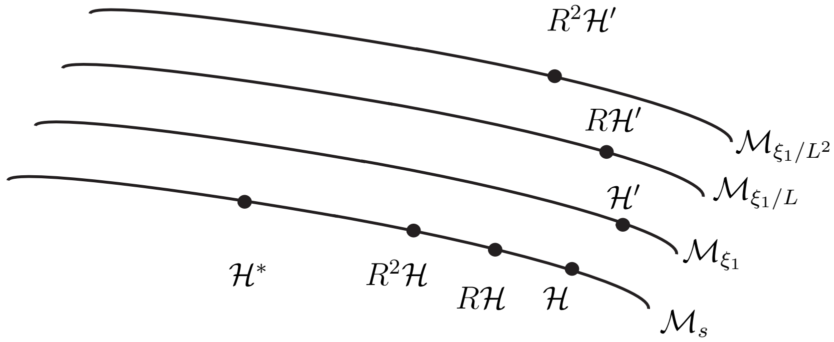
Then a) $\mathcal{M}_s = \mathcal{M}_\infty$

b) $R_L : \mathcal{M}_\xi \to \mathcal{M}_{\xi/L}$

b) follows since R_L scales by L:

$$\langle\phi(0)\phi(x)\rangle_{R_L\mathcal{H}} = L^{\alpha}\sum_{u,v}\langle\phi(u)\phi(Lx+v)\rangle L^{-2d} \sim e^{-L|x|/\xi}.$$

Thus, we have a picture of $\mathcal{K}$:



R_L takes :
– critical $\mathcal{H}$ to $\mathcal{H}^*$ upon iteration
– noncritical $\mathcal{H}$ away from $\mathcal{H}^*$ upon iteration.

13.2 Linear Analysis around Fixed Points

Let us study R_L near $\mathcal{H}^*$. We assume that the usual dynamical systems ideas are applicable i.e. that R_L is, say, a smooth map in $\mathcal{K}$[6]. Thus, let $\mathcal{L} = DR_{\mathcal{H}^*}$ be the derivative of R at $\mathcal{H}^*$. This means that, for any $\mathcal{H} \in \mathcal{K}$,

$$R(\mathcal{H}^* + \epsilon\mathcal{H}) = \mathcal{H}^* + \epsilon\mathcal{L}\mathcal{H} + \mathcal{O}(\epsilon^2).$$

The spectrum of $\mathcal{L}$ plays now a crucial rule in the analysis of R near $\mathcal{H}^*$.
In the known examples, the spectrum of $\mathcal{L}$ has the following structure :

a) It is discrete, consisting of, real, positive, eigenvalues λ_i of finite multiplicity.

b) There is a finite number of λ_i with $\lambda_i \geq 1$.

Let $\mathcal{H}_i$ be an eigenvector : $\mathcal{L}\mathcal{H}_i = \lambda_i\mathcal{H}_i$. We say that :

[6]If $\mathcal{K}$ is a Banach space, the derivative of R at $\mathcal{H}$ is defined as usual as the linear map $DR_{\mathcal{H}} : \mathcal{K} \to \mathcal{K}$:

$$DR_{\mathcal{H}}\mathcal{H}_1 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[R(\mathcal{H}+\epsilon\mathcal{H}_1) - R(\mathcal{H})].$$

1° $\mathcal{H}_i$ is *relevant* if $\lambda_i > 1$

2° $\mathcal{H}_i$ is *marginal* if $\lambda_i = 1$

3° $\mathcal{H}_i$ is *irrelevant* if $\lambda_i < 1$

Note that $DR^n_{\mathcal{H}^*} = D(R \circ R \circ R \cdots R)_{\mathcal{H}^*} = (DR_{\mathcal{H}^*})^n = \mathcal{L}^n$ (since $R(\mathcal{H}^*) = \mathcal{H}^*$), so

$$R^n(\mathcal{H}^* + \epsilon\mathcal{H}_i) = \mathcal{H}^* + \epsilon\lambda_i^n\mathcal{H}_i + \dots$$

i.e. the $\mathcal{H}_i$ perturbation increases ($\lambda_i > 1$) or decreases ($\lambda_i < 1$) exponentially in n or stays constant ($\lambda_i = 1$). Let us interpret this. Suppose first the case where there is only one $\lambda > 1$, with multiplicity 1, say λ_1. Then for $|x - y| >> 1$

$$\langle\phi(L^n x)\phi(L^n y)\rangle_{\mathcal{H}^*+\epsilon\mathcal{H}_1} \approx L^{-na}\langle C^n_L\phi(x)C^n_L\phi(y)\rangle_{\mathcal{H}^*+\epsilon\mathcal{H}_1}$$

$$= L^{-na}\langle\phi(x)\phi(y)\rangle_{R^n(\mathcal{H}^*+\epsilon\mathcal{H})} = L^{-na}\langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*+\epsilon\lambda_1^n\mathcal{H}_1+\mathcal{O}((\epsilon\lambda_1^n)^2)}.$$

Take n such that $\epsilon\lambda_1^n = \mathcal{O}(1)$. Recall that we had one direction in $\mathcal{K}$ where we *depart* from $\mathcal{H}^*$ upon iteration, namely the one parametrized by ξ. Thus, it is natural to assume that $\mathcal{H}^* + \epsilon\mathcal{H}_1$ has $\xi < \infty$ and $\mathcal{H}^* + \mathcal{O}(1)\mathcal{H}_1$ has $\xi = \mathcal{O}(1)$. From above, then

$$\begin{aligned}\xi(\mathcal{H}^* + \epsilon\mathcal{H}_1) &= L^n\xi(\mathcal{H}^* + \mathcal{O}(1)\mathcal{H}_1) \\ &= L^n\mathcal{O}(1).\end{aligned}$$

Write $\lambda_1 = L^\alpha$ (so $\alpha > 0$). Then $\epsilon \sim L^{-n\alpha} \Rightarrow \xi(\mathcal{H}^* + \epsilon\mathcal{H}_1) \sim \epsilon^{-1/\alpha}$.
Let us identify $\epsilon = 0$ with $T = T_c$ (T = temperature). Thus, our Hamiltonian $\mathcal{H}^*$ could be say Ising model $\beta_c\mathcal{H}_{Ising}$ and $\mathcal{H}^* + \epsilon\mathcal{H}_1$ would be $\beta\mathcal{H}_{Ising}$ with $(1+\epsilon)\beta^* = \beta$ or $\epsilon \sim T - T_c$. More generally, we could start with Ising and consider some effective long distance model (see below for precise definitions) for its correlations. Then again $\epsilon \sim T - T_c$. So then

$$\xi \sim (T - T_c)^{-1/\alpha}$$

and $-1/\alpha$ is the so called correlation length critical exponent describing how ξ diverges as $T \to T_c$[7].

Universality. Consider now $\mathcal{H} = \mathcal{H}^* + \epsilon\mathcal{H}_i, \lambda_i < 0$. As before,

$$\begin{aligned}L^{na}\langle C^n_L\phi(x)C^n_L\phi(y)\rangle_{\mathcal{H}} &= \langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*+\epsilon\lambda_i^n\mathcal{H}_i} \\ &= \langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*} + \mathcal{O}(\lambda_i^n)\end{aligned}$$

so, if $\lambda_i = L^{-\alpha_i}$, $\alpha_i > 0$, then, since $|u-v| \sim L^n \Rightarrow L^{-\alpha_i n} \sim |u-v|^{\alpha_i}$. Thus these directions give subleading corrections to the decay determined by $\mathcal{H}^*$. We have *universality* : the

[7]Note that we got $\langle\phi(x)\phi(y)\rangle_T \sim \frac{1}{|x-y|^a}e^{-\frac{|x-y|}{\xi(T)}}$ for all x, y.

leading assymptotics of $\langle\prod_{i=1}^{N}\phi(L^n x_i)\rangle_{\mathcal{H}}$ is $\langle\prod_{i=1}^{N}\phi(L^n x_i)\rangle_{\mathcal{H}^*}$ if $\mathcal{H}\in\mathcal{M}_s$ (as $n\to\infty$ and x_i are apart from each other) and the details of $\mathcal{H}$ are only seen in the $\mathcal{O}(L^{-\alpha_i n})$ corrections. We say that all $\mathcal{H}\in\mathcal{M}_s$ **have the same critical behaviour**.
Also, all the exponents α_i are determined by $\mathcal{H}^*$, which appear not to depend on which $\mathcal{H}$ we have (in $\mathcal{M}_s$).

14 The Gaussian Fixed Point

14.1 Definition of RG in Momentum Space

As we saw above, the RG drives upon iteration the Gaussian measure μ_C where $C=\frac{1}{-\Delta}$ (on $\mathbb{Z}^d$) to the fixed point

$$C^*(x-y)=\int_{\square}du\int_{\square}dv\int_{\mathbb{R}^d}e^{ip(x-y)}e^{ip(u-v)}\frac{1}{p^2}\frac{d^dp}{(2\pi)^d}\qquad x,y\in\mathbb{Z}^d$$

i.e.

$$\lim_{n\to\infty}\langle C_L^n\phi(x)C_L^n\phi(y)\rangle_{\mu_C}=C^*(x-y).$$

Since

$$\int_{\square}due^{ipu}=\prod_{\mu=1}^{d}\int_{-1/2}^{1/2}e^{ip_\mu u_\mu}du_\mu=\prod_{\mu=1}^{d}\frac{2\sin p_\mu/2}{p_\mu},$$

we have,

$$C^*(x-y)=\int_{\mathbb{R}^d}e^{ip(x-y)}\frac{1}{p^2}\prod_{\mu=1}^{d}\left(\frac{2\sin p_\mu/2}{p_\mu}\right)^2\frac{d^dp}{(2\pi)^d}$$

(note ! this converges absolutely !) and writing $\int_{\mathbb{R}}dp_\mu f(p_\mu)=\sum_{n\in\mathbb{Z}}\int_{[-\pi,\pi]}dp_\mu f(p_\mu+2\pi n)$, we have

$$\begin{aligned}C^*(x-y)&=\int_{[-\pi,\pi]^d}e^{ip(x-y)}\hat{C}^*(p)\frac{d^dp}{(2\pi)^d}\\ \hat{C}^*(p)&=\sum_{n\in\mathbb{Z}^d}\frac{1}{(p+2\pi n)^2}\prod_{\mu=1}^{d}\left(\frac{2\sin p_\mu/2}{p_\mu+2\pi n_\mu}\right)^2.\end{aligned}$$

This is the Gaussian fixed point of the block spin RG. We should now like to perform the linear analysis near C^*. We will do this in a slightly different model, where it is less messy (everything can be done in the case of block spin with identical conclusions).

We will consider, instead of a spin model on $\mathbb{Z}^d$ (lattice cutoff), a "spin model" on $\mathbb{R}^d$ where we use a momentum space cutoff. A priori this is as good from the physics point of view.

Thus, let $d\mu_G$ be the Gaussian measure on $S'(\mathbb{R}^d)$ with covariance

$$G(x-y) = \int_{\mathbb{R}^d} e^{ip(x-y)} \frac{\chi(p)}{p^2} \frac{d^d p}{(2\pi)^d}$$

where $d > 2$, $\chi(p)$ is a cutoff, say

$$\chi(p) = e^{-p^2}$$

but anything with $\chi(0) = 1$ and fast decay at ∞ will do (say $\chi \in S(\mathbb{R}^d)$). A priori we know that only smeared fields $\varphi(f), f \in S(\mathbb{R}^d)$ are integrable w.r.t. μ_G and

$$\int e^{i\varphi(f)} d\mu_G = e^{-\frac{1}{2}(f,Gf)}$$

so that e.g. $\langle \varphi(f)\varphi(g) \rangle = (f, Gg) = \int f(x)g(y)G(x-y)dxdy$. However, now $\varphi(x)$ itself makes sense (defined as $\lim_{\epsilon \to 0} \varphi(f_\epsilon)$ where $\lim_{\epsilon \to 0} f_\epsilon = \delta_x$ (e.g. $f_\epsilon(y) = (2\pi\epsilon)^{-d/2} e^{-\frac{1}{2\epsilon}(x-y)^2}$) and we have $\int \varphi(x)\varphi(y)d\mu_G = G(x-y)$ well defined, $\forall x, y$. (Technically, μ_G is supported on *distributions that are C^∞ functions*, see Homework).

Thus, let us consider the finite volume measures

$$d\nu_\Lambda(\varphi) = \frac{1}{Z_\Lambda} e^{-V_\Lambda(\varphi)} d\mu_G(\varphi)$$

where Z_Λ normalizes $\int d\nu_\Lambda = 1, V_\Lambda$ is a function of φ's localized in Λ, e.g., $V_\Lambda = \int_\Lambda dx P(\varphi(x))$ P e.g. a polynomial.

We now describe the RG in this setup. We need to split φ into "local" and "global" or "high momentum" and "low momentum" parts as in the block spin case. We do this by first splitting G : write, for $L > 1$,

$$\hat{G}(p) = \frac{1}{p^2}\chi(p) = \frac{1}{p^2}\chi(Lp) + \frac{1}{p^2}(\chi(p) - \chi(Lp)).$$

Or

$$\begin{aligned} G(x-y) &= \int_{\mathbb{R}^d} e^{ip(x-y)} \frac{1}{p^2}\chi(Lp)\frac{d^d p}{(2\pi)^d} + \int_{\mathbb{R}^d} e^{ip(x-y)} \frac{1}{p^2}(\chi(p) - \chi(Lp)) \\ &= L^{2-d} \int_{\mathbb{R}^d} e^{ip\frac{x-y}{L}} \frac{1}{p^2}\chi(p)\frac{d^d p}{(2\pi)^d} + \int_{\mathbb{R}^d} e^{ip(x-y)} \frac{1}{p^2}(\chi(p) - \chi(Lp)) \\ &= L^{2-d} G(\frac{x-y}{L}) + \Gamma(x-y) \end{aligned} \tag{14.1}$$

Here $\Gamma(x-y)$ is the "high momentum" or "fluctuation within blocks" part : $\chi(p) - \chi(Lp) = e^{-p^2} - e^{-L^2p^2} = \mathcal{O}(p^2)$ for $p \approx 0$ and it gets its main contribution if $|p| \in [\frac{1}{L}, 1]$. In particular, for χ analytic (prove !) $|\Gamma(x-y)| \leq Ae^{-|x-y|/L}$, i.e. it has correlation length $< \infty$.

Lemma 14.1 Suppose $\mu_{G_i}, i = 1, 2$ are two Gaussian measures on $S'(\mathbb{R}^d)$ and $G = G_1+G_2$. Then for all $F \in L^\infty(\mu_G)$,

$$\int F(\varphi)d\mu_G(\varphi) = \int F(\varphi_1 + \varphi_2)d\mu_{G_1}(\varphi_1)d\mu_{G_2}(\varphi_2).$$

Remark. Hence φ is the sum of independent Gaussians.
Proof $1°$. Holds for $F(\varphi) = e^{i\varphi(f)}$, $f \in S(\mathbb{R}^d) : e^{-\frac{1}{2}(f,Gf)} = \int e^{i\varphi(f)}d\mu_G$ and

$$\int e^{i(\varphi_1(f)+\varphi_2(f))}d\mu_{G_1}d\mu_{G_2} = \int e^{i\varphi_1(f)}d\mu_{G_1} \int e^{i\varphi_2(f)}d\mu_{G_2} = e^{-\frac{1}{2}[(f,G_1f)+(f,G_2f)]} = e^{-\frac{1}{2}(f,Gf)}.$$

$2°$. Suffices to check for F a cylinder-function i.e. recall that cylinder sets

$$C(f_1, \cdots, f_n, A_1, \cdots, A_n) = \{\varphi \in S'(\mathbb{R}^d)|\varphi(f_i) \in A_i, i = 1, \cdots, n\}$$

where $f_i \in S(\mathbb{R}^d), A_i$ Borel sets in $\mathbb{R}$. Let χ_C be the characteristic function of this C. Then our claim follows (why ?) if we prove it for $g = \chi_C$ for all cylinder sets C. But $\varphi(f_1), \cdots, \varphi(f_n)$ are Gaussian with covariance

$$\langle\varphi(f_i)\varphi(f_j)\rangle = (f_i, Gf_j) \equiv A_{ij}$$

i.e. for $g : \mathbb{R}^n \to \mathbb{R}$ Borel measurable,

$$\int g(\varphi(f_1), \cdots, \varphi(f_n))d\mu_G(\varphi) = \int g(x_1, \cdots, x_n)d\mu_A(x) \tag{14.2}$$

and similarity for $\varphi_\alpha(f_i)$ $\alpha = 1, 2$:

$$\int g(\varphi_1(f_1) + \varphi_2(f_1), \cdots, \varphi_1(f_n) + \varphi_2(f_n)d\mu_{G_1}d\mu_{G_2}$$
$$= \int g(y_1 + z_1, \cdots, y_n + z_n)d\mu_{A_1}(y)d\mu_{A_2}(z). \tag{14.3}$$

But (14.2) = (14.3) if we can show it for $g(x) = e^{i(\lambda,x)}$ for all $\lambda \in \mathbb{R}^n$ i.e. $\int e^{i(\lambda,x)}d\mu_A(x) = \int e^{i(\lambda,y+z)}d\mu_{A_1}(y)d\mu_{A_2}(z)$ which is a special case of $1°$. □

Let us apply the Lemma to (14.1). We note that the covariance of $L^{-\frac{d-2}{2}}\varphi(\frac{x}{L})$ is $L^{2-d}G(\frac{x-y}{L})$ if φ has covariance G. Hence :

$$\int F(\varphi)d\mu_G(\varphi) = \int F(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z)d\mu_G(\varphi)d\mu_\Gamma(Z).$$

This gives our RG :

$$\langle F\rangle_{\nu_\Lambda} = \int F(\varphi)d\nu_\Lambda(\varphi) = \frac{\int F(\varphi)e^{-V_\Lambda(\varphi)}d\mu_G(\varphi)}{\int e^{-V_\Lambda(\varphi)}d\mu_G(\varphi)}$$
$$= \frac{\int F(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z)e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_G(\varphi)d\mu_\Gamma(Z)}{\int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_G d\mu_\Gamma}$$
$$= \langle\widetilde{F}\rangle_{\tilde{\nu}}$$

with

$$\begin{aligned}
d\tilde{\nu}(\varphi) &= \frac{1}{Z} e^{-\widetilde{V}(\varphi)} d\mu_G(\varphi) \qquad Z = \int e^{-\widetilde{V}} d\mu_G \\
e^{-\widetilde{V}(\varphi)} &= \int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)} d\mu_\Gamma(Z) \\
\widetilde{F}(\varphi) &= \frac{\int F(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z) e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)} d\mu_\Gamma(Z)}{\int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)} d\mu_\Gamma(Z)}.
\end{aligned}$$

Hence, we got a *renormalized measure* $\tilde{\nu}$ which is again of same form as ν and we can define the RG as a map of V's by

$$R_L V(\varphi) = \widetilde{V}(\varphi) = -\log \int e^{-V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)} d\mu_\Gamma(Z). \tag{14.4}$$

Compare with block-spin ! The latter can actually, with a suitable definition of φ on the RHS, be written exactly as (14.4).

Remark. If $V = \int_\Lambda P(\varphi(x))dx$ then

$$\begin{aligned}
& V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z) = \int_\Lambda P(L^{-\frac{d-2}{2}}\varphi(\frac{x}{L}) + Z(x))dx \\
& = L^d \int_{L^{-1}\Lambda} P(L^{-\frac{d-2}{2}}\varphi(y) + Z(Ly))dy
\end{aligned}$$

i.e. it depends on φ in $L^{-1}\Lambda$. So we may call $\widetilde{V}$ by $\widetilde{V}_{L^{-1}\Lambda}$ if we wish. Of course this will not matter in the limit $\Lambda \nearrow \mathbb{Z}^d$.

We have now a *fixed point* $V = 0$, i.e. $d\mu_G \to d\mu_G$ under our RG. Call this the *Gaussian fixed point.*

14.2 Compute $DR_{V=0} = \mathcal{L}$

From (14.4) we get immediately

$$(\mathcal{L}V)(\varphi) = \int V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z) d\mu_\Gamma(Z).$$

This starts to look not too bad ! We will study the spectrum of this linear map. Let us first see some examples.

Example 1. Let $V=\int_\Lambda \varphi(x)^2dx$. Then

$$
\begin{aligned}
\mathcal{L}V(\varphi) &= \int\int_\Lambda \Big(L^{-\frac{d-2}{2}}\varphi\Big(\frac{x}{L}\Big)+Z(x)\Big)^2 dx d\mu_\Gamma(Z) = \\
&= L^{2-d}\int_\Lambda \varphi\Big(\frac{x}{L}\Big)^2 dx + \int_\Lambda dx \int Z(x)^2 d\mu_\Gamma \\
&= L^2\int_{\Lambda/L}\varphi(x)^2dx + |\Lambda|\Gamma(0) \ \ \Big(\text{since } \int Z(x)^2d\mu_\Gamma=\Gamma(0)\Big) \\
&= L^2\int_{\Lambda/L}\Big(\varphi(x)^2+L^{d-2}\Gamma(0)\Big)dx.
\end{aligned}
$$

Hence, since, from (14.1), $G(0)=L^{2-d}G(0)+\Gamma(0)$, we get $L^{d-2}\Gamma(0)=L^{d-2}G(0)-G(0)$, and

$$
\mathcal{L}\int_\Lambda(\varphi(x)^2-G(0))dx = L^2\int_{\Lambda/L}(\varphi(x)^2-G(0))dx
$$

i.e. we found an *eigenfunction* (as $\Lambda\to\mathbb{R}^d$) $\int(\varphi(x)^2-G(0))dx$ with eigenvalue L^2 i.e. with exponent $\alpha=2$. Note that this corresponds to a mass term. In the lattice cutoff case we have

$$
\begin{aligned}
&\lim_{\Lambda\to\mathbb{R}^d}\frac{1}{Z_\Lambda}e^{-\frac{m^2}{2}\sum_{x\in\Lambda}\varphi(x)^2}d\mu_G(\varphi) = d\mu_{G_{m^2}}(\varphi) \\
&G_{m^2}(x-y) = \int e^{ip(x-y)}e^{-p^2}\frac{1}{\mu(p)+m^2}.
\end{aligned}
$$

(Can you figure out what happens in the momentum space cutoff case?).This is a relevant eigenvector.

Example 2. Consider $V(\varphi)=\int_\Lambda\Big(\nabla\varphi(x)\Big)^2dx$. In the same way, using $\nabla\Big(\varphi(\frac{x}{L})\Big)=L^{-1}(\nabla\varphi)(\frac{x}{L})$ we get : $\mathcal{L}V=\int_{\Lambda/L}\Big(\nabla\varphi(x)\Big)^2dx-|\Lambda|\nabla^2\Gamma(0)$ and

$$
\mathcal{L}\left(\int_\Lambda\Big[\Big(\nabla\varphi(x)\Big)^2+\nabla^2G(0)\Big]\right)dx = \int_{\Lambda/L}\Big(\Big(\nabla\varphi(x)\Big)^2+\nabla^2G(0)\Big)dx
$$

so this is a *marginal* eigenvector : note that it corresponds to a change of the fixed point which is corresponding to $\mathcal{H}=\int(\nabla\varphi)^2$: in fact we have a one-parameter family of fixed points $d\mu_{\alpha G}$ $\alpha>0$.

Example 3. Consider $V(\varphi)=\int_\Lambda\varphi(x)^4dx$. Now (using $\int d\mu_\Gamma Z^{2m+1}=0$)

$$
\begin{aligned}
&\int d\mu_\Gamma(Z)\left[L^{2(2-d)}\varphi\left(\frac{x}{L}\right)^4+6L^{2-d}\varphi\left(\frac{x}{L}\right)^2Z(x)^2+Z(x)^4\right] \\
&= L^{2(2-d)}\varphi\left(\frac{x}{L}\right)^4+6L^{2-d}\varphi\left(\frac{x}{L}\right)^2\Gamma(0)+3\Gamma(0)^2
\end{aligned}
$$

so

$$
\mathcal{L}V = L^{4-d}\int_{\Lambda/L}\varphi(x)^4dx+6L^2\Gamma(0)\int_{\Lambda/L}\varphi(x)^2dx+3\Gamma(0)^2|\Lambda|
$$

so, putting

$$V_4 = \int_\Lambda (\varphi(x)^4 - 6G(0)\varphi(x)^2 + 3G(0)^2)dx$$

we get $\mathcal{L}V_4 = L^{4-d}V_4$, i.e. V_4 is *relevant* if $d < 4$, *marginal* if $d = 4$ and *irrelevant* if $d > 4$. It is obvious now that we have eigenvectors V_n, polynomial of degree n, with eigenvalues $L^d L^{n\frac{2-d}{2}}$. A simple way to get the formula for these eigenvectors is to consider

$$\mathcal{L}e^{\varphi(f)} = e^{L^{\frac{2-d}{2}}\int\varphi(\frac{x}{L})f(x)dx}\int d\mu_\Gamma e^{Z(f)} = e^{\varphi(\tilde{f})+\frac{1}{2}(f,\Gamma f)}$$

where $\tilde{f}(x) = L^{\frac{2-d}{2}}L^d f(Lx) = L^{\frac{2+d}{2}}f(Lx)$. Since

$$(\tilde{f}, G\tilde{f}) = L^{2+d}\int f(Lx)G(x-y)f(Ly)dxdy = L^{2-d}\int f(x)G\Big(\frac{x-y}{L}\Big)f(y)dxdy$$

we have, see (14.1), $(f,\Gamma f) = (f, Gf) - (\tilde{f}, G\tilde{f})$ and so

$$\mathcal{L}e^{-\frac{1}{2}(f,Gf)+\varphi(f)} = e^{-\frac{1}{2}(\tilde{f},G\tilde{f})+\varphi(\tilde{f})}.$$

This is a kind of generating functional of eigenvectors. Let

$$H_n(t,a) = \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} e^{-\frac{1}{2}a\lambda^2+\lambda t}$$

be the n^{th} Hermite polynomial (polynomial of degree n in t, homogenous of degree n in $t, \sqrt{a}$). Then, let

$$V_n(\varphi, f) = H_n\Big(\varphi(f), (f, Gf)\Big) = \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} e^{-(f,Gf)\lambda^2+\lambda\varphi(f)}$$

so,

$$\mathcal{L}V_n(\varphi, f) = V_n(\varphi, \tilde{f}).$$

Taking the limit $\varphi(f) \to \varphi(x)$ (i.e. letting f tend to a delta function), $(f, Gf) \to G(x - x) = G(0)$, $V_n(\varphi, f)$ becomes a polynomial $P_n(\varphi(x)$ in $\varphi(x)$, $P_n(\varphi(x) = H_n(\varphi(x), G(0)) = \varphi(x)^n + a_1\varphi(x)^{n-2}G(0) + \cdots$ and

$$\mathcal{L}\int_\Lambda P_n\Big(\varphi(x)\Big)dx = L^{d+n\frac{2-d}{2}}\int_{\Lambda/L} P_n\Big(\varphi(x)\Big)dx.$$

where P_n are Hermite polynomials.

A **local** V is an integral of a polynomial in $\varphi(x)$ and its derivatives. Clearly we get eigenvectors

$$\int_\Lambda \varphi(x)^n \prod_{k=1}^{\ell}\Big(\nabla^k\varphi(x)\Big)^{n_k} dx + \text{lower order polynomial}$$

with eigenvalue

$$L^d L^{\frac{2-d}{2}(n+\sum n_k)} L^{-(\sum k n_k)}$$

i.e. the more there are φ's or derivatives, the more V is irrelevant. In the class of even local V's the relevant ones are, for $d \geq 3$:

$$\begin{array}{lll} \int \varphi^2 & L^2 & \text{always} \\ \int \varphi^4 & L^{4-d} & d < 4 \end{array}$$

the marginal ones are :

$$\begin{array}{ll} \int (\nabla\varphi)^2 & \text{always} \\ \int \varphi^6 & d = 3 \\ \int \varphi^4 & d = 4 \end{array}$$

In case of odd V, we have $\int \varphi$ with eigenvalue $L^{\frac{d+2}{2}}$ always relevant (this corresponds to a magnetic field), $\int \varphi^3$ with eigenvalue $L^{\frac{6-d}{2}}$ relevant for $d < 6$, $\int \varphi^5$ with eigenvalue $L^{5-\frac{3d}{2}}$, relevant $3d < 10$. (note that $\int \nabla\varphi, \int \nabla\varphi\varphi^2$ are boundary terms).

14.3 The space $\mathcal{K}$ for $\mathcal{L}$

We can set up a space $\mathcal{K}$ so that $\mathcal{L} : \mathcal{K} \to \mathcal{K}$ and study the spectrum of $\mathcal{L}$.
Basically, we want to consider V's of the form

$$\int dx_1 \cdots dx_n K(x_1, \cdots, x_n) \prod_{i=1}^n \varphi(x_i)$$

and specify the K's. For this, generalize a bit what we found above. Define the *normal ordered* product :

$$: \prod_{i=1}^n \varphi(x_i) := \sum_{I \subset \{1,\cdots,n\}} (-1)^{\frac{|I|}{2}} \prod_{j \notin I} \varphi(x_j) \langle \prod_{i \in I} \varphi(x_i) \rangle.$$

Homework.

$$\int : \prod_{i=1}^n \varphi(x_i) : \prod_{i=1}^n f(x_i) dx_i = \prod_{i=1}^n \frac{\partial}{\partial \lambda_i}\Big|_{\lambda_i=0} \exp\Big[-\frac{1}{2}(f, Gf) + \varphi(f)\Big]$$

$$f = \sum_{i=1}^n \lambda_i f_i, \;\; f_i \in S(\mathbb{R}^d).$$

This implies :

$$\mathcal{L} : \prod_{i=1}^n \varphi(x_i) := L^{\frac{2-d}{2}n} : \prod_{i=1}^n \varphi\Big(\frac{x_i}{L}\Big) : .$$

The space $\mathcal{K}_\gamma$ consists of *finite* sums of the form. (Let $\varphi \in S(\mathbb{R}^d)$)

$$V(\varphi) = \sum_{\underset{\sim}{n}\ \underset{\sim}{m}} \int K_{\underset{\sim}{n}\ \underset{\sim}{m}}(x_1 \cdots x_N) : \prod_{i=1}^{N} \varphi(x_i)^{n_i} \nabla\varphi(x_i)^{m_i} : \prod_{i=1}^{N} dx_i \tag{14.5}$$

where $\underset{\sim}{n} = (n_1, \cdots, n_N)$, $\underset{\sim}{m} = (m_1, \cdots, m_N), n_i \in \mathbb{N}, m_i \in \mathbb{N}^d$, $(\nabla\varphi)^{m_i} = \prod_{\alpha=1}^{d} (\nabla_\alpha \varphi)^{m_{i\alpha}}$, $m_i = (m_{i1}, \cdots, m_{id})$ and $K_{\underset{\sim}{n}\underset{\sim}{m}}(x_1 \cdots x_N) = K_{\underset{\sim}{n}\underset{\sim}{m}}(x_1 + x, x_2 + x, \cdots, x_N + x)$ $\forall x \in \mathbb{R}^d$, (translation invariance) and

$$\int |K_{\underset{\sim}{n}\underset{\sim}{m}}(0, x_2, \cdots, x_N)| e^{\gamma \mathcal{L}(0, x_2, \cdots, x_N)} \equiv \|K_{\underset{\sim}{n}\underset{\sim}{m}}\|_\gamma < \infty$$

where $\mathcal{L}(x_1, \cdots, x_N) \equiv d(\underset{\sim}{x})$ is the length of the shortest connected path Γ in $\mathbb{R}^d$ such that all x_i belong to Γ; here $\gamma > 0$.

Then $\mathcal{L}V$ has kernels K':

$$K'_{\underset{\sim}{n}\underset{\sim}{m}}(\underset{\sim}{x}) = L^{\frac{2-d}{2}\sum_{i=1}^{N}(n_i+m_i)} L^{-\sum m_i} L^{Nd} K_{\underset{\sim}{n}\underset{\sim}{m}}(L\underset{\sim}{x})$$

and

$$\begin{aligned} \|K'_{\underset{\sim}{n}\underset{\sim}{m}}\|_{\gamma'} &= L^{\frac{2-d}{2}\sum_{i=1}^{N}(n_i+m_i)} L^{-\sum_{i=1}^{N} m_i} L^{Nd} \int |K_{\underset{\sim}{n}\underset{\sim}{m}}(L\underset{\sim}{x})| e^{\gamma' d(\underset{\sim}{x})} dx_2 \cdots dx_n \\ &= L^{\frac{2-d}{2}\sum_{i=1}^{N}(n_i+m_i)} L^{-\sum_{i=1}^{N} m_i} L^{d} \int |K_{\underset{\sim}{n}\underset{\sim}{m}}(\underset{\sim}{x})| e^{\gamma' d(L^{-1}\underset{\sim}{x})} dx_2 \cdots dx_n. \end{aligned}$$

But $d(L^{-1}\underset{\sim}{x}) = L^{-1} d(\underset{\sim}{x})$, so

$$\|K'_{\underset{\sim}{n}\underset{\sim}{m}}\|_{L\gamma} \leq L^d L^{\frac{2-d}{2}\sum_{i=1}^{N}(n_i+m_i)} L^{-\sum_{i=1}^{N} m_i} \|K_{\underset{\sim}{n}\underset{\sim}{m}}\|_\gamma$$

Hence $\mathcal{L}V \in \mathcal{K}_{L\gamma}$ and thus also $\mathcal{L}V \in \mathcal{K}_\gamma$.

Thus only (for V even)

$$\begin{aligned} &\int K(x-y)\varphi(x)\varphi(y), \int K_{\alpha\beta}(x-y)\nabla_\alpha\varphi(x)\nabla_\beta\varphi(y), \int K(x-y)\varphi(x)\Delta\varphi(y) \\ &\int K(x_1 \cdots x_4)\varphi(x_1)\cdots\varphi(x_4), \int K(x-y)\varphi(x)^2\varphi(y)^2 \\ &\int K(x_1x_2x_3)\varphi(x_1)\varphi(x_2)\varphi(x_3)^2, \int K(x-y)\varphi(x)\varphi(y)^3 \end{aligned} \tag{14.6}$$

are not irrelevant in addition to $\int \varphi^2, \int \varphi^4, \int (\nabla\varphi)^2$. These can be split into relevant/marginal and irrelevant as follows. Consider e.g. the first term in (14.6). Let $\widehat{K}(p)$ be the Fourier transform of K. Since $\int |K(x)| e^{\gamma|x|} dx < \infty$ we get $\widehat{K}(p)$ is analytic in $|\mathrm{Im} p| < \gamma$. For simplicity we consider only rotation invariant V's i.e. $K(x) = K(|x|)$.

Lemma 14.2 Let $K(x)$ be rotation invariant with $\|K\|_\gamma = \int |K(x)|e^{\gamma|x|}dx < \infty$. Then for all $\varphi \in S(\mathbb{R}^d)$)

$$\int K(x-y)\varphi(x)\varphi(y)dxdy = \alpha \int \varphi(x)^2 + \beta(\nabla\varphi(x))^2 + \widetilde{V}(\varphi)$$

where $|\alpha|, |\beta| \le C\|K\|_\gamma$ and $\widetilde{V} \in \mathcal{K}_{\gamma/2}$ is irrelevant with kernel $\le C\|K\|_\gamma$.

Proof. Since $\int |K(x)|e^{\gamma|x|}dx < \infty$ we get

$$|\widehat{K}(p)| = |\int e^{ipx}K(x)dx| \le \int e^{|\mathrm{Im}p||x|}|K(x)|dx$$

is analytic in $|\mathrm{Im}p| < \gamma$. Taylor expand

$$\widehat{K}(p) = \widehat{K}(0) + \beta p^2 + \widehat{R}(p) \tag{14.7}$$

(we used the rotation symmetry to get the second term of the form βp^2) where

$$\begin{aligned}\widehat{R}(p) &= \frac{1}{2}\int_0^1 dt(1-t)^2 \frac{d^3}{dt^3}\widehat{K}(tp) \\ &= \sum_{\alpha\beta\gamma} p_\alpha p_\beta p_\gamma \widehat{H}_{\alpha\beta\gamma}(p)\end{aligned}$$

we have

$$|\alpha| = |\widehat{K}(0)| \le \|K\|_\gamma$$

and using Cauchy's theorem

$$|\beta| = |\partial^2\widehat{K}(0)| \le \gamma^{-2}\|K\|_\gamma$$

It remains to study the remainder. We have

$$\frac{d^3}{dt^3}\widehat{K}(tp) = \sum_{\alpha\beta\gamma} p_\alpha p_\beta p_\gamma (\partial_\alpha\partial_\beta\partial_\gamma\widehat{K})(tp)$$

and so

$$\begin{aligned}\widehat{H}_{\alpha\beta\gamma}(p) &= \int_0^1 dt(1-t)^2(\partial_\alpha\partial_\beta\partial_\gamma\widehat{K})(tp) \\ &= i^3\int_0^1 dt(1-t)^2(\widehat{x_\alpha x_\beta x_\gamma K}(x))(tp).\end{aligned}$$

Taking inverse Fourier transform and changing variables we get

$$\begin{aligned}H_{\alpha\beta\gamma}(x) &= i^3\int_0^1 dt(1-t)^2 t^{-d}\int e^{ipx/t}(\widehat{x_\alpha x_\beta x_\gamma K}(x))(p)dp \\ &= i^3\int_0^1 dt(1-t)^2 t^{-d-3}x_\alpha x_\beta x_\gamma K(x/t).\end{aligned}$$

Hence

$$\|H_{\alpha\beta\gamma}\|_{\gamma/2} \leq \int_0^1 dtt^{-d} \int dx(|x|/t)^3 |K(x/t)| e^{\gamma|x|/2}$$
$$= \int_0^1 dt \int dx|x|^3 |K(x)| e^{\gamma t|x|/2}.$$

Since $|x|^3 \leq \frac{48}{\gamma^3} e^{\gamma|x|/2}$ we get finally

$$\|H_{\alpha\beta\gamma}\|_{\gamma/2} \leq \frac{48}{\gamma^3} \|K\|_\gamma.$$

Thus

$$\tilde{V} = i^3 \int H_{\alpha\beta\gamma}(x-y) \partial_\alpha \phi(x) \partial_\alpha \partial_\beta \phi(y) dx dy$$

is irrelevant and satisfies the claim. $\square$

The other terms can be analyzed similarly, e.g.

$$\int K(x_1, \cdots, x_4) \prod \varphi(x_i) = a \int \varphi^4 + \widetilde{V}$$

where $a = \int K(0, x_2, x_3, x_4) dx_2 dx_3 dx_4$ and $\widetilde{V}$ is irrelevant. Thus every $V \in \mathcal{K}_\gamma$ can be written as

$$V = r \int : \varphi^2 : + z \int : (\nabla\varphi)^2 : + \lambda \int : \varphi^4 : + \widetilde{V} \quad , \quad \widetilde{V} \in \mathcal{K}_{\gamma/2}$$

and then

$$\mathcal{L}V = L^2 r \int : \varphi^2 : + z \int : (\nabla\varphi)^2 : + L^{4-d} \lambda \int : \varphi^4 : + \widetilde{V}'$$

with $\widetilde{V}' \in \mathcal{K}_{L\gamma/2}$, $\|\widetilde{V}'\|_{L\gamma/2} \leq L^{-2} \|\widetilde{V}\|$. These are our "coordinates" in $\mathcal{K}$.

15 Perturbative analysis of RV

15.1 General formalism

Consider now the full R:

$$RV(\varphi) = -\log \int d\mu_\Gamma(Z) \exp\Big[-V(L^{\frac{2-d}{2}} \varphi(\frac{\cdot}{L}) + Z)\Big]. \tag{15.1}$$

Suppose V is in $\mathcal{K}$ as above. There are several problems :

1. φ in (15.1) lives on $\mathbb{R}^d$ and is not in $S(\mathbb{R}^d)$, rather it is in $S' \cap C^\infty$ i.e. it may increase as $x \to \infty$. This is the **IR**-problem. We used Λ for that, so we may replace the kernels $K_{\underset{\sim}{n}\,\underset{\sim}{m}}$ above by $K_{\underset{\sim}{n}\,\underset{\sim}{m}}\,\chi$ where χ restricts $x_i \in \Lambda$.

2. More serious problem is the convergence of (15.1) as Z becomes large. This is the *stability problem.* Say for $V = \lambda\varphi^4$ we need at least Re $\lambda > 0$. For a V as in the previous section, this positivity property is not easy to state. Instead of trying to address the stability problem, we will calculate RV perturbatively in λ if $V = \lambda\varphi^4$. With a lot of work, RV can actually be analyzed rigorously. One does the following :

   a) For $|\varphi(x)| < C$ one uses the representation in terms of $K_{\underset{\sim}{n}\ \underset{\sim}{m}}$, actually an infinite series.

   b) For $|\varphi(x)| > C$ one uses another representation where positivity is explicit.

   The problem is to combine a) and b) : $\varphi(x)$ can be small and $\varphi(y)$ large nearby. One uses expansion ideas from statistical mechanics to decouple these problems.

Thus, let us start with $V(\varphi) = \lambda \int_\Lambda \varphi(x)^4 dx$ and see what RV looks like. Expand in powers of λ:

$$(RV)(\varphi) = \sum_{n=0}^{N} \lambda^n (RV)_n(\varphi) + \text{Remainder}$$

$$(RV)_n(\varphi) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big|_{\lambda=0}(RV)(\varphi) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big|_{\lambda=0}\Big(-\log\int e^{-V(L^{\frac{2-d}{2}}\varphi(\frac{\cdot}{L}+Z)}d\mu_\Gamma(Z)\Big).$$

Hence,

$$\begin{aligned}
\lambda(RV)_1(\varphi) &= \lambda\int\int_\Lambda\Big(L^{\frac{2-d}{2}}\varphi\Big(\frac{x}{L}\Big)+Z(x)\Big)^4 dx d\mu_\Gamma = \mathcal{L}V(\varphi)\\
\lambda^2(RV)_2(\varphi) &= -\frac{1}{2}\Big[\int V^2 d\mu_\Gamma - \int V d\mu_\Gamma \int V d\mu_\Gamma\Big]\\
&= -\frac{1}{2}\langle V(L^{\frac{2-d}{2}}\varphi(\frac{\cdot}{L})+Z); V(L^{\frac{2-d}{2}}\varphi(\frac{\cdot}{L})+Z)\rangle_\Gamma
\end{aligned}$$

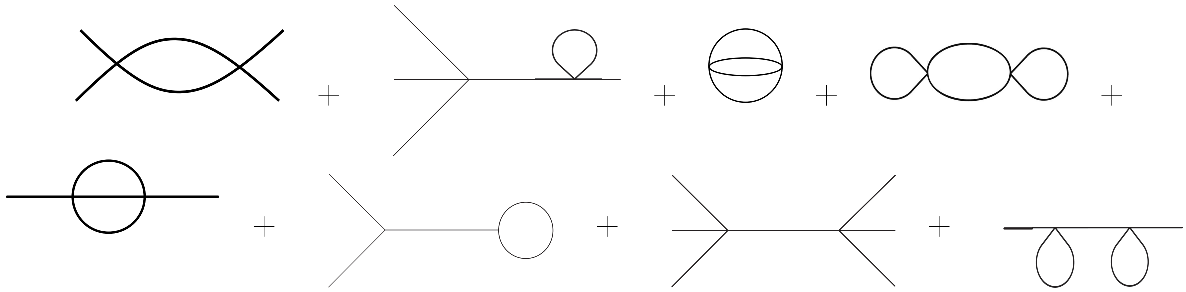
where we use the notation of truncated expectations.
In general

$$\lambda^n(\text{RV})_n(\varphi) = (-)^{n-1}\frac{1}{n!}\langle V; V; \cdots; V\rangle_\Gamma.$$

Consider $(\text{RV})_2$: Denote $L^{\frac{2-d}{2}}\varphi(\frac{x}{L}) \equiv \Psi(x)$. So

$$\begin{aligned}
&\Big\langle\Big(\Psi(x)+Z(x)\Big)^4\Big(\Psi(y)+Z(y)\Big)^4\Big\rangle_\mu - \Big\langle\Big(\Psi(x)+Z(x)\Big)^4\Big\rangle_\mu\Big\langle\Big(\Psi(y)+Z(y)\Big)^4\Big\rangle_\mu\\
&= 36\cdot 2\cdot \Psi(x)^2\Psi(y)^2\Gamma(x-y)^2 + 16\cdot 3\Big(\Psi(x)\Psi(y)^3+\Psi(x)^3\Psi(y)\Big)\Gamma(x-y)\Gamma(0)\\
&+4\,!\,\Gamma(x-y)^4 + \begin{pmatrix}4\\2\end{pmatrix}^2\cdot 2\Gamma(x-y)^2\Gamma(0)^2\\
&+16\cdot 3\,!\,\Psi(x)\Psi(y)\Gamma(x-y)^3 + 6\cdot 4\cdot 3(\Psi(x)^2+\Psi(y)^2)\Gamma(x-y)^2\Gamma(0)\\
&+16\Psi(x)^3\Psi(y)^3\Gamma(x-y) + 16\cdot 3\cdot 3\Psi(x)\Psi(y)\Gamma(x-y)\Gamma(0)^2
\end{aligned}$$

or graphically ONE LINE IS MISSING FROM SIXTH GRAPH!



where we have same rules as before except that :
- edges are $\Gamma(x-y)$
- external edges have $\Psi(x) = L^{\frac{2-d}{2}}\varphi(\frac{x}{L})$.

Thus we end up with the representation

$$(RV)_n(\varphi) = \sum_{\mathcal{G}} W_{\mathcal{G}}(\varphi)$$

where the sum runs through all *connected* graphs $\mathcal{G}$ with $n+m$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_m$. Vertices y_i have four edges ($\{z, z\}$ allowed) attached and vertices x_j have $4 - n_j$ edges attached where $n_j > 0$. The $W_{\mathcal{G}}(\varphi)$ is equals

$$n(\mathcal{G}) \int \left(\int \prod_{\{z,z'\} \in e(\mathcal{G})} \Gamma(z - z') \prod_{i=1}^{m} dy_j \right) \prod_{i=1}^{n} \Psi(x_i)^{n_i} dx_i \equiv \int K_{\mathcal{G}}(x_1, \ldots, x_n) \prod_{i=1}^{n} \Psi(x_i)^{n_i} dx_i$$

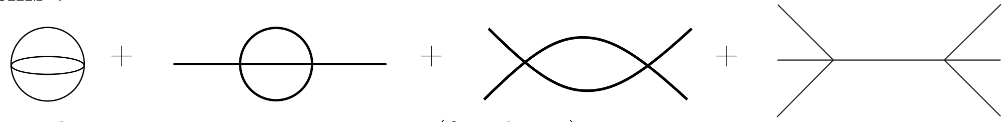
Recalling that $|\Gamma(x-y)| \leq Ce^{-\alpha|x-y|}$, for some α and the fact that the graph is connected we get

Exercise. $\|K_{\mathcal{G}}\|_\gamma < \infty$, for some $\gamma > 0$.

Thus $\sum_{n=0}^{N} \lambda^n (RV)_n(\varphi) \in \mathcal{K}_\gamma$ for some γ. It is more convenient to consider $V(\varphi) = \lambda \int_\Lambda : \varphi(x)^4 : dx$ where, recall, $: \varphi^4(x) := \varphi(x)^4 - 6\varphi(x)^2 G(0) + 3G(0)^2$. Then

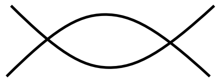
$$\lambda \ (\mathrm{RV})_1(\varphi) = \mathcal{L}V = \lambda L^{4-d} \int_{\Lambda/L} : \varphi(x)^4 :$$

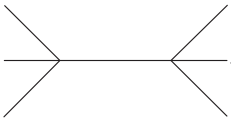
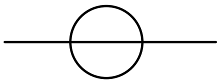
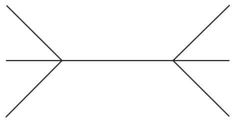
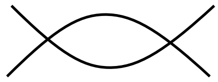
Exercise. Prove this !

Thus, e.g. $\mathrm{RV}_2 =$ 

From our analysis in the previous section, we get (for $d = 4$)

$$(RV)(\varphi) = (\lambda + a\lambda^2 + \mathcal{O}(\lambda^3)) \int_{\Lambda/L} : \varphi(x)^4 : + (b\lambda^2 + \mathcal{O}(\lambda^3)) \int_{\Lambda/L} : \varphi^2 :$$
$$+ (c\lambda^2 + \mathcal{O}(\lambda^3)) \int_\Lambda (\nabla\varphi)^2 + \widetilde{V}$$

where $\widetilde{V}$ is *irrelevant* i.e. $\mathcal{L}\widetilde{V}$ contracts and $\widetilde{V} = \mathcal{O}(\lambda^2)$. Here a comes from 

and , $a = -\frac{72}{2}\int \Gamma(x)^2 dx + \frac{16.9}{2}\int \Gamma(x-y) L^{2-d} C(\frac{x}{L} - \frac{y}{L})$ (see computation for $d = 4$ below); b, c from  and  and 

(see computation for $d = 4$ below).

15.2 Iteration

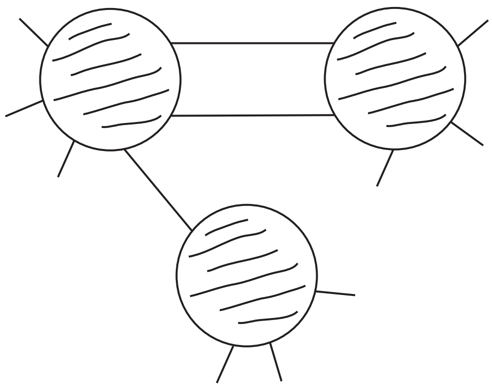
Suppose now $V = \lambda \int : \varphi^4 : + r \int : \varphi^2 : + z \int : (\nabla\varphi)^2 : + \widetilde{V}$ where $\widetilde{V} \in K$ is irrelevant, $\|\mathcal{L}\widetilde{V}\|_\gamma \leq L^{-\alpha}\|\widetilde{V}\|_\gamma$ for $\alpha > 0$. Calculate perturvatively RV:

$$(RV)(\varphi) \;=\; \sum_{n=1}^{N} (-)^{n+1} \langle V; V; \cdots; V\rangle_\Gamma + \text{Remainder} \tag{15.2}$$

where $V = V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z)$

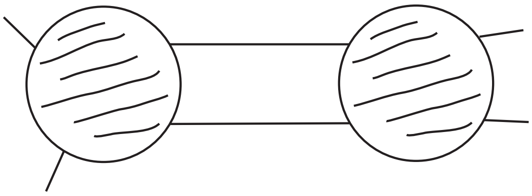
$$\equiv \sum_{n=0}^{N} \ (\text{RV})_n(\varphi) + \text{Remainder}$$

The point is now that $RV_n \in K_\gamma$ again. Indeed, we get graphs as before and also from r, z and $\widetilde{V}$. Pictorially, the $\widetilde{V}$ ones are



where  are $K_{\underset{\sim}{n}\,\underset{\sim}{m}}$ and lines are $\Gamma(x - y)$.
It is not hard to show that the exponential decay of $K_{\underset{\sim}{n}\,\underset{\sim}{m}}$ and Γ give rise to exponential decay of these graphs. Thus the claim.

Example E.g. $\left\langle \left(\int K(x_1, \cdots, x_4) \prod_{i=1}^{4} \varphi(x_i) \right)^2 \right\rangle$ gives for instance



$$
\begin{aligned}
&= \int \prod_{i=1}^{4} dx_i L^{\frac{2-d}{2}} \varphi\Big(\frac{x_i}{L}\Big) \int \prod_{i=1}^{4} dy_i K(x_1,x_2,y_1,y_2)\Gamma(y_1-y_3)\Gamma(y_2-y_4)K(y_3,y_4,x_3,x_4) \\
&\equiv \int \widetilde{K}(x_1,\cdots,x_4)\prod \varphi(x_i)dx_i
\end{aligned}
$$

where

$$
\widetilde{K}(\underset{\sim}{x}) = L^{4-2d}L^{4d}\int \prod_{i=1}^{4} dy_i K(Lx_1,Lx_2,y_1,y_2)\Gamma(y_1-y_3)\Gamma(y_2-y_4)K(y_3,y_4,Lx_3,Lx_4)
$$

and

$$
\begin{aligned}
&\int e^{\gamma\mathcal{L}(0,x_2,x_3,x_4)}|\widetilde{K}(0,\cdots,x_4)|dx_2\cdots dx_4 \\
&= L^{-3d}\int e^{L^{-1}\gamma\mathcal{L}(0,x_2,x_3,x_4)}|\widetilde{K}(0,L^{-1}x_2,L^{-1}x_3,L^{-1}x_4)|dx_2\cdots dx_4 \qquad (15.3)
\end{aligned}
$$

Use $\mathcal{L}(0,x_2,x_3,x_4)\le\mathcal{L}(0,x_2,y_1,y_2)+|y_1-y_3|+|y_2-y_4|+\mathcal{L}(y_3,y_4,x_3,x_4)$ (since the RHS is the length of a particular path Γ such that $0,x_2,x_3,x_4\in\Gamma$).
So,

$$
(15.3) \le L^{4-d}\int \prod_{i=1}^{4} dy_i \prod_{i=2}^{4} dx_i \bar{K}(0,x_2,y_1y_2)\bar{\Gamma}(y_1-y_3)\bar{\Gamma}(y_2-y_4)\bar{K}(y_3,\cdots,x_4)
$$

with $\bar{K}=e^{L^{-1}\gamma\mathcal{L}}K$, $\bar{\Gamma}(x)=e^{L^{-1}\gamma|x|}\Gamma(x)$.
Use $|\bar{\Gamma}(y_2-y_4)|\le C$, integrate over y_4,x_3,x_4 :

$$
(15.3) \le L^{4-d}\int \prod_{i=1}^{3} dy_i dx_2 |\bar{K}(0,x_2,y_1,y_2)\bar{\Gamma}(y_1-y_3)| \|K\|_{L^{-1}\gamma}.
$$

Integrate y_3 : $\int |\bar{\Gamma}(y_1-y_3)|dy_3 < C'$ and then the rest :

$$
(15.3) \le CL^{4-d}\|K\|^2_{L^{-1}\gamma} \le CL^{4-d}\|K\|^2_{\gamma}.
$$

Summary. Perturbatively, the R^nV retains the form :

$$
V_n \equiv R^nV(\varphi) = z_n\int_{L^{-n}\Lambda} :(\nabla\varphi)^2: + r_n\int_{L^{-n}\Lambda} :\varphi^2: + \lambda_n\int_{L^{-n}\Lambda} :\varphi^4: + \widetilde{V}_n
$$

where $\widetilde{V}_n \in \mathcal{K}$ and $\|\mathcal{L}\widetilde{V}_n\| \leq L^{-\alpha}\|V_n\|, \quad \alpha > 0$.
We get the recursion (denote $g_n = (z_n, r_n, \widetilde{V}_n)$)

$$\begin{aligned}
z_{n+1} &= z_n + \mathcal{O}(V_n^2) \\
r_{n+1} &= L^2 r_n + \mathcal{O}(V_n^2) \\
\lambda_{n+1} &= L^{4-d}\lambda_n + a\lambda_n^2 + \mathcal{O}(\lambda_n^3, \lambda_n g_n, g_n^2) \\
\widetilde{V}_{n+1} &= \mathcal{L}\widetilde{V}_n + \mathcal{O}(V_n^2).
\end{aligned}$$

Let us see how these behave as $n \to \infty$.

15.3 $\quad d > 4$

We want to determine the critical point, i.e. take

$$V = z \int :(\nabla\phi)^2: + r \int :\varphi^2: + \lambda \int :\varphi^4:$$

and find $r(\lambda)$ such that $V \in \mathcal{M}_s$. Since $4 - d < 0$ we expect the relevant fixed point to be the Gaussian $V = \tilde{z} \int :(\nabla\varphi)^2:$, for some constant $\tilde{z} \neq z$, noted z_∞ below.
Hence, the problem is to find $r_0 = r(\lambda)$ such that

$$r_n, \lambda_n, \widetilde{V}_n \to 0 \quad \text{as} \quad n \to \infty.$$

Since $r_{n+1} = L^2 r_n + \cdots$ tends to increase, we need to be careful. Let us construct r_0 inductively. Given λ (small), take $r_0 \in [-A\lambda^2, A\lambda^2] \equiv I_0, A$ chosen below. Then $|r_1 - L^2 r_0| \leq C\lambda^2 + C(A)\lambda^3$ where C is A-independent. Consider r_1 as a function of r_0. For A large enough, r_1 maps our interval

$$r_1(I_0) \supset \left[-\frac{L^2}{2}A\lambda^2, \frac{L^2}{2}A\lambda^2\right]$$

i.e., since $\lambda_1 \leq L^{4-d}\lambda + \mathcal{O}(\lambda^2) < \lambda$, we can, by continuity, find an interval

$$I_1 \subset I_0$$

such that

$$r_1(I_1) = [-A\lambda_1^2, A\lambda_1^2].$$

Now, keep on iterating. We find intervals $I_n \subset I_{n-1} \subset \cdots I_0$ such that r_n as a function of r_0 satisfies

$$r_n(I_n) = [-A\lambda_n^2, A\lambda_n^2]$$

and $\lambda_n \leq L^{(4-d-\epsilon)n}\lambda$, for $\epsilon > 0$, $\|\widetilde{V}_n\| \to 0$.

Thus, $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$ (since I_n closed $\subset I_{n-1}$) and, for $r_0 \in \bigcap_{n=0}^{\infty} I_n$, we have

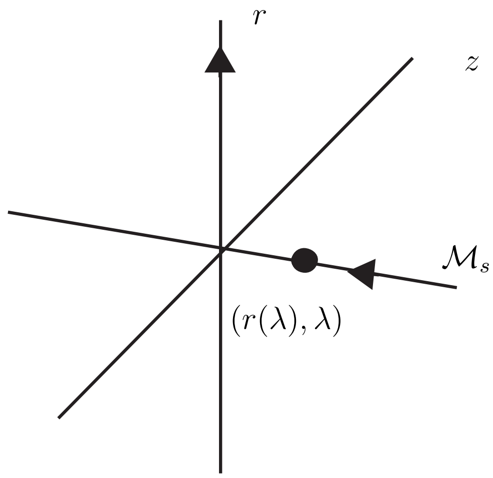
$$\widetilde{V}_n, \lambda_n, r_n \to 0 \quad \text{as} \quad n \to \infty.$$

Actually it is readily seen that $|I_n| \underset{n\to\infty}{\longrightarrow} 0$ so $\bigcap_{n=0}^{\infty} I_n$ = one point $\equiv r(\lambda)$, the critical r value. Clearly $r(\lambda) = \mathcal{O}(\lambda^2)$.

How about z_n ? Since $z_{n+1} = z_n + \mathcal{O}(\lambda_n^2)$ and $z_0 = 0$ we have $z_n = \sum_{m=0}^{n-1} \mathcal{O}(\lambda_m^2) = \sum_{m=0}^{n-1} \mathcal{O}((L^{(4-d)m}\lambda)^2) \underset{n\to\infty}{\longrightarrow} z_\infty = \mathcal{O}(\lambda^2)$. So

$$R^n V \underset{n\to\infty}{\longrightarrow} z_\infty \int :(\nabla\varphi)^2.$$

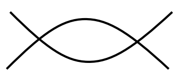
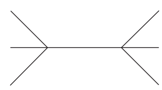
Hence, we have the one-parameter family of Gaussian fixed points $e^{-z\int:(\nabla\varphi)^2:}\, d\mu_G$ and $e^{-r(\lambda)\int:\varphi^2:-\lambda\int:\varphi^4:}\, d\mu_G \underset{R^n}{\longrightarrow} e^{-z_\infty\int:(\nabla\varphi)^2:}\, d\mu_G$.
The Gaussian line has one unstable direction $:\varphi^2:$ and, once that is fixed to the critical value, we flow to the line :



15.4 $\quad d = 4$

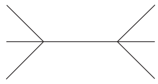
Since $\lambda_{n+1} = \lambda_n$ to leading order we need to go to $\mathcal{O}(\lambda^2)$. The sign of a in $\lambda_{n+1} = \lambda_n + a\lambda_n^2 + \cdots$ is very important so let us calculate it.
We will prove that $\lambda_n \to 0$ and $r_n, z_n, \widetilde{V}_n = \mathcal{O}(\lambda_n^2)$. Thus a gets contribution from the term

$$-\frac{1}{2}\int\Big\langle :\Big(L^{\frac{2-d}{2}}\varphi\Big(\frac{x}{L}\Big)+Z(x)\Big)^4 : : \Big(L^{\frac{2-d}{2}}\varphi\Big(\frac{y}{L}\Big)+Z(y)\Big)^4 : \Big\rangle\, dxdy.$$

This gives graph a)  which contributes directly and b)  which contributes indirectly.

a) Equals

$$\begin{aligned}
&-\frac{72}{2}L^{4-2d}\int dxdy\varphi\Big(\frac{x}{L}\Big)^2\varphi\Big(\frac{y}{L}\Big)^2\Gamma(x-y)^2 dxdy\\
&= -\frac{72}{2}L^{4-2d}\int dx\varphi\Big(\frac{x}{L}\Big)^4\Big(\int dy\Gamma(y)^2\Big) + \text{irrelevant (goes into } \widetilde{V}).
\end{aligned}$$

b) We have to normal order  since $\widetilde{V}_{n+1}$ is normal ordered. This graph equals

$$-\frac{1}{2}4^2 L^{-\frac{2-d}{2}-6}\int :\varphi\Big(\frac{x}{L}\Big)^3 : \Gamma(x-y) : \varphi\Big(\frac{y}{L}\Big)^3 : .$$

Use

$$\begin{aligned} :\varphi\Big(\frac{x}{L}\Big)^3 :: \varphi\Big(\frac{y}{L}\Big)^3 : &=: \varphi\Big(\frac{x}{L}\Big)^3 \varphi\Big(\frac{y}{L}\Big)^3 : \\ &+ 9G\Big(\frac{x-y}{L}\Big) : \varphi\Big(\frac{x}{L}\Big)^2 \varphi\Big(\frac{y}{L}\Big)^2 : + \text{ quadratic term } + \text{ constant} \end{aligned}$$

and write

$$\int G\Big(\frac{x-y}{L}\Big)\Gamma(x-y) : \varphi\Big(\frac{x}{L}\Big)^2 \varphi\Big(\frac{y}{L}\Big)^2 : dxdy = \int dx : \varphi\Big(\frac{x}{L}\Big)^4 : \int dy\ G\ \Big(\frac{y}{L}\Big)\Gamma(y) + \text{irrelevant.}$$

Thus a) and b) together give

$$-36L^{4-d}\int dx : \varphi(x)^4 : \int \Big(\Gamma(y)^2 + 2L^{2-d}G\Big(\frac{y}{L}\Big)\Gamma(y)\Big) = -b(L)\int : \varphi^4 : dx$$

(using $L^{4-d}=1$, $\Gamma(y)^2 + 2L^{2-d}G(\frac{y}{L})\Gamma(y) = G(y)^2 - (L^{2-d}G(\frac{y}{L}))^2)$, where

$$b(L) = 36\int dy\Big[G(y)^2 - (L^{-2}G\Big(\frac{y}{L}\Big))^2\Big]. \tag{15.4}$$

Therefore our recursion now is

$$\begin{aligned} z_{n+1} &= z_n + \mathcal{O}(\lambda_n^2) \\ r_{n+1} &= L^2 r_n + \mathcal{O}(\lambda_n^2) \\ \lambda_{n+1} &= \lambda_n - b(L)\lambda_n^2 + \mathcal{O}(\lambda_n^3) \\ \widetilde{V}_{n+1} &= \mathcal{L}\widetilde{V}_n + \mathcal{O}(\lambda_n^2) \\ \|\mathcal{L}\widetilde{V}_n\| &\le L^{-2}\|\widetilde{V}_n\| \end{aligned} \tag{15.5}$$

How does (15.5) behave? Consider first the differential equation

$$\frac{d\lambda}{dn} = -\beta\lambda^2$$

so

$$\frac{d\lambda}{\lambda^2} = -\beta dn,\ \frac{1}{\lambda_n} = \frac{1}{\lambda} + \beta n,\ \lambda_n = \frac{\lambda}{1+n\beta\lambda} \to 0$$

i.e. for large $n, \lambda_n \sim \frac{1}{n\beta}$.

Homework. Prove that (15.5) $\Rightarrow \lambda_n < \frac{c}{n}$.

Actually there is some interest in calculating $b(L)$. We have

$$b(L) = -36\int d^4y \int_0^{\log L} ds \frac{d}{ds}(e^{-2s}G(e^{-s}\vec{y}))^2$$

but
$$\frac{d}{ds}\Big(e^{-2s}G(e^{-s}\vec{y}\,)\Big)^2 = (-4 - \vec{y}\cdot\vec{\nabla}_y)\Big(e^{-2s}G(e^{-s}\vec{y}\,)\Big)^2$$
where we used the fact that $G(\vec{x}\,)$ depends on $|\vec{x}| : G(\vec{x}) = \widetilde{G}(r)\ r = |\vec{x}|$ and
$$\frac{d}{ds}\widetilde{G}(e^{-s}r) = -e^{-s}r\widetilde{G}'(e^{-s}r) = -r\frac{d}{dr}\widetilde{G}(e^{-s}r) = -\vec{x}\cdot\vec{\nabla}_x G(e^{-s}\vec{x}\,).$$
Thus, (with $A = \int_{S^3} d\Omega$)
$$\begin{aligned} b(L) &= +36\int_0^{\log L} ds \int_0^\infty dr\ r^3\Big(4 + r\frac{d}{dr}(e^{-2s}G(e^{-s}r))^2\Big)\int_{S^3} d\Omega \\ &= 36A\int_0^{\log L} ds \int_0^\infty dr\frac{d}{dr}\Big(r^4(e^{-2s}G(e^{-s}r))^2\Big) \\ &= 36A\int_0^{\log L} ds \lim_{r\to\infty} r^4\Big(e^{-2s}G(e^{-s}r)\Big)^2. \end{aligned}$$
But
$$\begin{aligned} G(r) &= \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip_1 r}}{p^2}e^{-p^2} \\ &= r^{-2}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip_1}}{p^2}e^{-p^2/r^2} \underset{r\to\infty}{\longrightarrow} \quad r^{-2}\int\frac{d^3p}{(2\pi)^3}\frac{e^{-\sqrt{p^2}}}{2\sqrt{p^2}}. \end{aligned} \tag{15.6}$$
So
$$\lim_{r\to\infty} r^4\Big(e^{-2s}G(e^{-s}r)\Big)^2 = \Big(\int_0^\infty \frac{d\rho}{2\pi^2}\rho e^{-\rho}\Big)^2 = \Big(\frac{1}{2\pi^2}\Big)^2.$$
Hence
$$b(L) = 9A\pi^{-4}\log L \equiv \beta_2 \log L$$
A = area of unit sphere in $\mathbb{R}^4$.

Remark. β_2 is *universal*, it did not depend on our cutoff e^{-p^2}. We could have used <u>any</u> cutoff $\chi(p)$! This is a deep fact, see below.

Summary. For $d = 4$, again we find a $r_0(\lambda)$ such that
$$R^n(r_0\int :\varphi^2: + \lambda\int :\varphi^4:) = V_n$$
has $\lambda_n \to 0$ like $\frac{1}{n}, r_n = \mathcal{O}(\lambda_n^2) = \mathcal{O}(\frac{1}{n^2}) \to 0\ \widetilde{V}_n = \mathcal{O}(\lambda_n^2) \to 0$ and
$$z_n \to z_\infty = \sum_{n=0}^\infty \mathcal{O}(\lambda_n^2) < \infty$$
since $\sum \frac{1}{n^2}$ converges. We go to the Gaussian fixed point, but only *logarithmically* in the scale : $n \sim \log L^n$.

15.5 $d < 4$

Now $\lambda_n = L^{4-d}\lambda_n + \mathcal{O}(\lambda_n^2)$ *increases*, so sooner or later λ_n is big enough so we can not trust our perturbative analysis that holds only if λ_n is small enough.

It is very instructive to pretend that d is a continuous variable and take $d = 4-\epsilon$, ϵ small. Then

$$\lambda_{n+1} = L^{\epsilon}\lambda_n - b(L)\lambda_n^2 + \mathcal{O}(\lambda_n^3).$$

Let us look for a fixed point $\lambda^* = a\epsilon + \mathcal{O}(\epsilon^2)$. We get $1 = L^{\epsilon} - b(L)a\epsilon + \mathcal{O}(\epsilon^2)$ or $a = \frac{\log L}{b(L)} = \beta_2^{-1}$ above (we used $b(L)\Big|_{d=4-\epsilon} = b(L)\Big|_{d=4} + \mathcal{O}(\epsilon) = \beta_2 \log L + \mathcal{O}(\epsilon)$). Thus a fixed point would be

$$\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2).$$

However, now $z_{n+1} = z_n + \mathcal{O}(\lambda_n^2) = z_n + \mathcal{O}(\lambda^{*2})$ *increases.* What went wrong ? We have indeed a new fixed point $\lambda^* \neq 0$ and we need a different scaling of φ in our RG :

$$L^{-\frac{d-2}{2}} \to L^{-\frac{d-2+\eta}{2}}.$$

Let us proceed formally. Now

$$\begin{aligned}
&e^{-\mathcal{H}} = e^{-\frac{1}{2}(\varphi,G^{-1}\varphi)-V} \approx e^{-\frac{1}{2}\int(\nabla\varphi)^2-V} \\
&\to e^{-L^{-\eta}\frac{1}{2}\int(\nabla\varphi)^2-RV(L^{-\frac{\eta}{2}}\varphi)} \\
&= e^{-\frac{1}{2}(L^{-\eta}+\mathcal{O}(\lambda^2))\int(\nabla\varphi)^2-r_1\int:\varphi^2:-\lambda_1\int:\varphi^4:-\widetilde{V}}.
\end{aligned}$$

Thus, $1 + z_0 = 1$ ($z_0 = 0$)

$$1 + z_{n+1} = L^{-\eta}(1+z_n) + \mathcal{O}(\lambda_n^2)$$

$\Rightarrow \eta = \mathcal{O}(\epsilon^2)$ and

$$\begin{aligned}
\lambda_{n+1} &= L^{\epsilon-2\eta}\lambda_n - \beta_2 \log L\lambda_n^2 + \mathcal{O}(\lambda_n^3) \\
&= L^{\epsilon}\lambda_n + \mathcal{O}(\epsilon^2)\lambda_n - \beta_2\log L\lambda_n^2 + \mathcal{O}(\lambda_n^3)
\end{aligned}$$

so λ^* is still $\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2)$.
We got also $r_{n+1} = L^{2-\eta}r_n + \mathcal{O}(\lambda_n^2)$, $\|\mathcal{L}\widetilde{V}_n\| \leq L^{-2-\eta}\|\widetilde{V}_n\|$.

Summary $d = 4-\epsilon$

Now we have a new fixed point

$$\mathcal{H}^* = \frac{1}{2}\int(\nabla\varphi)^2 + r^*\int :\varphi^2: + \lambda^*\int :\varphi^4: + \widetilde{V}^*$$

with $\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2)$, $r^* = \mathcal{O}(\epsilon^2)$, $\widetilde{V}^* = \mathcal{O}(\epsilon^2)$.
Also

$$\Big\langle \varphi(x)\varphi(y)\Big\rangle_{\mathcal{H}^*} \sim \frac{1}{|x-y|^{d-2+\eta}}$$

$\eta = \mathcal{O}(\epsilon^2)$.

Remark $d = 3$ is $\epsilon = 1$ i.e. is non-perturbative. We expect, say for block-spin RG, that for all λ there is $r(\lambda)$ such that $\mathcal{H} = \sum(\varphi(x) - \varphi(y))^2 + r(\lambda)\sum\varphi(x)^2 + \lambda\sum\varphi(x)^4$ is critical and $\exists\eta$ such that

$$R^n\mathcal{H} \to \mathcal{H}^*$$

where R^n has $C\varphi(y) = L^{+\frac{d-2+\eta}{2}}\frac{1}{L^d}\sum\varphi(Lx+y)$ $(d=3)$ and $\eta, \mathcal{H}^*$ are λ-independent. $\mathcal{H}^*$ has only one relevant direction, namely the correlation length and no marginal ones (for small ϵ, one sees that the marginal $(\nabla\varphi)^2$ becomes irrelevant).

16 The Continuum Limit

16.1 Effective Field Theories

We finally consider the QFT problem from the RG point of view. Suppose we have a scalar QFT i.e. a measure ν on $S'(\mathbb{R}^d)$. Let σ_i^n be L^{-n} cube centered at $L^{-n}i, i \in \mathbb{Z}^d$. Suppose $L^{+nd}\int_{\sigma_i^n}\varphi(x)dx \equiv \varphi_n(L^{-n}i)$ are well defined random variables (for the free field they are, the covariance is just $L^{2dn}\int_{\sigma_i^n}dx\int_{\sigma_j^n}dyG(x-y)$), i.e. given a $f \in \mathbf{s}(\mathbb{Z}^d)$ suppose

$$S_n(f) = \int e^{i\sum f_j\varphi_n(L^{-n}j)L^{-nd}}d\nu(\varphi)$$

is a positive definite function (this means that $S(f) = \int e^{i\varphi(f)}d\nu$ extends from $f \in S(\mathbb{R}^d)$ to $f = \sum f_j\chi_{\sigma_j^n}$, χ characteristic function). Then S_n defines a measure on $\mathbf{s}'(L^{-n}\mathbb{Z}^d)$, call it ν_n which is the joint distribution of the variables $\varphi_n(L^{-n}j)$. We call ν_n the *effective QFT on scale* L^{-n} corresponding to ν. Thus, the correlation functions of the effective theory are just averages of those of ν over cubes σ_i^n :

$$\Big\langle\prod_{\alpha=1}^N\varphi_n(L^{-n}i_\alpha)\Big\rangle_{\nu_n} = L^{Ndn}\prod_{\alpha=1}^N\int_{\sigma_{i_\alpha}^n}dx_{i_\alpha}\Big\langle\prod_{\alpha=1}^N\varphi(x_{i_\alpha})\Big\rangle_\nu. \tag{16.1}$$

It is convenient to relate all of these effective theories to a fixed lattice $\mathbb{Z}^d$. For this, define a $\mathbb{Z}^d$ field $\phi_n(i) = L^{-an}\varphi_n(L^{-n}i)$ where a is a parameter to be fixed later, and let μ_n be the distribution of ϕ_n, i.e.

$$\Big\langle\prod_\alpha\phi_n(i_\alpha)\Big\rangle_{\mu_n} = \Big\langle\prod_\alpha L^{-an}\varphi_n(L^{-n}i_\alpha)\Big\rangle_{\nu_n}. \tag{16.2}$$

How are the different ϕ_n's related ? Easy :

$$\begin{aligned}\phi_{n-1}(i) &= L^{-a(n-1)}\varphi_{n-1}(L^{-n+1}i) = L^{-a(n-1)}L^{(n-1)d}\int_{\sigma_i^{n-1}}\varphi(x)dx \\ &= L^{-a(n-1)}L^{-d}\sum_{|j_\alpha|<L/2}L^{nd}\int_{\sigma_{Li+j}^n}\varphi(x)dx = L^{a-d}\sum_j\phi_n(Li+j) \\ &= (C\phi_n)(i)\end{aligned} \tag{16.3}$$

where C is the block spin operator. Thus

$$\mu_{n-1} = C^* \mu_n.$$

Suppose μ_n are given by Hamiltonians $\mathcal{H}_n$. Then

$$\mathcal{H}_{n-m} = R^m \mathcal{H}_n \qquad \forall n, m.$$

Definition 16.1 Let $R\mathcal{H}^* = \mathcal{H}^*$. The *unstable manifold* $\mathcal{M}_u$ of $\mathcal{H}^*$ is

$$\mathcal{M}_u = \Big\{ \mathcal{H} \Big| \exists\, \mathcal{H}_n \underset{n\to\infty}{\longrightarrow} \mathcal{H}^*, R^n \mathcal{H}_n = \mathcal{H} \Big\}.$$

Note that $R : \mathcal{M}_u \to \mathcal{M}_u$. Thus, we are led to the conjecture :

There exists $a \in \mathbb{R}$ and $\mathcal{H}^*$ such that $R\mathcal{H}^* = \mathcal{H}^*$ and $\mathcal{H}_n \underset{n\to\infty}{\longrightarrow} \mathcal{H}^*$. Then all $\mathcal{H}_n$ are on the unstable manifold of $\mathcal{H}^*$. Hence we got a nice picture: critical phenomena are on $\mathcal{M}_S$, QFT on $\mathcal{M}_u$!

Example ν = Gaussian measure, covariance $\widehat{G}(p) = \frac{1}{p^2+m^2}$ (free field, mass m). Then ν_n has covariance

$$G_n(L^{-n}i, L^{-n}j) = \int_{\mathbb{R}^d} e^{ipL^{-n}(i-j)} \widehat{G}(p) \Big| L^{nd} \int_{\sigma_0^n} e^{ipx} dx \Big|^2 \frac{d^d p}{(2\pi)^d}.$$

Let $\rho_n(p) = \Big| L^{nd} \int_{\sigma_0^n} e^{ipx} dx \Big|^2$ and, for $a = \frac{d-2}{2}$, μ_n has covariance C_n :

$$\begin{aligned} C_n(i,j) &= L^{(2-d)n} \int e^{ipL^{-n}(i-j)} \frac{1}{p^2+m^2} \rho_n(p) \frac{d^d p}{(2\pi)^d} \\ &= \int e^{ip(i-j)} \frac{1}{p^2 + L^{-2n} m^2} \rho_n(L^n p) \frac{d^d p}{(2\pi)^d} \end{aligned}$$

where

$$\rho_n(L^n p) = \Big| \int_{\sigma_0^0} e^{ipx} dx \Big|^2 = \prod_{\mu=1}^{d} \left(\frac{2 \sin p_{\mu/2}}{p_\mu} \right)^2 \equiv \rho(p)$$

is n-independent. Here C_n are covariances that parametrize the Gaussian part of the unstable manifold of the Gaussian fixed point $C_{n=\infty}$ of the block spin RG. They have correlation length $L^n m^{-1}$.

Suppose now that we have a RG $R : \mathcal{K} \to \mathcal{K}$ and a fixed point $\mathcal{H}^*$. How would we get a QFT from these data ?

Easiest way is the *scaling limit* : For $x_i \neq x_j$ $i \neq j$, consider the limit

$$\lim_{n\to\infty} L^{Nan}\Big\langle \prod_{i=1}^{N} \phi(L^n x_i)\Big\rangle_{\mathcal{H}^*} \equiv \lim_{n\to\infty} G^{(n)}(x_1, \cdots, x_N) \equiv G(x_1, \cdots, x_N). \tag{16.4}$$

(to be more precise, suppose $x_i \in (L^{-M}\mathbb{Z})^d$ for some M).

We expect this limit to exist. We expect that for $|y_i - y_i|$ large $\forall i \neq j$, we have for $C(y_1, \cdots, y_N) \equiv \langle \prod \phi(y_i)\rangle_{\mathcal{H}^*}$:

$$\begin{aligned} &\Big|C(y_1+z, y_2, \cdots, y_N) - C(y_1, y_2, \cdots, y_N)\Big| \\ &\approx (\min_{i>1} |y_1 - y_i|)^{-\epsilon}\Big|C(y_1 \cdots y_N)\Big| \end{aligned}$$

$\epsilon > 0$, if $|z| < \mathcal{O}(1)$ (say L). Then

$$\Big\langle \prod_{i=1}^{N} \phi(L^n x_i)\Big\rangle_{\mathcal{H}^*} = \Big\langle \prod_{i=1}^{N} \phi_{av}(L^n x_i)\Big\rangle_{\mathcal{H}^*}(1+\mathcal{O}(L^{-n\epsilon}))$$

where $\phi_{av}(Lx) = L^{-d} \sum_{|y_\alpha|<\frac{L}{2}} \phi(Lx+y)$. Thus,

$$\begin{aligned} G^{(n)}(x_1, \cdots, x_N) &= L^{Nan}\Big\langle \prod_i \phi_{av}(L(L^{n-1}x_i))\Big\rangle_{\mathcal{H}^*}(1+\mathcal{O}(L^{-n\epsilon})) \\ &= G^{(n-1)}(x_1, \cdots, x_N)(1+\mathcal{O}(L^{-n\epsilon})) \end{aligned}$$

since $(C\phi)(x) = L^a\phi_{av}(Lx)$ and $R\mathcal{H}^* = \mathcal{H}^*$. Thus, one expects the limit in (16.4) to exist. The distributions $G(x_1, \cdots, x_N)$ are correlations of a QFT ($x_i \in \mathbb{R}^d$ now). What are the effective field theories ν_m or μ_m or $\mathcal{H}_m$ of this QFT ?

Answer All $\mathcal{H}_m$ are just $\mathcal{H}^*$!

Proof This is just bookkeeping. By definition (16.1), (16.2),

$$\Big\langle \prod_{\alpha=1}^{N} \phi(i_\alpha)\Big\rangle_{\mu_m} = \int dx_1 \cdots dx_N \chi(x_\alpha \in \sigma^m_{i_\alpha}) G(x_1, \cdots, x_N) L^{N(d-a)m}$$

where $\sigma^m_{i_\alpha} = L^{-m}$ cube at $L^{-m}i_\alpha$. Write the integral as a Riemann sum and use the definition (16.4) of G:

$$\int \prod dx_\alpha \chi G = \lim_{n\to\infty} \sum_{\{x_\alpha \in L^{-n}\mathbb{Z}^d\}} L^{-Ndn}\chi(x_\alpha \in \sigma^m_{i_\alpha}) L^{N(d-a)m} L^{Nan}\Big\langle \prod_{\alpha=1}^{N} \phi(L^n x_\alpha)\Big\rangle_{\mathcal{H}^*}.$$

Write $x_\alpha = L^{-n} y_\alpha$. Then y_α is summed over $y_\alpha \in \mathbb{Z}^d, y_\alpha \in L^{n-m}$-cube at $L^{n-m} i_\alpha$ (take $n > m$). Thus, using (16.3),

$$\begin{aligned}\left\langle \prod_{\alpha=1}^{N} \phi(i_\alpha) \right\rangle_{\mu_m} &= \lim_{n\to\infty} \left\langle \prod_{\alpha} (C^{n-m}\phi)(i_\alpha) \right\rangle_{\mathcal{H}^*} \\ &= \left\langle \prod_{\alpha=1}^{N} \phi(i_\alpha) \right\rangle_{\mathcal{H}^*}. \qquad \square\end{aligned}$$

The scaling limit is a scale-invariant, massless, field theory :

$$G_N(\lambda x_1, \cdots, \lambda x_N) = \lambda^{-Na} G(x_1 \cdots x_N).$$

In particular (assuming translation invariance):

$$G_2(x_1, x_2) = \frac{\text{const}}{|x_1 - x_2|^a}.$$

16.2 Non-Scale Invariant Theories

How to get a QFT with effective theories $\mathcal{H}_n \neq \mathcal{H}^*$? E.g. we would like to get a massive theory. We would like to pose this constructively : find (lattice) cutoff-L^{-n} theories that converge as $n \to \infty$ to a QFT.

Example Look at free field again. Consider Ginzburg-Landau model on $\mathbb{Z}^d$ with correlation length $\sim L^n$:

$$\mathcal{H}_n = \frac{1}{2} \sum_{\langle xy \rangle} \Big(\phi(x) - \phi(y)\Big)^2 + \frac{L^{-2n} m^2}{2} \sum_x \phi(x)^2$$

i.e.

$$\begin{aligned} e^{-\mathcal{H}_n} D\phi &= d\mu_{C_n}(\phi) \\ C_n &= (-\Delta + L^{-2n} m^2)^{-1} \end{aligned}$$

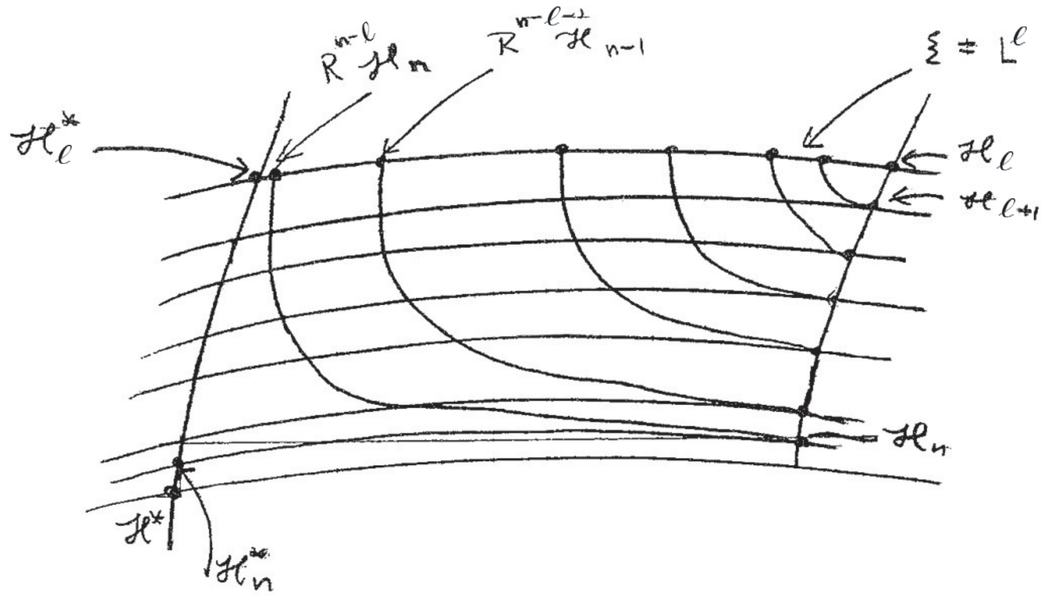
on $\mathbb{Z}^d$. We know that

$$L^{2n(\frac{d-2}{2})} \Big\langle \phi(L^n x)\phi(L^n y) \Big\rangle_{C_n} \underset{n\to\infty}{\longrightarrow} \left(\frac{1}{-\Delta_{\mathbb{R}^d} + m^2} \right)(x, y).$$

From RG we find

$$\begin{aligned} R^{n-\ell} \mathcal{H}_n &\underset{n\to\infty}{\longrightarrow} \frac{1}{2}(\phi, G_\ell^{-1}\phi) = \mathcal{H}_\ell^* \\ G_\ell(x, y) &= \int_{\mathbb{R}^d} e^{ip(x-y)} \frac{1}{p^2 + L^{-2\ell} m^2} \rho(p) \frac{d^d p}{(2\pi)^d} \qquad (x, y \in \mathbb{Z}^d) \end{aligned}$$

which are on the unstable manifold : $R\mathcal{H}_\ell^* = \mathcal{H}_{\ell-1}^*$. Pictorially,
So, in general, we should look for a 1-parameter family of Hamiltonians $\mathcal{H}_{L^n}$ such that $R^{n-\ell}\mathcal{H}_{L^n}$ converge as $n \to \infty$ for all m. It is easy to see that $\mathcal{H}_{L^n}$ must tend to $\mathcal{M}_S$ as $n \to \infty$ and $R^{n-\ell}\mathcal{H}_{L^n}$ to $\mathcal{M}_u$.
As an example we consider the φ^4 QFT in various dimensions.



16.3 φ_d^4

We consider a momentum cutoff for simplicity and let

$$d\nu_\epsilon(\varphi) = \frac{1}{Z} e^{-z_\epsilon \int_\Lambda :(\nabla\varphi)^2: - \rho_\epsilon \int_\Lambda :\varphi^2: - g_\epsilon \int_\Lambda :\varphi^4:} d\mu_{G_\epsilon}.$$

Let $z_\epsilon \int_\Lambda : (\nabla\varphi)^2 : + \rho_\epsilon \int_\Lambda : \varphi^2 : + g_\epsilon \int_\Lambda : \varphi^4 := W_\epsilon(\varphi)$; here[8] $\widehat{G}_\epsilon(p) = \frac{1}{p^2+m^2}\chi(\epsilon p)$ and let, e.g., $\chi = e^{-p^2}$. We ask : are there functions of ϵ $z_\epsilon, \rho_\epsilon, g_\epsilon$ such that ν_ϵ converges to a measure ν on S' as $\epsilon \to 0$? We know that $\lambda = 0$ is a solution, and ask if a non-Gaussian ν is possible. The effective measures are defined as above, only easier : Write, for $\ell > \epsilon$

$$\begin{aligned} \widehat{G}_\epsilon(p) = \frac{1}{p^2+m^2}\chi(\epsilon p) &= \frac{1}{p^2+m^2}\chi(\ell p) + \frac{1}{p^2+m^2}\Big(\chi(\epsilon p) - \chi(\ell p)\Big) \\ &\equiv \widehat{G}_\ell(p) + \widehat{\Gamma}_{\epsilon\ell}(p) \end{aligned}$$

and define

$$d\nu_\ell^{(\epsilon)}(\varphi) = \left(\int e^{-W_\epsilon(\varphi+z)} d\mu_{\Gamma_{\epsilon\ell}}(z)\right) d\mu_{G_\ell}(\varphi)$$

which has cutoff ℓ. To connect to Statistical Mechanics and RG, we need to scale as in (16.2) :

$$\phi_\epsilon(x) = \epsilon^{\frac{(d-2)}{2}} \varphi(\epsilon x)$$

(we take $a = \frac{d-2}{2}$ to anticipate what comes later). Then ν_ϵ becomes μ_ϵ:

$$d\mu_\epsilon(\phi) = e^{-W_\epsilon(\epsilon^{\frac{2-d}{2}}\phi(\frac{\cdot}{\epsilon}))} d\mu_{C_\epsilon}(\phi) \equiv e^{-V_\epsilon(\phi)} d\mu_{C_\epsilon}$$

[8] We put m^2 into the covariance for simplicity : we are concerned with the **UV**-problem, not the **IR**.

where $\widehat{C}_\epsilon(p) = \frac{1}{p^2+\epsilon^2 m^2}\chi(p)$, has **UV**-cutoff 1, and is Gaussian. (Note the **IR** cutoff $\epsilon^2 m^2$). Here

$$V_\epsilon(\phi) = z_\epsilon \int :(\nabla\varphi)^2: + \epsilon^2 \rho_\epsilon \int :\varphi^2: + \epsilon^{4-d} g_\epsilon \int :\varphi^4:$$

note the powers of ϵ (so called canonical dimensions of mass, coupling). Then we have

$$V_\ell^{(\epsilon)} \equiv R_{\ell/\epsilon} V_\epsilon$$

where R_L is as before :

$$R_L V = -\log \int e^{-V(L^{\frac{2-d}{2}}\phi(\frac{\cdot}{L})+Z)} d\mu_\Gamma(Z)$$

$\widehat{\Gamma}(p) = \frac{1}{p^2}(\chi(p) - \chi(Lp))$. Let us put

$$r_\epsilon = \epsilon^2 \rho_\epsilon, \quad \lambda_\epsilon = \epsilon^{4-d} g_\epsilon.$$

As before, we could study the iteration $r_n, z_n, \lambda_n \to r_{n+1}, z_{n+1}, \lambda_{n+1}$ but here we can also do it infinitesimally, i.e. as a differential equation. We put $L^n = e^s$ and get, for $R_{e^s} V_\epsilon = \Big(z_\epsilon(s), r_\epsilon(s), \lambda_\epsilon(s), \widetilde{V}_\epsilon(s)\Big) \equiv V_\epsilon(s)$:

$$\begin{aligned}
\frac{dz}{ds} &= \mathcal{O}(V^2) \\
\frac{dr}{ds} &= 2r + \mathcal{O}(V^2) \\
\frac{d\lambda}{ds} &= (4-d)\lambda - \beta_2 \lambda^2 + \mathcal{O}\Big(\lambda^3, \lambda(z+r+\widetilde{V}), (z+r+\widetilde{V})^2\Big) \\
\frac{d\widetilde{V}}{ds} &= \mathcal{M}\widetilde{V} + \mathcal{O}(V^2)
\end{aligned}$$

where

$$\mathcal{M}\widetilde{V}(\phi) = \frac{d}{ds}\Big|_0 \int \widetilde{V}(e^{\frac{2-d}{2}s}\phi(e^{-s}\cdot) + Z) d\mu_\Gamma(Z).$$

We want to choose $r_\epsilon(0), z_\epsilon(0), \lambda_\epsilon(0)$ such that $\lim_{t\to\infty} V_{e^{-t}}(t+s) \equiv V^{\text{eff}}(s)$ exists, $\forall s \in \mathbb{R}$.

d=2. Take $r_\epsilon = 0, \lambda_\epsilon = \epsilon^2 g, z_\epsilon = 0$.

Put

$$\lambda_\epsilon(s) = \epsilon^2 e^{2s} g_t(s),\ r_\epsilon(s) = \epsilon^2 e^{2s} \rho_t(s),\ \epsilon = e^{-t}.$$

We get

$$\frac{dg}{ds} = \mathcal{O}(e^{-2(t-s)} g^2),\ \frac{d\rho}{ds} = \mathcal{O}(e^{-2(t-s)} g^2).$$

Hence the limits $\lim_{t\to\infty} g_t(t+s)$, $\lim_{t\to\infty} r_t(t+s)$ exist. This is a super renormalizable theory : the "bare" coupling g_ϵ can be taken ϵ-independent and $z_\epsilon, \rho_\epsilon$ also.

d=3. Now $\lambda_\epsilon(s) = \epsilon e^s g_t(s)$ and we have a "resonance" in the ρ-equation:

$$\frac{dg}{ds} = \mathcal{O}(e^{-(t-s)}g^2)$$
$$\frac{d\rho}{ds} = \mathcal{O}(g^2).$$

More precisely, write $\frac{dr}{ds} = 2r + \gamma\lambda^2 + \mathcal{O}(\lambda^3)$, then,

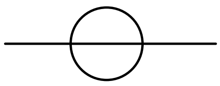
$$\frac{d\rho}{ds} = \gamma g^2 + \mathcal{O}(e^{-(t-s)}g^3).$$

Since g stays almost constant, we get

$$\rho_t(s) = \rho_t(0) + s\gamma g^2 + \text{bounded}.$$

Thus, we need to take $\rho_t(0) = -t\gamma g^2$ i.e.

$$g_\epsilon = g, \ \rho_\epsilon = \gamma g^2 \log\epsilon$$

this is a "mass counterterm" of $\mathcal{O}(g^2)$, coming from the graph  that diverges logarithmically in **UV**.

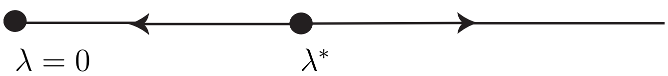
d=4. Now we stay in the (r, λ) picture. λ *decreases* logarithmically :

$$\frac{d\lambda}{ds} = -\beta_2\lambda^2 + \mathcal{O}(\lambda^3)$$
$$\lambda(s) \sim \frac{\lambda(0)}{1 + \beta_2 s\lambda_\epsilon(0)}.$$

If $\lambda_\epsilon(0)$ stays bounded in ϵ, as we have to assume in our perturbative analysis, then $\lambda(s) \underset{s\to\infty}{\longrightarrow} 0$. Thus the only perturbative theory (= the one where the coupling constant remains small on all scales) is the Gaussian. Same holds for

d > 4. $\frac{d\lambda}{ds} = -(d-4)\lambda + \mathcal{O}(\lambda^2)$ i.e. $\lambda(s) \to 0$.

One might hope that, by taking $\lambda_\epsilon(0)$ large enough, things might change. E.g. there might be another fixed point λ^* for the RG, i.e. the flow in λ would be



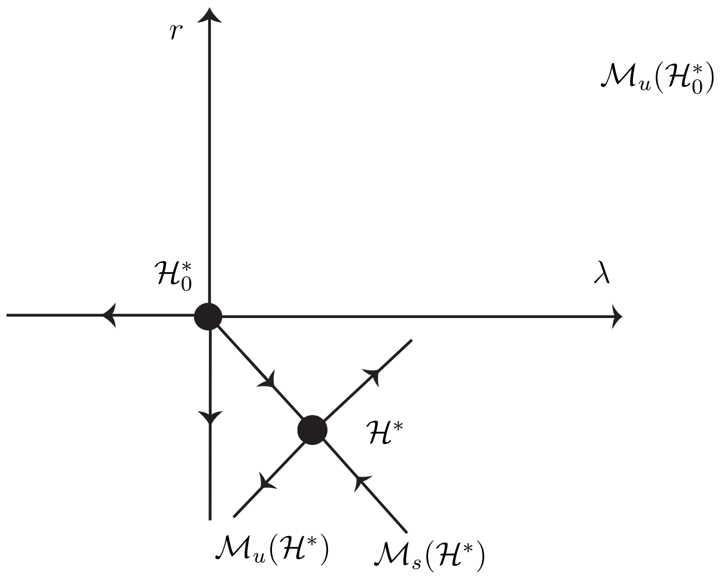
Unfortunately, this is unlikely to happen : in $d > 4$ it is rigorously proven that for <u>all</u> $z(\epsilon), r(\epsilon), \lambda(\epsilon)$ the only $\epsilon \to 0$ limits are Gaussian. For $d = 4$ there are strong theoretical and numerical arguments supporting the same conjecture.

d<4: nontrivial fixed points.

Consider the $d = 3$ QFT constructed above. This has mass $\sim m^2$, coupling λ. What about $m \to 0$? Above we could have taken $m = 0$ and started with $r_\epsilon, \lambda_\epsilon$ to produce an effective Hamiltonian on scale 1 of the form

$$z \int : \nabla\varphi^2 : +r \int : \varphi^2 : +\lambda \int : \varphi^4 : +\widetilde{V}$$

with various values of r. To solve for the **IR** we would need to keep iterating the RG. Now λ increases and eventually, if we really choose $r = r(\lambda) =$ critical point, we would tend to the nontrivial fixed point $\mathcal{H}^*$. Thus, **UV** is controlled by the Gaussian fixed point, **IR** by the non-Gaussian.
We could construct a scaling limit at $\mathcal{H}^*$: this would be a scale-invariant QFT with non-Gaussian **IR** and **UV**. $\mathcal{H}^*$ has (at least) one relevant direction : the correlation length. We could construct, as above, a *massive* QFT corresponding to the unstable manifold of $\mathcal{H}^*$. We expect the unstable manifold of the Gaussian fixed point $\mathcal{H}_0^*$ to be 2-dimensional (r, λ), and the unstable manifold of $\mathcal{H}^*$ to be one dimensional. So the flow is



Thus, a "typical" QFT is a typical point of the r, λ plane with Gaussian **UV** and finite correlation length. The ones on $\mathcal{M}_s(\mathcal{H}^*)$ have non-Gaussian **IR**, Gaussian **UV** and the ones on $\mathcal{M}_u(\mathcal{H}^*)$ have non-Gaussian **UV**, $\xi < \infty$. (We should really include also the $r = \infty$ ($\xi = 0$) "high-temperature" fixed point).

16.4 Asymptotic Freedom

What would have happened if β_2 above had been *negative* at $d=4$? Now

$$\frac{d\lambda}{ds}=|\beta_2|\lambda^2+\mathcal{O}(\lambda^3)\Rightarrow\lambda_\epsilon(0)=\frac{\lambda_\epsilon(s)}{1+|\beta_2|s\lambda_\epsilon(s)}$$

i.e. if $\epsilon=e^{-t}, s=t$, and we want to fix $\lambda_{e^{-t}}(t)=g$, then the "bare coupling" λ_ϵ is

$$\lambda_\epsilon=\frac{g}{1+|\beta_2|\log\epsilon^{-1}g}\underset{\epsilon\to 0}{\longrightarrow}0$$

i.e. to have a non-Gaussian theory, we need to let $\lambda_\epsilon\to 0$ as $\epsilon\to 0$. This is obvious, since λ is now *unstable.* Actually, it is instructive to calculate exactly λ_ϵ : we need the λ-equation to $\mathcal{O}(\lambda^3)$ (i.e. again, we need to consider $z,r,\widetilde{V}$ too, to $\mathcal{O}(\lambda^3)$). So consider

$$\frac{d\lambda}{ds}=\beta(\lambda)=\beta_2\lambda^2+\beta_3\lambda^3+\tilde{\beta}(\lambda),\ \tilde{\beta}=\mathcal{O}(\lambda^4).$$

So

$$\begin{aligned}
&\int_{\lambda_\epsilon}^{g}\frac{d\lambda}{\beta(\lambda)}=\log\epsilon^{-1}\\
&=\int_{\lambda_\epsilon}^{g}d\lambda\frac{1}{\beta_2\lambda^2}\frac{1}{1+\frac{\beta_3}{\beta_2}\lambda+\mathcal{O}(\lambda^2)}=\int_{\lambda_\epsilon}^{g}d\lambda\left(\frac{1}{\beta_2\lambda^2}-\frac{\beta_3}{\beta_2^2\lambda}+\mathcal{O}(1)\right)\\
&=\frac{1}{\beta_2\lambda_\epsilon}-\frac{1}{\beta_2 g}-\frac{\beta_3}{\beta_2^2}\log\frac{g}{\lambda_\epsilon}+\mathcal{O}(g)=\log\epsilon^{-1}\\
&\Rightarrow\lambda_\epsilon=\frac{g}{1+\beta_2 g\log\epsilon^{-1}-\frac{g\beta_3}{\beta_2}\log\log\epsilon^{-1}+\mathcal{O}(1)}.
\end{aligned}$$

This is how the bare coupling should be chosen to have a continuum limit.

Remark. We have seen that β_2 is universal (independent on regularization). It can be shown that β_3 is also. (β_4 is not). They are intrinsic, depend only on the continuum limit (which does not depend on the regularization). An example of asympotically free theory is the Ginzburg-Landau model with λ *negative.* This is not stable, but can be rigorously defined by analytic continuation. Since $\lambda=-g,\ g>0$ we get $\frac{dg}{ds}=\beta_2 g^2+\cdots\ \beta_2>0$. More interesting one is the $SU(N)$ gauge theory. There one has two couplings z and λ, both marginal, with

$$\begin{aligned}
\frac{dz}{ds}&=\gamma\lambda^2+\cdots\\
\frac{d\lambda}{ds}&=\beta_3\lambda^3+\cdots
\end{aligned}$$

so $\lambda(s)^2\approx\frac{g^2}{1+\beta_3 g^2(t-s)}$ where $\epsilon=e^{-t},\lambda_\epsilon^2(t)=g^2$ and $z_\epsilon(t)-z_\epsilon(0)=\gamma\int_0^t\lambda(s)^2\sim\frac{\gamma}{\beta_3}\log t$ so $z_\epsilon(0)$ must diverge as $-\frac{\gamma}{\beta_3}\log\log\epsilon^{-1}$.

Remarks.

1. Thus, we expect to find QFT's at fixed points and on their unstable manifolds. For Ginzburg-Landau model, $d < 4$ we find several non-Gaussian ones. For $d \geq 4$ the fixed points <u>and</u> unstable manifolds are Gaussian.

2. Asymptotically free theories have Gaussian fixed points with marginally unstable non-Gaussian direction. Here we have a non-Gaussian continuum limit.

Examples : *Non Abelian gauge theories* with not too many fermions (QCD is one). *Non-linear σ-models in $d = 2$* (e.g. $O(N), N > 2$ spin systems we discussed earlier). Certain $d = 2$ fermionic models (Gross-Neveu model).

3. **IR** behaviour of asymptotically free models is interesting. Now $\lambda(s) \nearrow$ as s increases. E.g. in non-Abelian gauge and σ-models. These theories have no relevant variable like r in Ginzburg-Landau model that would provide a mass (correlation length). Nevertheless, they are massive ($\xi < \infty$) : ξ with be $\sim e^{-\mathrm{const}/\lambda^2}$ (so called dimensional transmutation). There is no other critical point than $\lambda = 0$. (λ^2 is the temperature in Heisenberg model).

4. Theories that are expected to be Gaussian (trivial) are : Ginzburg-Landau model, QED, Standard Model of weak and EM interactions. This sounds paradoxical since QED has a very successful renormalized perturbation theory and is the most accurate known physical theory. The point is that, as long as we do not take ϵ to 0, we can keep $\lambda_\epsilon > 0$ and so $\lambda_\epsilon(s) > 0$. Now if one calculates a correlation function in perturbation theory, the graphs that are convergent integrals of the form $\int_{|p|<\epsilon^{-1}} I(p)dp$ depend on the cutoff like ϵ to some power. But we may take (recall, $\lambda_\epsilon = \frac{\lambda}{1-\beta_2 \log \epsilon^{-1}\lambda}$ has to be small so $((\log \epsilon^{-1})^{-1} \sim \lambda)$

$$\epsilon \sim e^{-\mathrm{const}/\lambda}.$$

   Hence, the dependence on cutoff is $\sim e^{-\mathrm{const}/\lambda}$, very small for λ small ($\frac{1}{\lambda} = 137$ for QED). We say that all the above theories are good *effective theories* for distances $>> \epsilon$ (in practice, energies $<< 10^{15}$ GeV). For smaller distances one needs to find a new theory (say string theory).

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