MODEL THEORY

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In these notes I will look at that part of "pre-Morley's theorem era" model theory that I feel is most relevant for the current trends in model theory. Some of the topics are not chosen because of the theorems but because of the methods behind the proofs. Only the surface of each topic chosen is scratched, for more see [CK] or [Ho]. When we attach some name(s) of persons to theorems, we just indicate the name of the theorem commonly used, the person(s) are not always the one(s) that actually proved the theorem originally. For the history, see the historical notes in [CK].

Contents

1. First-order logic

In this section we recall the basic definitions of the first-order logic.

1.1 Definition. *A vocabulary L is a collection of relation, function and constant symbols. Each relation symbol R and function symbol f come with the arity $\#R, \#f \in \mathbb{N} - \{0\}$.*

We let $L = \{R_i, f_j, c_k| \ i \in I^*, \ j \in J^*, \ k \in K^*\}$ be a fixed but arbitrary vocabulary.

1.2 Definition. *The collection of $(L$-$)$terms is defined as follows:*
(i) variables v_i, $i \in \mathbb{N}$, are terms,
(ii) constant symbols c_k, $k \in K^*$, are terms,
(iii) if $n = \#f_j$, $j \in J^*$, and $t_1, ..., t_n$ are terms, then $f_j(t_1, ..., t_n)$ is a term.

1.3 Definition. *The collection of atomic $(L$-$)$formulas is defined as follows:*
(i) if t and u are terms, then $t = u$ is an atomic formula,
(ii) if $n = \#R_i$, $i \in I^*$, and $t_1, ..., t_n$ are terms, then $R_i(t_1, ..., t_n)$ is an atomic formula,
(iii) $\top$ is an atomic formula.

The formula $\top$ is needed for the elimination of quantifiers in the case L does not contain constant symbols, see section 5.

1.4 Definition. *The collection of $(L$-$)$formulas is defined as follows:*
(i) atomic formulas are formulas,
(ii) if ϕ is a formula, then $\neg\phi$ is a formula,
(iii) if ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula,
(iv) if ϕ is a formula and $i \in \mathbb{N}$, then $\exists v_i \phi$ is a formula.
By $L_{\omega\omega}$ we denote the set of all L-formulas.

The following notation is used:

$$\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$$

$$\phi \rightarrow \psi = \neg\phi \vee \psi$$

$$\phi \leftrightarrow \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$$

$$\forall v_i \phi = \neg\exists v_i \neg\phi.$$

1.5 Definition. *The notion v_i is free in ϕ is defined as follows:*
(i) ϕ is atomic: v_i is free in ϕ if v_i appears in ϕ,
(ii) $\phi = \neg\psi$: v_i is free in ϕ if it is free in ψ,
(iii) $\phi = \psi \wedge \theta$: v_i is free in ϕ if it is free in ψ or θ,
(iv) $\phi = \exists v_j \psi$: v_i is free in ϕ if it is free in ψ and $i \neq j$.

A sentence is a formula in which no v_i is free.

If $x=(x_1,...,x_n)$ is a sequence of variables (when we write like this we assume that for $k\neq m$, $x_k\neq x_m$), then the notation $\phi(x)$ means that if v_i is free in ϕ then $v_i\in\{x_1,...,x_n\}$. Similarly for a term t, $t(x)$ means that if v_i appears in t, then $v_i\in\{x_1,...,x_n\}$. Often we split x into two (or more) sequences y and z and write $\phi(y,z)$ in place of $\phi(x)$.

1.6 Definition. *A (L-)structure (i.e. model) is a sequence*

$$\mathcal{A}=(\mathcal{A},R_i^{\mathcal{A}},f_j^{\mathcal{A}},c_k^{\mathcal{A}})_{i\in I^*,j\in J^*,k\in K^*}$$

where

(i) $\mathcal{A}$ is a non-empty set (the universe of $\mathcal{A}$, when we want to make a distinction between the model and its universe, we write $dom(\mathcal{A})$ for the universe),

(ii) $R^{\mathcal{A}_i}\subseteq\mathcal{A}^{\#R_i}$,

(iii) $f_j^{\mathcal{A}}:\mathcal{A}^{\#f_j}\rightarrow\mathcal{A}$,

(iv) $c_k^{\mathcal{A}}\in\mathcal{A}$.

When it does not risk confusion we write just $R_i=R_i^{\mathcal{A}}$ etc.

1.7 Definition. *For a term $t(x)$, $x=(x_1,...,x_n)$, structure $\mathcal{A}$ and $a=(a_1,...,a_n)\in\mathcal{A}^n$, $t^{\mathcal{A}}(a)$ is defined as follows:*

(i) $t=v_i$: $t^{\mathcal{A}}(a)=a_m$, where m is such that $v_i=x_m$,

(ii) $t=c_k$: $t^{\mathcal{A}}(a)=c_k^{\mathcal{A}}$,

(iii) $t=f_j(t_1,...,t_m)$: $t^{\mathcal{A}}(a)=f_j^{\mathcal{A}}(t_1^{\mathcal{A}}(a),...,t_m^{\mathcal{A}}(a))$.

1.8 Definition (Tarski). *For a formula $\phi(x)$, $x=(x_1,...,x_n)$, structure $\mathcal{A}$ and $a=(a_1,...,a_n)\in\mathcal{A}^n$, $\mathcal{A}\models\phi(a)$ is defined as follows:*

(i) $\phi=t=u$: $\mathcal{A}\models\phi(a)$ if $t^{\mathcal{A}}(a)=u^{\mathcal{A}}(a)$,

(ii) $\phi=R_i(t_1,...,t_m)$: $\mathcal{A}\models\phi(a)$ if $(t_1^{\mathcal{A}}(a),...,t_m^{\mathcal{A}}(a))\in R_i^{\mathcal{A}}$,

(iii) $\phi=\top$: $\mathcal{A}\models\phi(a)$ always,

(iv) $\phi=\neg\psi$: $\mathcal{A}\models\phi(a)$ if $\mathcal{A}\not\models\psi(a)$,

(v) $\phi=\psi\wedge\theta$: $\mathcal{A}\models\phi(a)$ if $\mathcal{A}\models\psi(a)$ and $\mathcal{A}\models\theta(a)$,

(vi) $\phi=\exists v_i\psi$: $\mathcal{A}\models\phi(a)$ if there is $b\in\mathcal{A}$ such that $\mathcal{A}\models\psi(b,a_1,...,a_n)$ for $\psi=\psi(v_i,x_1,...,x_n)$.

1.9 Remark. *In the Definition 1.8 (v) we assumed that $v_i\notin\{x_1,...,x_n\}$. This can be done without loss of generality, see the course Matemaattinen logiikka. This sloppy notation will be used regularly in these notes.*

1.10 Fact. *For all $\phi(x)$, $x=(x_1,...,x_n)$, and $y=(y_1,...,y_n)$, there is $\psi(y)$ such that for all $\mathcal{A}$ and $a\in\mathcal{A}^n$, $\mathcal{A}\models\phi(a)$ iff $\mathcal{A}\models\psi(a)$.*

Proof. See the course Matemaattinen logiikka. □

1.11 Definition. *For a structure $\mathcal{A}$, a relation $R \subseteq \mathcal{A}^n$ is definable (in the vocabulary L), if there are a formula $\phi(x, y)$, $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$, and $b \in \mathcal{A}^m$ such that for all $a \in \mathcal{A}^n$, $a \in R$ iff $\mathcal{A} \models \phi(a, b)$. The elements of b are called the parameters of the definition. If the parameters are not needed, we say that R is definable without parameters. A function $f : \mathcal{A}^n \to \mathcal{A}$ is definable if the relation $\{(a_1, ..., a_{n+1}) \in \mathcal{A}^{n+1} | \ f(a_1, ..., a_n) = a_{n+1}\}$ is definable.*

1.12 Exercise. *Show that the set of integers is definable without parameters in $(\mathbf{C}, +, \times, exp, 0, 1)$, where $+$ and $\times$ are the addition and multiplication of complex numbers and $exp(x) = e^x$.*

2. On ordinals and cardinals

In this section we recall some facts from set theory that are needed throughout these notes.

2.1 Definition. *Suppose $R \subseteq X^2$. We say that (X, R) is a well-ordering and alternatively that R well-orders X if*

(i) (X, R) is a linear ordering i.e. for all $x, y, z \in X$, (a)-(c) below holds:

(a) if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$,

(b) if $(x, y) \in R$, then $(y, x) \notin R$,

(c) $(x, y) \in R$ or $x = y$ or $(y, x) \in R$,

(ii) there are no $x_n \in X$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $(x_{n+1}, x_n) \in R$.

Notice that if $(X, <)$ is a well-ordering and $Y \subseteq X$ is not empty, then there is a $<$-least element in Y.

2.2 Definition. *We say that a set α is an ordinal if $(\alpha, \in)$ is a well-ordering and α is transitive i.e. for all x and y, if $y \in x \in \alpha$, then $y \in \alpha$. The class of all ordinals is denoted by On.*

2.3 Fact.

(i) $\in$ well-orders the class On (for ordinals α and β, instead of writing $\alpha \in \beta$ we write $\alpha < \beta$).

(ii) If α is an ordinal and $x \in \alpha$, then x is an ordinal i.e. $\alpha = \{\beta \in On | \ \beta < \alpha\}$.

(iii) For every well-ordering (X, R) there are a unique ordinal α and a unique bijection $\pi : X \to \alpha$ such that for all $x, y \in X$, $(x, y) \in R$ iff $\pi(x) < \pi(y)$.

(iv) For ordinals α and β, $\alpha \subseteq \beta$ iff $\alpha = \beta$ or $\alpha < \beta$.

(v) $\emptyset$ is an ordinal (usually denoted by 0), if α is an ordinal then $\alpha \cup \{\alpha\}$ is the least ordinal strictly greater than α (usually denoted by $\alpha + 1$) and if α_i, $i \in I$, are ordinals, then $\bigcup_{i \in I} \alpha_i$ is the least ordinal greater or equal to every α_i.

Proof. Basic set theory course or [Je]. □

Finite ordinals and natural numbers are often thought as the same, i.e. $0 = \emptyset$, $1 = 0+1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$, $2 = 1+1 = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$ etc. The set of all finite ordinals is called ω (by Fact 2.3 (v), ω is an ordinal) and so $\omega = \mathbb{N}$. An ordinal α

is a successor ordinal if there is an ordinal β such that $\alpha = \beta + 1$. Otherwise α is called a limit ordinal.

Many of our constructions and proofs are based on the following recursion and induction principles.

2.4 Fact.

(i) Suppose G is a function from sets to sets definable in the vocabulary of set theory (i.e. a class function). Then there is a unique class function F from On to sets such that for all ordinals α, $F(\alpha) = G(F \restriction \alpha)$.

(ii) Suppose P is a collection of sets definable in the vocabulary of set theory (i.e. a class) and $X \subseteq On$ is also a class. Then $X \subseteq P$ if for all ordinals $\alpha \in X$ the following holds:

(*) if for all $\beta \in X \cap \alpha$, $\beta \in P$, then $\alpha \in P$.

Proof. Basic set theory course or [Je]. ▫

2.5 Fact (Schröder-Bernstein). *If there are injections $f : X \to Y$ and $g : Y \to X$, then there is a bijection $\pi : X \to Y$.*

Proof. Basic set theory course or [Je]. ▫

2.6 Definition. *We say that an ordinal α is a cardinal if for all $\beta < \alpha$, there is no injection ($\Leftrightarrow$ bijection by Schröder-Bernstein) from α to β.*

Notice that every finite ordinal is a cardinal as well as ω and that infinite cardinals are limit ordinals.

2.7 Definition. *For all ordinals α, a cardinal ω_α (often also called $\aleph_\alpha$) is defined to be the least infinite cardinal strictly greater than ω_β for any $\beta < \alpha$.*

Notice that the class function $\alpha \mapsto \omega_\alpha$ exists by Fact 2.4 (i) and that $\omega_0 = \omega$ ($= \mathbb{N}$). For a cardinal κ, by κ^+ we denote the least cardinal $> \kappa$ (so if $\kappa = \omega_\alpha$, $\kappa^+ = \omega_{\alpha+1}$).

2.8 Fact.

(i) For every set X there are a cardinal κ and a bijection $\pi : X \to \kappa$. Furthermore such κ is unique and is denoted by $|X|$ and called the cardinality of X (or just the size or power of X).

(ii) Suppose that at least one of X and Y is infinite and that neither is empty. Then $|X \cup Y| = |X \times Y| = max\{|X|, |Y|\}$ (usually this cardinal is denoted by $|X| + |Y|$).

(iii) Suppose I is infinite and X_i, $i \in I$, are non-empty and distinct. Then $|\bigcup_{i\in I} X_i| = max\{\kappa, |I|\}$, where $\kappa = \bigcup_{i\in I} |X_i|$.

(iv) Suppose X is infinite and let $P(X)$ be the set of all subsets of X and X^X be the set of all function from X to X. Then $|P(X)| = |X^X| > |X|$. We denote $|P(X)|$ by $2^{|X|}$ (so 2^ω is the cardinality of the continuum).

(v) If κ is a cardinal and $\alpha < \kappa^+$, then there is no function $f : \alpha \to \kappa^+$ such that $\cup_{i<\alpha} f(i) = \kappa^+$ (i.e. successor cardinals are regular).

Proof. Basic set theory course or [Je]. ▫

3. Compactness

3.1 Definition. *Let $F \subseteq P(X)$.*

(i) F has the finite intersection property if for all $X_i \in F$, $i < n$, $\bigcap_{i<n} X_i \neq \emptyset$.

(ii) F is a filter if $X \in F$, $\emptyset \notin F$, if $Z, Y \in F$, then $Z \cap Y \in F$ and if $Z \in F$ and $Z \subseteq Y \subseteq X$, then $Y \in F$.

(iii) F is an ultrafilter if it is a filter and for all $Y \subseteq X$, either $Y \in F$ or $X - Y \in F$.

3.2 Lemma. *Suppose $F \subseteq P(X)$ has the finite intersection property. Then there is an ultrafilter $U \subseteq P(X)$ such that $F \subseteq U$.*

Proof. Let X_i, $i < \alpha$, enumerate all elements of $P(X)$. By recursion we define an increasing sequence of subsets U_i of $P(X)$ with the finite intersection property:

$i = 0$: $U_i = F$.

$i = j+1$: If $U_j \cup \{X_j\}$ has the finite intersection property, we let $U_i = U_j \cup \{X_j\}$. Otherwise $U_j \cup \{(X - X_j)\}$ has the finite intersection property and we let this be U_i. (If $(\bigcap_{k<n} Y_j) \cap X_i = \emptyset$ and $(\bigcap_{k<m} Z_j) \cap (X - X_i) = \emptyset$ then $(\bigcap_{k<n} Y_j) \cap (\bigcap_{k<m} Z_j) = \emptyset$.)

i is limit: $U_i = \bigcup_{j<i} U_j$.

It is easy to see that $U = \bigcup_{i<\alpha} U_i$ is as wanted (exercise) □

Suppose $\mathcal{A}_\eta$, $\eta \in X$, are models and $U \subseteq P(X)$ is an ultrafilter. By $\Pi_{\eta \in X} \mathcal{A}_\eta$ we mean the set of all $f : X \to \cup_{\eta \in X} \mathcal{A}_\eta$ such that for all $\eta \in X$, $f(\eta) \in \mathcal{A}_\eta$. Then $f \equiv g \ mod\ U$ if $\{\eta \in X |\ f(\eta) = g(\eta)\} \in U$ is an equivalence relation (exercise). By f/U we mean the equivalent class of f and let $\Pi_{\eta \in X} \mathcal{A}_\eta / U$ be the set of all these equivalence classes. We make $\Pi_{\eta \in X} \mathcal{A}_\eta / U$ into an L-structure $\mathcal{A}$ (also denoted by $\Pi_{\eta \in X} \mathcal{A}_\eta / U$) by adding the following interpretations:

$(g_1/U, ..., g_n/U) \in R_i^{\mathcal{A}}$ if $\{\eta \in X |\ (g_1(\eta), ..., g_n(\eta)) \in R_i^{\mathcal{A}_\eta}\} \in U$, where $n = \# R_i$,

$f_j^{\mathcal{A}}(g_1/U, ..., g_n/U) = g/U$, where $n = \# f_j$ and $g(\eta) = f_j^{\mathcal{A}_\eta}(g_1(\eta), ..., g_n(\eta))$,

$c_k^{\mathcal{A}} = g/U$, where $g(\eta) = c_k^{\mathcal{A}_n}$.

We notice that these definitions do not depend on the representatives of the equivalence classes g_1/U,...,g_n/U (exercise).

3.3 Lemma. *For all terms $t = t(x)$, $x = (x_1, ..., x_n)$, $\mathcal{A} = \Pi_{\eta \in X} \mathcal{A}_\eta / U$ and $a_i \in \Pi_{\eta \in X} \mathcal{A}_\eta$, $1 \leq i \leq n$, $t^{\mathcal{A}}(a_1/U, ..., a_n/U) = g/U$, where g is such that $g(\eta) = t^{\mathcal{A}_\eta}(a_i(\eta), ..., a_n(\eta))$.*

Proof. By induction on t: For variables and constants the claim is the definition of the interpretation. So suppose $t(x) = f(t_1(x), ..., t_m(x))$, $f \in L$ a function symbol. By the induction assumption, for $1 \leq k \leq m$, $t_k^{\mathcal{A}}(a_1/U, ..., a_n/U) = g_k/U$, where $g_k(\eta) = t_k^{\mathcal{A}_\eta}(a_1(\eta), ..., a_n(\eta))$. Then $t^{\mathcal{A}}(a_1/U, ..., a_n/U) = f^{\mathcal{A}}(g_1/U, ..., g_n/U)$. By the definition of $f^{\mathcal{A}}$,

$f^{\mathcal{A}}(g_1/U, ..., g_n/U) = g/U$, where

$g(\eta) = f^{\mathcal{A}_\eta}(g_1(\eta), ..., g_m(\eta)) =$

$f^{\mathcal{A}_\eta}(t_1^{\mathcal{A}_\eta}(a_1(\eta), ..., a_n(\eta)), ..., t_m^{\mathcal{A}_\eta}(a_1(\eta), ..., a_n(\eta))) = t^{\mathcal{A}_\eta}(a_i(\eta), ..., a_n(\eta))$. □

3.4 Theorem (Łos). *For all formulas ϕ, $\Pi_{\eta\in X}\mathcal{A}_\eta/U \models \phi(g_1/U, ..., g_n/U)$ iff $\{\eta \in X|\ \mathcal{A}_\eta \models \phi(g_1(\eta), ..., g_n(\eta))\} \in U$.*

Proof. Easy induction on ϕ (exercise)□

3.5 Definition. *A collection of sentences is called a theory. If T is a theory, we say that it is consistent if it has a model i.e. there is a structure $\mathcal{A}$ such that $\mathcal{A} \models \phi$ for all $\phi \in T$ (i.e. $\mathcal{A} \models T$). If $x = (x_1, ..., x_n)$ and Σ is a collection of formulas of the form $\psi(x)$, then we write that $\Sigma \models \phi(x)$ if for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ if $\mathcal{A} \models \psi(a)$ for all $\psi \in \Sigma$. In particular, for a theory T and a sentence ϕ, we write $T \models \phi$ if every model of T is a model of ϕ. If $T = \emptyset$, we write just $\models \phi$.*

3.6 Compactness theorem. *If every finite $T' \subseteq T$ is consistent, then T is consistent.*

Proof. We prove this by induction on $|T|$. For finite T the claim is trivial. So we may assume that $\kappa = |T|$ is infinite. Let ϕ_i, $i < \kappa$, enumerate T. By the induction assumption, for all $i < \kappa$, there is $\mathcal{A}_i$ such that $\mathcal{A}_i \models \phi_j$ for all $j \leq i$. Let $F = \{\kappa - \alpha|\ \alpha < \kappa\}$. Then F has the finite intersection property and thus there is an ultrafilter U extending F by Lemma 3.2. By Łos, since for all $\alpha < \kappa$, $\kappa - \alpha \subseteq \{i < \kappa|\ \mathcal{A}_i \models \phi_\alpha\}$ and $\kappa - \alpha \in U$, for all $\alpha < \kappa$, $\Pi_{\eta\in X}\mathcal{A}_\eta/U \models \phi_\alpha$. □

Notice that compactness theorem implies the following: if $\Sigma \models \phi$, then there is finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \phi$ (exercise, see fact 4.4 below).

In the following example, we let $L = \{+, \times, -, 0, 1\}$, where $+$ and $\times$ are 2-ary function symbols, $-$ is 1-ary function symbol and 0 and 1 are constant symbols. In stead of $+(x, y)$ we write $x + y$ and the same for $\times$. (The function symbol $-$ is included for convenience for section 6, it is not really needed.) We let T_f consist of the following sentences:

$\forall v_0\forall v_1\forall v_2(v_0 + (v_1 + v_2) = (v_0 + v_1) + v_2)$,
$\forall v_0\forall v_1(v_0 + v_1 = v_1 + v_0)$,
$\forall v_0(v_0 + 0 = v_0)$,
$\forall v_0(v_0 + (-v_0) = 0)$,
$\forall v_0\forall v_1\forall v_2(v_0 \times (v_1 \times v_2) = (v_0 \times v_1) \times v_2)$,
$\forall v_0\forall v_1(v_0 \times v_1 = v_1 \times v_0)$,
$\forall v_0(v_0 \times 1 = v_0)$,
$\neg 0 = 1$,
$\forall v_0\exists v_1((v_0 = 0) \vee (v_0 \times v_1 = 1))$,
$\forall v_0\forall v_1\forall v_2(v_0 \times (v_1 + v_2) = (v_0 \times v_1) + (v_0 \times v_2))$,
$\forall v_0\forall v_1\forall v_2((v_0 + v_1) \times v_2 = (v_0 \times v_2) + (v_1 \times v_2))$.

So $\mathcal{A} \models T_f$ iff $\mathcal{A}$ is a field.

For $n \in \mathbb{N}$, the notation nt, t a term, is defined as follows: $0t = 0$ (here the first 0 is the natural number and the second is the constant) and $(n+1)t = nt + t$. Similarly $t^0 = 1$ and $t^{n+1} = t^n \times t$. Then we let

$T_{f_0} = T_f \cup \{\neg p1 = 0|\ p \text{ a prime}\}$

and for primes p,

$T_{f_p} = T_f \cup \{p1 = 0\}$.

Then $\mathcal{A} \models T_{f_0}$ iff $\mathcal{A}$ is a field of characteristic 0 and $\mathcal{A} \models T_{f_p}$ iff $\mathcal{A}$ is a field of characteristic p.

3.7 Example. *For all L-sentences ϕ, if $T_{f_0} \models \phi$ then there is n such that $T_{f_p} \models \phi$ for all $p > n$.*

Proof. Suppose not. Let ϕ witness this. Then $X = \{p \in \mathbb{N} |\ T_{f_p} \not\models \phi\}$ is infinite. For all $p \in X$ choose $\mathcal{A}_p \models T_{f_p}$ so that $\mathcal{A}_p \not\models \phi$. Let F consist of sets $\{p \in X |\ p > m\}$, $m \in \mathbb{N}$. Then F has the finite intersection property and can be extended to an ultrafilter U. By Łos, $\mathcal{A} = \Pi_{p \in X} \mathcal{A}_p / U \models T_f$ and $\mathcal{A} \not\models \phi$. Also for all primes q, the set of $p \in X$ such that $\mathcal{A}_p \models q1 = 0$ contains at most q. Thus by Łos, $\mathcal{A} \models \neg q1 = 0$ for all primes q i.e. $\mathcal{A} \models T_{f_0}$, a contradiction. □

4. Tarski-Vaught

4.1 Definition. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures, $X \subseteq \mathcal{A}$ and $f : X \to \mathcal{B}$.*

(i) We say that f is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if for all atomic $\phi(x)$, $x = (x_1, ..., x_n)$ and $a \in X^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$.

(ii) We say that f is a partial elementary map from $\mathcal{A}$ to $\mathcal{B}$ if for all formulas $\phi(x)$, $x = (x_1, ..., x_n)$ and $a \in X^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$.

(iii) f is embedding if it is a partial isomorphism and $X = \mathcal{A}$.

(iv) f is elementary embedding if it is a partial elementary map and $X = \mathcal{A}$.

(v) f is isomorphism if it is an embedding and surjection. If in addition $\mathcal{B} = \mathcal{A}$, f is called an automorphism.

(vi) $\mathcal{A}$ is a submodel of $\mathcal{B}$ (denoted $\mathcal{A} \subseteq \mathcal{B}$) if the identity function $id_{\mathcal{A}}$ is an embedding. $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$ (denoted $\mathcal{A} \preceq \mathcal{B}$) if $id_{\mathcal{A}}$ is an elementary embedding.

Abusing the notation, if 4.1 (i) (4.1 (ii)) holds we write that $f : \mathcal{A} \to \mathcal{B}$ is a partial isomorphism (elementary map) although the domain of f may not be the whole $\mathcal{A}$. Notice that if $f : \mathcal{A} \to \mathcal{B}$ is a partial isomorphism (partial elementary map), then also $f^{-1} : \mathcal{B} \to \mathcal{A}$ is a partial isomorphism (partial elementary map).

4.2 Fact. *If f is an isomorphism, then f is an elementary embedding.*

Proof. The course Matemaattinen logiikka. □

Notice that $f : \mathcal{A} \to \mathcal{B}$ is an (elementary) embedding iff f is an isomorphism between $\mathcal{A}$ and some (elementary) substructure of $\mathcal{B}$. Also notice that if $\mathcal{A} \subseteq \mathcal{B}$ and $a \in \mathcal{A}^n$ and $\phi(x)$, $x = (x_1, ..., x_n)$ is quantifier free (i.e. no quantifier appear in ϕ), then $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(a)$ (exercise).

4.3 Definition. *Let $\mathcal{A}$ be a structure and $A \subseteq \mathcal{A}$. By $L(A)$ we mean a vocabulary we get from L by adding new constants $\underline{a}$ for all $a \in A$. $(\mathcal{A}, A)$ means an $L(A)$-structure we get from $\mathcal{A}$ by interpreting $\underline{a}^{\mathcal{A}} = a$. By $Th(\mathcal{A})$ we mean* *the set of all L-sentences true in $\mathcal{A}$ and so by $Th(\mathcal{A}, A)$ we mean the set of all $L(A)$-sentences true in $(\mathcal{A}, A)$.*

For a formula ϕ and constant c, $\phi(c/x_i)$ is defined as follows:

(i) if ϕ is atomic then $\phi(c/x)$ is what we get from ϕ by replacing x by c everywhere,

(ii) if $\phi = \neg\psi$ then $\phi(c/x) = \neg(\psi(c/x))$,

(iii) if $\phi = \psi \wedge \theta$ then $\phi(c/x) = \psi(c/x) \wedge \theta(c/x)$,

(iv) if $\phi = \exists v_i \psi$, then $\phi(c/x) = \phi$ if $v_i = x$ and otherwise $\phi(c/x) = \exists v_i(\psi(c/x))$.

For $\phi(y, x)$, $x = (x_1, ..., x_n)$ and $a = (a_1, ..., a_n) \in A^n$, we write $\phi(y, \underline{a})$ instead of $\phi(\underline{a_1}/x_1)...(\underline{a_n}/x_n)$.

4.4 Fact. $\mathcal{A} \models \phi(a)$ *iff* $(\mathcal{A}, A) \models \phi(\underline{a})$.

Proof. The course Matemaattinen logiikka. □

4.5 Lemma. *If $\mathcal{A}$ is infinite and κ is a cardinal, then there is $\mathcal{B}$ of cardinality $\geq \kappa$ such that $\mathcal{A} \preceq \mathcal{B}$. (In particular, the Hanf number of the first-order logic is ω.)*

Proof. Let $L^* = L(\mathcal{A}) \cup \{c_i |\ i < \kappa\}$, where c_i are new constant symbols. Let $T^* = Th(\mathcal{A}, \mathcal{A}) \cup T$ where $T = \{\neg c_i = c_j |\ i < j < \kappa\}$. Since $\mathcal{A}$ is infinite, for all finite $T' \subseteq T$, we can interpret the constants c_i in $(\mathcal{A}, \mathcal{A})$ so that T' is true in that model. Since $(\mathcal{A}, \mathcal{A}) \models Th(\mathcal{A}, \mathcal{A})$, by the compactness theorem T^* has a model $\mathcal{B}^*$. Clearly $|\mathcal{B}^*| \geq \kappa$. By renaming the elements of $\mathcal{B}^*$ (i.e. by taking an isomorphic copy of $\mathcal{B}^*$), we may assume that for all $a \in \mathcal{A}$, $\underline{a}^{\mathcal{B}} = a$. Let $\mathcal{B} = \mathcal{B} \restriction L$ i.e. what we get from $\mathcal{B}^*$ by dropping out the interpretations for the constants in $L^* - L$. We are left to show that $\mathcal{A} \preceq \mathcal{B}$. But $\mathcal{A} \models \phi(a)$ iff $(\mathcal{A}, \mathcal{A}) \models \phi(\underline{a})$ iff $\phi(\underline{a}) \in T^*$ iff $\mathcal{B}^* \models \phi(\underline{a})$ iff $\mathcal{B} \models \phi(a)$. □

4.6 Theorem (Tarski-Vaught). *$\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all formulas $\phi(v_i, x)$, $x = (x_1, ..., x_n)$ and $a \in \mathcal{A}^n$ the following holds: If $\mathcal{B} \models \exists v_i \phi(v_i, a)$, then there is $b \in \mathcal{A}$ such that $\mathcal{B} \models \phi(b, a)$.*

Proof. By induction on $\psi(y)$, $y = (y_1, ..., y_m)$, we show that for all $b \in \mathcal{A}^m$, $\mathcal{A} \models \psi(b)$ iff $\mathcal{B} \models \psi(b)$.

1. ψ is atomic: Immediate since $\mathcal{A} \subseteq \mathcal{B}$.
2. $\psi = \neg\phi$ or $\phi \wedge \theta$: Immediate by the induction assumption.
3. $\psi = \exists v_i \phi(v_i, y)$: Two directions:

"$\Rightarrow$": If $\mathcal{A} \models \psi(b)$, then there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \phi(c, b)$. By the induction assumption, $\mathcal{B} \models \phi(c, b)$ and thus $\mathcal{B} \models \psi(b)$.

"$\Leftarrow$": If $\mathcal{B} \models \psi(b)$, then by the assumption, there is $c \in \mathcal{A}$ such that $\mathcal{B} \models \phi(c, b)$. By the induction assumption, $\mathcal{A} \models \phi(c, b)$ and thus $\mathcal{A} \models \psi(b)$. □

Suppose that for all $\gamma < \beta < \alpha$, $\mathcal{A}_\gamma \subseteq \mathcal{A}_\beta$. Then $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ is the structure $\mathcal{B}$ such that $dom(\mathcal{B}) = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$, $R_i^{\mathcal{B}} = \bigcup_{\beta<\alpha} R_i^{\mathcal{A}_\beta}$, $f_j^{\mathcal{B}} = \bigcup_{\beta<\alpha} f_j^{\mathcal{A}_\beta}$ and $c_k^{\mathcal{B}} = c_k^{\mathcal{A}_0}$.

4.7 Corollary. *Suppose that for all $\gamma < \beta < \alpha$, $\mathcal{A}_\gamma \preceq \mathcal{A}_\beta$ and let $\mathcal{B} = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Then for all $\gamma < \alpha$, $\mathcal{A}_\gamma \preceq \mathcal{B}$. Furthermore, if for all $\beta < \alpha$, $\mathcal{A}_\beta \preceq \mathcal{C}$, then $\mathcal{B} \preceq \mathcal{C}$.*

Proof. We repeat the proof of Tarski-Vaught and proof by induction on $\phi(x)$, $x=(x_1,...,x_n)$, that for all $a\in\mathcal{B}^n$, $\mathcal{B}\models\phi(a)$ iff $\mathcal{A}_\gamma\models\phi(a)$ for all $\gamma<\alpha$ such that $a\in\mathcal{A}_\gamma^n$. We prove the case $\phi=\exists v_i\psi(v_i,x)$, the other cases are left as an exercise:

"$\Rightarrow$": Then there is $b\in\mathcal{B}$ such that $\mathcal{B}\models\psi(b,a)$. But then we can find $\gamma\le\beta<\alpha$ such that $b\in\mathcal{A}_\beta$. By the induction assumption, $\mathcal{A}_\beta\models\psi(b,a)$ and thus $\mathcal{A}_\beta\models\phi(a)$. Since $\mathcal{A}_\gamma\preceq\mathcal{A}_\beta$, $\mathcal{A}_\gamma\models\phi(a)$.

"$\Leftarrow$": Exactly as the direction "$\Rightarrow$" in the proof of Tarski-Vaught.

The furthermore part follows immediately from Tarski-Vaught (exercise). □

Recall that $L_{\omega\omega}$ is the set of all (L-)formulas. Notice that $|L_{\omega\omega}|=|L|+\omega$.

4.8 Lemma. *Suppose $A\subseteq\mathcal{A}$. Then there is $A\subseteq\mathcal{B}\preceq\mathcal{A}$ such that $|\mathcal{B}|\le|A|+|L_{\omega\omega}|$ (i.e. the Löwenheim-Skolem number of the first-order logic is $|L_{\omega\omega}|$).*

Proof. For every formula $\phi(v_i,x)$, $x=(x_1,...,x_n)$ we define a function $g_{\phi(v_i,x)}:\mathcal{A}^n\to\mathcal{A}$ so that if $\mathcal{A}\models\exists v_i\phi(v_i,a)$, then $\mathcal{A}\models\phi(g_{\phi(v_i,x)}(a),a)$. Then we close $C=A\cup\{c_k^{\mathcal{A}}|\ k\in K^*\}$ under these function and under the functions $f_j^{\mathcal{A}}$ (we could drop the elements $c_k^{\mathcal{A}}$ from C and forget the functions $f_j^{\mathcal{A}}$ and still get the same set) i.e. we let $B\subseteq\mathcal{A}$ be the intersection of all $D\subseteq\mathcal{A}$ such that $C\subseteq D$ and if $a\in D^n$ and g is an n-ary function as above, then $g(a)\in D$. Then $|B|\le|A|+|L_{\omega\omega}|$ (see basic set theory course or [Je]). We make a model $\mathcal{B}$ out of B by defining: $dom(\mathcal{B})=B$, $R_i^{\mathcal{B}}=R_i^{\mathcal{A}}\cap B^{\#R_i}$, $f_j^{\mathcal{B}}=f_j^{\mathcal{A}}\restriction B$ and $c_k^{\mathcal{B}}=c_k^{\mathcal{A}}$. By Tarski-Vaught, $\mathcal{B}\preceq\mathcal{A}$. □

4.9 Löwenheim-Skolem theorem. *If T is a theory and it has an infinite model, then it has a model in every cardinality $\kappa\ge|L_{\omega\omega}|$.*

Proof. By Lemma 4.5 T has a model $\mathcal{A}$ of cardinality $\ge\kappa$. By 4.8 we can find $\mathcal{B}\preceq\mathcal{A}$ of cardinality κ. Then $\mathcal{B}\models T$. □

5. Completeness and elimination of quantifiers

5.1 Definition.

(i) We say that a theory T is complete if for all sentences ϕ, either $T\models\phi$ or $T\models\neg\phi$.

(ii) We say that T is κ-categorical if upto isomorphisms T has exactly one model of cardinality κ.

Often when one talks about complete theories, one assumes also that T is consistent (inconsistent theories are not usually considered interesting). In fact unless otherwise stated, whenever we talk about a theory T, T is assumed to be consistent.

5.2 Lemma (Łos-Vaught). *If T is κ-categorical for some $\kappa\ge|L_{\omega\omega}|$ and T does not have finite models, then T is complete.*

Proof. Suppose not. Let ϕ be a sentence that witnesses this. Then by Löwenheim-Skolem both $T\cup\{\neg\phi\}$ and $T\cup\{\phi\}$ have a model of size κ. This contradicts the assumption that T is κ-categorical. □

5.3 Definition. *We say that T is closed under unions if for all $\mathcal{A}_i \models T$, $i < \alpha$, the following holds: If for all $i < j < \alpha$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, then $\bigcup_{i<\alpha} \mathcal{A}_i \models T$.*

5.4 Lemma. *If T consists of sentences of the form $\forall x_1 ... \forall x_n \exists y_1 ... \exists y_m \phi$, where ϕ is quantifier free, then T is closed under unions.*

Proof. Exercise. □

5.5 Definition.

(i) T has quantifier free set amalgamation (AP for short) if for all $\mathcal{A}, \mathcal{B} \models T$ and partial isomorphism $f : \mathcal{A} \to \mathcal{B}$ there are $\mathcal{B} \subseteq \mathcal{C} \models T$ and an embedding $g : \mathcal{A} \to \mathcal{C}$ such that $g \restriction dom(f) = f$.

(ii) T has quantifier free joint embedding (JEP for short) if for all $\mathcal{A}, \mathcal{B} \models T$ there are $\mathcal{B} \subseteq \mathcal{C} \models T$ and an embedding $f : \mathcal{A} \to \mathcal{C}$.

5.6 Lemma. *If T has AP and there is $\mathcal{A}$ (not necessarily a model of T) such that for all $\mathcal{B} \models T$ there is an embedding $f : \mathcal{A} \to \mathcal{B}$, then T has JEP.*

Proof. Exercise. □

5.7 Lemma. *Suppose $\kappa \geq |L_{\omega\omega}|$ and T has JEP and is closed under unions. Then there is a model $\mathcal{A}$ of T such that for all $\mathcal{B} \models T$ of power $\leq \kappa$, there is an embedding $f : \mathcal{B} \to \mathcal{A}$.*

Proof. Let $\mathcal{B}_i$, $i < \alpha$, list all models of T of power $\leq \kappa$ i.e. if $\mathcal{B}$ is a model of T of power $\leq \kappa$, then $\mathcal{B}$ is isomorphic with some $\mathcal{B}_i$, $i < \alpha$. By recursion on $i \leq \alpha$, we define models $\mathcal{A}_i$ of T as follows:

$i = 0$: $\mathcal{A}_i = \mathcal{B}_0$.

$i = j + 1$: By JEP we choose $\mathcal{A}_j \subseteq \mathcal{A}_i \models T$ such that there is an embedding $f : \mathcal{B}_j \to \mathcal{A}_i$.

i is limit: $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$.

Clearly $\mathcal{A} = \mathcal{A}_\alpha$ is as wanted. □

5.8 Lemma. *Suppose T has AP and is closed under unions and $\mathcal{A} \models T$. Then there is $\mathcal{A} \subseteq \mathcal{B} \models T$ such that for all partial isomorphisms $f : \mathcal{A} \to \mathcal{A}$ there is an embedding $g : \mathcal{A} \to \mathcal{B}$ such that $g \restriction dom(f) = f$.*

Proof. Let f_i, $i < \alpha$, list all partial isomorphism from $\mathcal{A}$ to $\mathcal{A}$. We define models $\mathcal{B}_i$ of T, $i \leq \alpha$, as follows:

$i = 0$: $\mathcal{B}_i = \mathcal{A}$.

$i = j + 1$: Now f_j is a partial isomorphism from $\mathcal{A}$ also to $\mathcal{B}_j$ since $\mathcal{A} \subseteq \mathcal{B}_j$ and thus by AP there is $\mathcal{B}_j \subseteq \mathcal{B}_i \models T$ and an embedding $g : \mathcal{A} \to \mathcal{B}_i$ such that $g \restriction dom(f_j) = f_j$.

i is limit: $\mathcal{B}_i = \bigcup_{j<i} \mathcal{B}_j$.

Clearly $\mathcal{B} = \mathcal{B}_\alpha$ is as wanted. □

5.9 Definition. *We say that $\mathcal{A} \models T$ is existentially closed if for all $\mathcal{A} \subseteq \mathcal{B} \models T$, atomic or negated atomic formulas $\phi_i(v_k, x)$, $i < n$ and $x = (x_1, ..., x_m)$, and $a \in \mathcal{A}^m$, the following holds: if $\mathcal{B} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$ then $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$.*

5.10 Lemma. *If $\mathcal{A} \models T$ is existentially closed, then for all $\mathcal{A} \subseteq \mathcal{B} \models T$, quantifier free formula $\phi(v_k, x)$, $x = (x_1, ..., x_m)$, and $a \in \mathcal{A}^m$, the following holds: if $\mathcal{B} \models \exists v_k \phi(v_k, a)$ then $\mathcal{A} \models \exists v_k \phi(v_k, a)$.*

Proof. Exercise. (Hint: By e.g. the course Logiikka I, every quantifier free formula $\phi(x)$ is equivalent with a formula of the form $\vee_{i<n} \wedge_{j<m} \phi_{ij}(x)$, where each ϕ_{ij} is atomic or negated atomic formula.) $\square$

5.11 Theorem. *Suppose $\kappa \geq |L_{\omega\omega}|$ and T has AP, JEP and is closed under unions. Then there is a model $\mathcal{A}$ of T such that for all $\mathcal{B} \models T$ of power $\leq \kappa$, there is an embedding $f : \mathcal{B} \to \mathcal{A}$ and for all partial isomorphisms $f : \mathcal{A} \to \mathcal{A}$ of power $\leq k$, there is an automorphism g of $\mathcal{A}$ such that $g \restriction dom(f) = f$. Furthermore, such a model is existentially closed.*

Proof. By recursion on $i \leq \kappa^+$ we define models $\mathcal{A}_i$ of T as follows:

$i = 0$: We let $\mathcal{A}_i$ be as given by Lemma 5.7.

$i = j + 1$: By Lemma 5.8 we can find $\mathcal{A}_j \subseteq \mathcal{A}_i \models T$ such that every partial isomorphism from $\mathcal{A}_j$ to $\mathcal{A}_j$ extends to an embedding from $\mathcal{A}_j$ to $\mathcal{A}_i$.

i is limit: $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$.

We show that $\mathcal{A} = \mathcal{A}_{\kappa^+}$ is as wanted. Clearly $\mathcal{A}$ has the first of the required properties. For the second, let $f : \mathcal{A} \to \mathcal{A}$ be a partial isomorphism of power $\leq \kappa$. Then by Fact 2.8 (v), there is $\alpha < \kappa^+$ such that $dom(f) \cup rng(f) \subseteq \mathcal{A}_\alpha$. By recursion on $\alpha \leq i < \kappa^+$ we define an increasing sequence of partial isomorphism $f_i : \mathcal{A}_{i+1} \to \mathcal{A}_{i+1}$ as follows:

$i = \alpha$: $f_i = f$.

$i = j + 1$: Since f_j is a partial isomorphism from $\mathcal{A}_i \to \mathcal{A}_i$, by the choice of $\mathcal{A}_{i+1}$, there is a an embedding $f_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$.

i is limit: Let $f_i^* = \bigcup_{\alpha \leq j < i} f_j$. Then $(f_i^*)^{-1}$ is a partial isomorphism from $\mathcal{A}_i$ to $\mathcal{A}_i$ and thus there is an embedding $g : \mathcal{A}_i \to \mathcal{A}_{i+1}$ such that $g \restriction rng(f_i^*) = (f_i^*)^{-1}$. We let $f_i = g^{-1}$.

Then $g = \bigcup_{\alpha \leq i < \kappa^+} f_i$ is a partial isomorphism, $dom(g) = \mathcal{A}$ and $rng(g) = \mathcal{A}$ i.e. g is an isomorphism and clearly it extends f.

To prove the furthermore part, let $\mathcal{B}$, $a = (a_1, ..., a_m)$ and $\phi_i(v_k, x)$, $i < n$, be as in the definition of existentially closed. Choose $b \in \mathcal{B}$ such that $\mathcal{B} \models \wedge_{i<n} \phi_i(b, a)$ and by Lemma 4.8 choose $\mathcal{C} \preceq \mathcal{B}$ of power $\leq \kappa$ such that $a \in \mathcal{C}^m$ and $b \in \mathcal{C}$. Then $\mathcal{C} \models T$ and there is an embedding $f : \mathcal{C} \to \mathcal{A}$. Now $g = (f \restriction \{a_1, ..., a_m\})^{-1}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{A}$ and thus there is an automorphism h' of $\mathcal{A}$ such that $h' \restriction dom(g) = g$. Let $h = h' \restriction rng(f)$. Then $h \circ f$ is an embedding of $\mathcal{C}$ to $\mathcal{A}$ and for all $1 \leq j \leq m$, $(h \circ f)(a_j) = a_j$. Since ϕ_i, $i < n$, are atomic or negated atomic, $\mathcal{A} \models \phi_i((h \circ f)(b), (h \circ f)(a))$. Thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. $\square$

5.12 Definition. *We say that T has the elimination of quantifiers if for all formulas $\phi(x)$, $x = (x_1, ..., x_n)$, there is a quantifier free formula $\psi(x)$ such that $T \models \forall x_1 ... \forall x_n (\phi(x) \leftrightarrow \psi(x))$.*

5.13 Theorem. *Suppose T has AP, JEP and is closed under unions. If T^* is such a theory that its models are exactly the existentially closed models of T, then T^* is complete and it has the elimination of quantifiers.*

Proof. Let $\kappa \geq |L_{\omega\omega}|$ and $\mathcal{A}$ be as in Theorem 5.11. Since $\mathcal{A}$ is existentially closed, $\mathcal{A} \models T^*$. The completeness of T^* follows easily from the elimination of quantifiers and the existence of $\mathcal{A}$ (exercise). To prove the elimination of quantifiers, we prove by a simultaneous induction on $\phi(x)$, $x = (x_1, ..., x_n)$ that

(i) if $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \models T^*$ and $a \in \mathcal{B}^n$, then $\mathcal{B} \models \phi(a)$ iff $\mathcal{A} \models \phi(a)$,

(ii) there is quantifier free $\psi(x)$ such that $T^* \models \forall x_1 ... \forall x_n(\phi(x) \leftrightarrow \psi(x))$.

The steps ϕ atomic, $\phi = \neg\theta$ and $\phi = \theta \wedge \theta'$ are trivial. So we assume that $\phi(x) = \exists v_i \theta(v_i, x)$.

Proof of (i): If $\mathcal{B} \models \phi(a)$ then for some $b \in \mathcal{B}$, $\mathcal{B} \models \theta(b, a)$ and so by (i) of the induction assumption, $\mathcal{A} \models \theta(b, a)$, so $\mathcal{A} \models \phi(a)$. Then suppose $\mathcal{A} \models \phi(a)$. By (ii) in the induction assumption, let ψ be quantifier free such that

(*) $T^* \models \forall v_i \forall x_1 ... \forall x_n(\theta(v_i, x) \leftrightarrow \psi(v_i, x))$.

Now $\mathcal{A} \models \exists v_i \psi(v_i, a)$ and since $\mathcal{B}$ is existentially closed $\mathcal{B} \models \exists v_i \psi(v_i, a)$. By (*), $\mathcal{B} \models \phi(a)$.

Proof of (ii): For $a \in \mathcal{A}^n$ and $x = (x_1, ..., x_n)$, let

$$t^x_{at}(a/\emptyset; \mathcal{A}) = \{\theta(x) |\ \theta \text{ atomic or negated atomic, } \mathcal{A} \models \theta(a)\}.$$

We write $(\mathcal{B}, b) \models t^x_{at}(a/\emptyset; \mathcal{A})$ if $\mathcal{B} \models \theta(b)$ for all $\theta(x) \in t^x_{at}(a/\emptyset; \mathcal{A})$.

1 Claim. *Suppose $\mathcal{A} \models \phi(a)$, $\mathcal{B} \models T^*$ and $(\mathcal{B}, b) \models t^x_{at}(a/\emptyset; \mathcal{A})$. Then $\mathcal{B} \models \phi(b)$.*

Proof of Claim 1: Suppose not. Let $\mathcal{B}$ and b witness this. By Lemma 4.8 we may assume that $|\mathcal{B}| \leq \kappa$. By the choice of $\mathcal{A}$, there is an embedding of $\mathcal{B}$ to $\mathcal{A}$ and so we may assume that $\mathcal{B} \subseteq \mathcal{A}$. Then $(\mathcal{A}, b) \models t^x_{at}(a/\emptyset; \mathcal{A})$ and thus $b_i \mapsto a_i$, $1 \leq i \leq n$, is a partial isomorphism and thus there is an automorphism f of $\mathcal{A}$ such that $f(b_i) = a_i$ for all $1 \leq i \leq n$. But then $\mathcal{A} \models \neg\phi(a)$, since $\mathcal{A} \models \neg\phi(b)$ by (i) and the choice of $\mathcal{B}$ and b, a contradiction. □ Claim 1.

2 Claim. *Suppose $\mathcal{A} \models \phi(a)$. Then there is finite $q \subseteq t^x_{at}(a/\emptyset; \mathcal{A})$ such that $T^* \models \forall x_1 ... \forall x_n((\wedge q) \rightarrow \phi)$.*

Proof of Claim 2: By Claim 1, $T^* \cup t^x_{at}(a/\emptyset; \mathcal{A}) \models \phi(x)$ and thus the claim follows from compactness. □ Claim 2.

Let p_i, $i < \alpha$, enumerate the set $\{t^x_{at}(a/\emptyset; \mathcal{A}) |\ \mathcal{A} \models \phi(a)\}$. Let $q_i \subseteq p_i$ be as in Claim 2 and $\psi_i(x) = \wedge q_i$.

Claim 3. *If $\mathcal{B} \models T^*$ and $\mathcal{B} \models \phi(b)$, then for some $i < \alpha$, $\mathcal{B} \models \psi_i(b)$.*

Proof of Claim 3: Suppose not. Then as in the proof of Claim 1, we can find $\mathcal{B}$ and b witnessing this so that $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{A} \models \neg\psi_i(b)$ for all $i < \alpha$ and by (i), $\mathcal{A} \models \phi(b)$. This contradicts the fact that p_i, $i < \alpha$, enumerates the set $\{t^x_{at}(a/\emptyset; \mathcal{A}) |\ \mathcal{A} \models \phi(a)\}$. □ Claim 3.

So $T^* \cup \{\neg\psi_i(x) |\ i < \alpha\} \models \neg\phi(x)$. By compactness, there is finite $X \subseteq \alpha$ such that $T^* \models \forall x_1 ... \forall x_n (\wedge_{i \in X} \neg\psi_i(x) \rightarrow \neg\phi(x))$ i.e. $T^* \models \forall x_1 ... \forall x_n (\phi(x) \rightarrow \vee_{i \in X} \psi_i(x))$. Since for all $i \in X$, $T^* \models \forall x_1 ... \forall x_n (\psi_i(x) \rightarrow \phi(x))$,

$$T^* \models \forall x_1 ... \forall x_n (\phi(x) \leftrightarrow \vee_{i \in X} \psi_i(x)).$$

□

5.14 Definition. *We say that T is model complete if for all $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{A} \preceq \mathcal{B}$.*

5.15 Lemma. *If T has the elimination of quantifiers, then T is model complete.*

Proof. Exercise. □

5.16 Exercise. *In section 1 we said that the atomic formula $\top$ is needed in the proof of Theorem 5.13. Where was it needed in the proof?*

6. Example: Algebraically closed fields

We return to the example from section 3. So in this section $L = \{+, \times, -, 0, 1\}$, and we study the theory T_{f_0}. Instead of 0 we could work also with any positive characteristic p, only changes needed would be that we should replace $\mathbf{Q}$ and $\mathbf{Z}$ by the p element field F_p.

6.1 Lemma. *For all polynomials $P \in \mathbf{Z}[x_1, ..., x_n]$ there is a term $t(x_1, ..., x_n)$ such that for all $\mathcal{A} \models T_{f_0}$ and $a \in \mathcal{A}^n$, $P(a) = t^{\mathcal{A}}(a)$ (and vice versa).*

Proof. Exercise. □

So for every atomic formula $\phi(x)$, $x = (x_1, ..., x_n)$, there is a polynomial $P \in \mathbf{Z}[x_1, ..., x_n]$ such that for all $\mathcal{A} \models T_{f_0}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $P(a) = 0$.

T_{acf_0} is the theory we get from T_{f_0} by adding the sentences

$\forall v_0 ... \forall v_n (\neg v_n = 0 \rightarrow \exists v_{n+1} \Sigma_{i=0}^{i=n} v_i \times v_{n+1}^i = 0)$,

for all $n \in \mathbb{N} - \{0\}$. Then $\mathcal{A} \models T_{acl_0}$ if $\mathcal{A}$ is an algebraically closed closed field of characteristic 0.

We need few facts from algebra.

6.2 Fact.

(i) Every field $\mathcal{A}$ can be extended to an algebraically closed field $\mathcal{B}$. Furthermore, this can be done so that there is $a \in \mathcal{B}$ such that for all non-zero $P \in \mathcal{A}[X]$, $P(a) \neq 0$ (i.e. a is not algebraic over $\mathcal{A}$).

(ii) If $\mathcal{A}, \mathcal{B} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$ (i.e. $\mathcal{C}$ and $\mathcal{D}$ are subrings) and $f : \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism, then there is an isomorphism g between the fields generated by $\mathcal{C}$ and $\mathcal{D}$ such that $g \restriction \mathcal{C} = f$.

(iii) If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are algebraic over $\mathcal{C}$ and $\mathcal{D}$, respectively, $P \in \mathcal{C}[X]$ is the minimal polynomial of a, $f : \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism and $f(P)(b) = 0$, then there is an isomorphism $g : \mathcal{C}(a) \rightarrow \mathcal{D}(b)$ such

that $g \restriction \mathcal{C} = f$ and $g(a) = b$. (Here $\mathcal{C}(a)$ is the field generated by $\mathcal{C} \cup \{a\}$ and $f(\Sigma_{i=0}^{i=n} c_i X^i) = \Sigma_{i=0}^{i=n} f(c_i) X^i$.)

(iv) If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are not algebraic over $\mathcal{C}$ and $\mathcal{D}$ respectively and $f : \mathcal{C} \to \mathcal{D}$ is an isomorphism, then there is an isomorphism $g : \mathcal{C}(a) \to \mathcal{D}(b)$ such that $g \restriction \mathcal{C} = f$ and $g(a) = b$.

Proof. See the course Algebra II. $\square$

6.3 Lemma. *T_{f_0} has AP, JEP and is closed under unions.*

Proof. T_{f_0} is closed under unions by Lemma 5.4 and since for all $\mathcal{A} \models T_{f_0}$, there is an embedding $f : (\mathbf{Z}, +, \times, -, 0, 1) \to \mathcal{A}$, JEP follows from AP and Lemma 5.6. So we are left to prove AP.

So suppose $\mathcal{A}, \mathcal{B} \models T_{f_0}$ and $f : \mathcal{A} \to \mathcal{B}$ is a partial isomorphism. By recursion on ordinals i we define subfields $\mathcal{A}_i$ of $\mathcal{A}$, algebraically closed fields $\mathcal{C}_i \supseteq \mathcal{B}$ and embeddings $f_i : \mathcal{A}_i \to \mathcal{C}_i$ as follows:

$i = 0$: We start by letting $C = \{t^{\mathcal{A}}(a) | \ t(x_1, ..., x_n)$ a term, $a \in dom(f)^n\}$ and $D = \{t^{\mathcal{B}}(b) | \ t(x_1, ..., x_n)$ a term, $b \in rng(f)^n\}$. When equipped with the induced structure, C and D are subrings of $\mathcal{A}$ and $\mathcal{B}$, respectively, and since f is a partial isomorphism $g(t^{\mathcal{A}}(a)) = t^{\mathcal{B}}(f(a))$ is an isomorphism from C to D such that $g \restriction dom(f) = f$ (exercise). Let $\mathcal{A}_0$ be the subfield generated by C and $\mathcal{D}$ be the subfield generated by D. By Fact 6.2 (ii), there is an isomorphism $f_0 : \mathcal{A}_0 \to \mathcal{D}$ such that $f_0 \restriction C = g$. By Fact 6.2 (i), we let $\mathcal{C}_0$ be any algebraically closed field containing $\mathcal{B}$.

$i = j + 1$: If $\mathcal{A}_j = \mathcal{A}$, then f_j is the require embedding of $\mathcal{A}$ to $\mathcal{C}_j \supseteq \mathcal{B}$.

So suppose $a \in \mathcal{A} - \mathcal{A}_j$. There are two cases:

(a) a is algebraic over $\mathcal{A}_j$: We let $\mathcal{C}_i = \mathcal{C}_j$, $\mathcal{A}_i = \mathcal{A}_j(a)$ and $P \in \mathcal{A}_j[X]$ be the minimal polynomial of a over $\mathcal{A}_j$. Since $\mathcal{C}_i$ is algebraically closed, there is $b \in \mathcal{C}_i$ such that $f(P)(b) = 0$. By Fact 6.2 (iii) there is an isomorphism $f_i : \mathcal{A}_i \to rng(f_j)(b)$ such that $f_i \restriction \mathcal{A}_j = f_j$.

(b) a is not algebraic: Let $\mathcal{A}_i = \mathcal{A}_j(a)$ and choose an algebraically closed $\mathcal{C}_i \supseteq \mathcal{C}_j$ such that some $b \in \mathcal{C}_i$ is not algebraic over $\mathcal{C}_j$. Then b is not algebraic over $rng(f_j)$ and thus by Fact 6.2 (iv) there is an isomorphism $f_i : \mathcal{A}_i \to rng(f_j)(b)$ such that $f_i \restriction \mathcal{A}_j = f_j$.

i is limit: Let $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$, $f_i = \bigcup_{j<i} f_j$, $\mathcal{C}_i = \bigcup_{j<i} \mathcal{C}_j$. $\square$

6.4 Theorem. *T_{acf_0} is complete and has the elimination of quantifiers.*

Proof. By Lemma 6.3 and Theorem 5.13, it is enough to show that for all $\mathcal{A} \models T_{f_0}$, $\mathcal{A}$ is existentially closed iff $\mathcal{A} \models T_{acf_0}$. By Fact 6.2 (i), the claim from left to right is clear. So we prove the other direction. So suppose $\mathcal{A} \models T_{acf_0}$, $\mathcal{A} \subseteq \mathcal{B} \models T_{f_0}$, $\phi_i(v_k, x)$, $i < n$, are atomic or negated atomic formulas, $a \in \mathcal{A}^m$ and $\mathcal{B} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. Let $b \in \mathcal{B}$ be such that $\mathcal{B} \models \wedge_{i<n} \phi_i(b, a)$. There are two cases:

1. b is algebraic over $\mathcal{A}$: Since $\mathcal{A}$ is algebraically closed, $b \in \mathcal{A}$ and thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$.

2. b is not algebraic over $\mathcal{A}$: Then w.o.l.g. each $\phi_i(v_k, a)$ is of the form $\neg P_i(v_k, a) = 0$ (or $0 = 0$), where $P_i(v_k, a) = \Sigma_{i=0}^{i=l} b_i v_k^i$ where each $b_i \in \{a_1, ..., a_m\}$. Since each polynomial $P_i(v_k, a)$ has only finitely many roots and $\mathcal{A}$ is infinite, there is $c \in \mathcal{A}$ such that $P_i(c, a) \neq 0$ for all $i < n$. Thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. □

6.5 Remark. *So $\{\phi | \ (\mathbf{C}, +, \times, -, 0, 1) \models \phi\}$ is recursive (i.e. computer can be programmed to tell for all sentences ϕ, whether $(\mathbf{C}, +, \times, -, 0, 1) \models \phi$ or not, see the course Matemaattinen logiikka). Similarly one (Tarski) can show that $\{\phi | \ (\mathbf{R}, +, \times, -, 0, 1) \models \phi\}$ is recursive. The following is a famous open question: Is $\{\phi | \ (\mathbf{R}, +, \times, -, exp, 0, 1) \models \phi\}$ recursive? Schanuel's conjecture implies that the answer is yes. By Exercise 1.12, $\{\phi | \ (\mathbf{C}, +, \times, -, exp, 0, 1) \models \phi\}$ knows which Diophantine equations $P(X_1, ..., X_n) = 0$, $P(X_1, ..., X_n) \in \mathbf{Z}[X_1, ..., X_n]$, have an integer root and thus by the negative answer to Hilbert's 10th problem (due to M.Davis, Y.Matiyasevich, H.Putnam and J.Robinson) $\{\phi | \ (\mathbf{C}, +, \times, -, exp, 0, 1) \models \phi\}$ is not recursive.*

6.6 Exercise. *Let $L = \{<\}$, $<$ is a 2-ary predicate symbols, and let T_{lo} (lo for linear ordering) consist of the following sentences:*

$\forall v_0 \forall v_1 \forall v_2 ((v_0 < v_1 \wedge v_1 < v_2) \rightarrow v_0 < v_2)$

$\forall v_0 \forall v_1 (v_0 < v_1 \rightarrow \neg v_1 < v_0)$

$\forall v_0 \forall v_1 (v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0)$.

Show that T_{lo} has AP, JEP and is closed under unions and find a theory T so that the models of T are exactly the existentially closed models of T_{lo}.

7. Ehrenfeucht-Fraïssé games

7.1 Definition.

(i) We say that ϕ is a relational atomic formula if it is of the form $R_i(x_1, ..., x_n)$ or $x_p = f_j(x_1, ..., x_n)$ or $f_j(x_1, ..., x_n) = x_p$ or $x_1 = c_k$ or $c_k = x_1$ or $x_1 = x_2$.

(ii) Relational formulas are defined as follows: Relational atomic formulas are relational formulas and if ϕ and ψ are relational then so are $\neg\phi$, $\phi \wedge \psi$ and $\exists x \phi$.

Notice that if L is finite, then for all $n \in \mathbb{N} - \{0\}$, the number of relational atomic formulas of the form $\phi(v_1, ..., v_n)$ is finite (the same is true for atomic formulas only if $J^* = \emptyset$).

7.2 Definition.

(i) For terms t, the relationality rank $rr(t)$ is defined as follows: If $t = v_i$, then $rr(t) = 0$, if $t = c_k$, then $rr(t) = 1$ and if $t = f_j(t_1, ..., t_n)$, then $rr(t) = max\{rr(t_1), ..., rr(t_n)\} + 1$.

(ii) For atomic formulas ϕ, the relationality rank $rr(\phi)$ is defined as follows: If $\phi = R_i(t_1, ..., t_n)$, then $rr(\phi) = max\{rr(t_1), ..., rr(t_n)\}$ and if $\phi = t = u$, then $rr(\phi) = rr(t) + rr(u) - 1$.

Notice that atomic ϕ is relational iff $rr(\phi) \leq 0$.

7.3 Lemma. *For all atomic formulas $\phi(x)$, $x = (x_1, ..., x_n)$, there is a relational formula $\psi(x)$ such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a) \leftrightarrow \psi(a)$ (i.e. $\phi(x)$ and $\psi(x)$ are equivalent).*

Proof. Easy induction on $rr(\phi)$: If $rr(\phi) \leq 0$, the claim is clear and for $rr(\phi) = p+1 > 0$ e.g. if $\phi = R_i(t_1(x), ..., t_m(x))$, we observe that ϕ is equivalent with $\exists y_1 ... \exists y_m (R_i(y_1, ..., y_m) \wedge \bigwedge_{1 \leq j \leq m} y_j = t_j)$ and that the relationality ranks of formulas $y_j = t_j$ are $\leq p$ and thus the claim follows from the induction assumption. $\square$

7.4 Lemma. *For all formulas $\phi(x)$, there is a relational formula $\psi(x)$ such that it is equivalent with $\phi(x)$.*

Proof. By Lemma 7.3, trivial induction on ϕ. $\square$.

7.5 Definition. *$f : A \to \mathcal{B}$ is a relational partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if $A \subseteq \mathcal{A}$ and for all relational atomic formulas $\phi(x_1, ..., x_n)$ and $a \in A^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$.*

7.6 Definition. *Suppose $a \in \mathcal{A}^n$ and $b \in \mathcal{B}^n$. In order to simplify the notation, we assume that $\mathcal{A} \cap \mathcal{B} = \emptyset$.*

(i) Ehrenfeucht-Fraïssé game $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ of length $k \leq \omega$ is a game played by two players, I and II. At each round $m < k$, first I chooses $c_m \in \mathcal{A} \cup \mathcal{B}$ and then II chooses a relational partial isomorphism $f_m : \mathcal{A} \to \mathcal{B}$ such that $c_m \in dom(f_m) \cup rng(f_m)$, for all $1 \leq i \leq n$, $a_i \in dom(f_m)$, $f_m(a_i) = b_i$ and if $m > 0$, then $f_m \restriction dom(f_{m-1}) = f_{m-1}$. For $k = 0$, II wins if $a_i \mapsto b_i$ is a relational partial isomorphism and for $k > 0$, the first who breaks the rules looses and if neither break the rules, II wins.

(ii) A strategy for a player II in $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ is a sequence $(g_i)_{i<k}$ such that for all $i < k$, g_i is an $i+1$-ary function from $\mathcal{A} \cup \mathcal{B}$ to partial maps from $\mathcal{A}$ to $\mathcal{B}$. The strategy is winning if II always wins the game by choosing $f_i(c_0, .., c_i)$ on each round i.

(iii) We say that II wins $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ ($II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$) if II has a winning strategy in the game.

If $a = b = \emptyset$, we write $EF_k(\mathcal{A}, \mathcal{B})$ for $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$.

Notice that in (i) above we could require that $|dom(f_m)| = n + m + 1$ and this would not change the winner of the game.

7.7 Definition. *The quantifier rank $qr(\phi)$ of a formula ϕ is defined as follows: If ϕ is atomic, $qr(\phi) = 0$, $qr(\neg\psi) = qr(\psi)$, $qr(\psi \wedge \theta) = max\{qr(\psi), qr(\theta)\}$ and $qr(\exists x \psi) = qr(\psi) + 1$.*

7.8 Theorem. *Suppose L is finite, $a \in \mathcal{A}^n$ and $b \in \mathcal{B}^n$, Then for all $k \in \mathbb{N}$ the following are equivalent:*

(i) $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$.

(ii) For all relational formulas $\phi(x)$, $x = (x_1, ..., x_n)$, of quantifier rank $\leq k$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(b)$.

Proof. (i)$\Rightarrow$(ii): We prove this by induction on k. The case $k = 0$ is immediate by the definitions. So we assume that $k = p + 1 > 0$. We prove (ii) by induction on ϕ. The cases when ϕ is atomic or $\neg\psi$ or $\psi \wedge \theta$ are trivial. So we suppose $\phi(x) = \exists y \psi(y, x)$. Clearly we may assume that $qr(\psi) \leq p$. By symmetry it is enough to show that if $\mathcal{A} \models \phi(a)$ then $\mathcal{B} \models \phi(b)$. Since $\mathcal{A} \models \phi(a)$, there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \psi(c, a)$. We let this c be the first choice of I in $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$. Let f_0 be the answer given by the winning strategy of II and let $d = f_0(c)$. Then $II \uparrow EF_p((\mathcal{A}, c, a), (\mathcal{B}, d, b))$ and so by the induction assumption $\mathcal{A} \models \psi(c, a)$ iff $\mathcal{B} \models \psi(d, b)$. So $\mathcal{B} \models \psi(d, b)$ thus $\mathcal{B} \models \phi(b)$.

(ii)$\Rightarrow$(i): Clearly it is enough to prove the following claim:

1 Claim. *Suppose $k \in \mathbb{N}$. For all $n \in \mathbb{N}$ there is a finite set F^k_n of relational formulas $\phi(x)$, $x = (v_1, ..., v_n)$, (so for $n = 0$ the formulas are sentences) of quantifier rank $\leq k$ such that*

(a) for all $\mathcal{A}$ and $a \in \mathcal{A}^n$ there is $\phi(x) \in F^k_n$ such that $\mathcal{A} \models \phi(a)$

(b) if $\mathcal{A} \models \phi(a)$ and $\phi \in F^k_n$, then the following holds: For all $\mathcal{B}$ and $b \in \mathcal{B}^n$

(*) $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ iff $\mathcal{B} \models \phi(b)$.

Proof. By induction on k.

$k = 0$: Let $\psi_i(x)$, $x = (v_1, ..., v_n)$ and $i < m$, list all relational atomic formulas in which only variables $v_0, ..., v_n$ appear. For $Y \subseteq m$, let $\phi_Y(x) = \bigwedge_{i \in Y} \psi_i(x) \wedge \bigwedge_{i \in m-Y} \neg\psi_i(x)$. Let $F^k_n = \{\phi_Y | \ Y \subseteq m\}$. Clearly F^k_n is as required.

$k = p+1$: Let $\psi_i(x, v_{n+1})$, $x = (v_1, ..., v_n)$ and $i < m$, enumerate the set F^p_{n+1}. For all non-empty $Y \subseteq m$, let

$$\phi_Y(x) = \bigwedge_{i \in Y} \exists v_{n+1} \psi_i(x, v_{n+1}) \wedge \forall v_{n+1} \bigvee_{i \in Y} \psi_i(x, v_{n+1}).$$

We show that $F^k_n = \{\phi_Y(x) | \ Y \subseteq m, \ Y \neq \emptyset\}$ is as required.

By the induction assumption, each ϕ_Y is relational and of quantifier rank $\leq p + 1 = k$. So let $\mathcal{A}$ and $a \in \mathcal{A}^n$ be given. Let Y be the set of all $i < m$ such that $\mathcal{A} \models \exists v_{n+1} \psi_i(a, v_{n+1})$. By the induction assumption, $Y \neq \emptyset$ and so $\phi_Y \in F^k_n$. Furthermore, $\mathcal{A} \models \phi_Y(a)$. Thus (a) holds.

For (b), suppose $\mathcal{A} \models \phi_Y(a)$, $Y \subseteq m$ non-empty. By (i)$\Rightarrow$(ii), if $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$, then $\mathcal{B} \models \phi_Y(b)$.

For the other direction in (*), suppose $\mathcal{B} \models \phi_Y(b)$ and we describe a winning strategy for II. Let $c_0 \in \mathcal{A} \cup \mathcal{B}$ be the first move of I. We suppose $c_0 \in \mathcal{B}$, the other case is similar. Since $\mathcal{B} \models \phi_Y(b)$, there is $i \in Y$ such that $\mathcal{B} \models \psi_i(b, c_0)$. Since $\mathcal{A} \models \exists v_{n+1} \psi_i(a, v_{n+1})$, there is $d \in \mathcal{A}$ such that $\mathcal{A} \models \psi_i(a, d)$. The first move of II is f_0, where $dom(f_0) = \{a_1, ..., a_n, d\}$, $f_0(a_i) = b_i$ for $1 \leq i \leq n$ and $f_0(d) = c_0$. By the induction assumption $II \uparrow EF_p((\mathcal{A}, a, d), (\mathcal{B}, b, c_0))$ and thus the rest of the moves, II can play according to this winning strategy and win the game. □ Claim 1.

□

7.9 Definition. *We say that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent ($\mathcal{A} \equiv \mathcal{B}$) if for all sentences ϕ, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.*

7.10 Corollary. *The following are equivalent:*
(i) $\mathcal{A} \equiv \mathcal{B}$.
(ii) For all finite $L^* \subseteq L$ *and* $k \in \mathbb{N}$, $II \uparrow EF_k(\mathcal{A} \restriction L^*, \mathcal{B} \restriction L^*)$.

Proof. Immediate by Theorem 7.8, Lemma 7.4 and the fact that every L-formula is L^*-formula for some finite $L^* \subseteq L$. □

7.11 Remark. *There are a vocabulary L and L-structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv \mathcal{B}$ but II does not win $EF_1(\mathcal{A}, \mathcal{B})$.*

Proof. Exercise. □

8. Types and saturated models

8.1 Lemma. *Suppose T is a complete theory and $\mathcal{A}, \mathcal{B} \models T$.*
(i) If $f : \mathcal{A} \to \mathcal{B}$ is an partial elementary map, then there is $\mathcal{C} \models T$ and an elementary embedding $g : \mathcal{A} \to \mathcal{C}$ such that $\mathcal{B} \preceq \mathcal{C}$ and $g \restriction dom(f) = f$.
(ii) There are $\mathcal{B} \preceq \mathcal{C}$ and an elementary embedding $f : \mathcal{A} \to \mathcal{C}$.

Proof. Since T is complete, the empty function from $\mathcal{A}$ to $\mathcal{B}$ is elementary and thus (ii) follows from (i) (compare Lemma 5.6). We prove (i). Without loss of generality, we may assume that $\mathcal{A} \cap \mathcal{B} = \emptyset$ (this simplifies the notation).

As e.g. in the proof of Lemma 4.5, it suffices to show that

$$T^* = Th(\mathcal{A}, \mathcal{A}) \cup Th(\mathcal{B}, \mathcal{B}) \cup \{\underline{d} = \underline{e} \mid d \in dom(f),\ e = f(a)\}$$

is consistent (exercise). Let $\phi(\underline{c}, \underline{b}) \in Th(\mathcal{B}, \mathcal{B})$, where $\phi = \phi(x, y)$, $c \in (\mathcal{B} - rng(f))^m$ and $b \in rng(f)^n$. Let $a \in dom(f)^n$ be such that $f(a_i) = b_i$ for $1 \le i \le n$. Since $Th(\mathcal{B}, \mathcal{B})$ is closed under conjunctions, by compactness it is enough to prove that

$$T' = Th(\mathcal{A}, \mathcal{A}) \cup \{\phi(\underline{c}, \underline{b})\} \cup \{\underline{d} = \underline{e} \mid d \in dom(f),\ e = f(a)\}$$

is consistent. Now $\mathcal{B} \models \exists x_1 ... \exists x_m \phi(x, b)$ and since f is elementary,

$$\mathcal{A} \models \exists x_1 ... \exists x_m \phi(x, a).$$

Let $c' = (c'_1, ..., c'_m) \in \mathcal{A}^m$ be such that $\mathcal{A} \models \phi(c', a)$. Let $\mathcal{C}$ be a model we get from $(\mathcal{A}, \mathcal{A})$ by adding the interpretations for $\underline{e}$, $e \in \mathcal{B}$, as follows: If $e = f(d)$ for some $d \in dom(f)$, then $\underline{e}^{\mathcal{C}} = d$, if $e = \underline{c_i}$ for some $1 \le i \le m$, then $\underline{e}^{\mathcal{C}} = c'_i$ and otherwise we choose the interpretations freely. Clearly $\mathcal{C} \models T'$. □

8.2 Definition. *Suppose $A \subseteq \mathcal{A}$ and $n > 0$.*

(i) $L_{\omega\omega}(A,n)$ is the set of all $\phi(x,a)$, where $\phi(x,y)$ is a formula, $x = (v_1,...,v_n)$ and a is a sequence of elements of A.

(ii) An n-type over A is a non-empty subset of $L_{\omega\omega}(A,n)$.

(iii) An n-type p over A is complete if for all $\phi \in L_{\omega\omega}(A,n)$, $\phi \in p$ or $\neg\phi \in p$.

(iv) $b \in \mathcal{A}^n$ realizes an n-type p over A if $\mathcal{A} \models \phi(b,a)$ for all $\phi(x,a) \in p$. $t(b/A) = t(b/A;\mathcal{A})$ is the set of all $\phi(x,a) \in L_{\omega\omega}(A,n)$ such that $\mathcal{B} \models \phi(b,a)$ i.e. the unique complete n-type over A realized by b.

(v) An n-type p over A is consistent (in $\mathcal{A}$) if there is $\mathcal{B}$ and $b \in \mathcal{B}^n$ such that $\mathcal{A} \preceq \mathcal{B}$ and b realizes p.

(vi) $S_n(A) = S_n(A;\mathcal{A})$ is the set of all complete consistent n-types over A.

8.3 Lemma. *Suppose $A \subseteq \mathcal{A}$ and p is an n-type over A. Then the following are equivalent.*

(i) p is consistent.

(ii) For all $\phi_i(x,a^i) \in p$, $i < m \in \mathbb{N}$, $\mathcal{A} \models \exists v_1...\exists v_n \bigwedge_{i<m} \phi_i(x,a^i)$.

(iii) $T = Th((\mathcal{A},A)) \cup \{\phi(c,\underline{a}) |\ \phi(x,a) \in p\}$ is consistent, where $c = (c_1,...,c_n)$ are new constant symbols.

Proof. (i)$\Rightarrow$(ii): Let $\mathcal{B}$ and b witness the consistency. Then $\mathcal{B} \models \bigwedge_{i<m} \phi_i(b,a^i)$ and thus $\mathcal{B} \models \exists v_1...\exists v_n \bigwedge_{i<m} \phi_i(x,a^i)$. Since $\mathcal{A} \preceq \mathcal{B}$, $\mathcal{A} \models \exists v_1...\exists v_n \bigwedge_{i<m} \phi_i(x,a^i)$.

(ii)$\Rightarrow$(iii): Let $\phi_i(c,\underline{a^i}) \in \{\phi(c,\underline{a}) |\ \phi(x,a) \in p\}$, $i < m \in \mathbb{N}$. By compactness, it suffices to show that $T' = Th((\mathcal{A},A)) \cup \{\phi_i(c,\underline{a^i}) |\ i < m\}$ is consistent. By (ii), there is $b \in \mathcal{A}^n$ such that $(\mathcal{A},\mathcal{A}) \models \phi_i(b,\underline{a^i})$ for all $i < m$. Thus by interpreting c_i to b_i, we get a model for T'.

(iii)$\Rightarrow$(i): Let $\mathcal{B}^*$ be a model of T and $\mathcal{B} = \mathcal{B}^* \restriction L$. Clearly we may choose $\mathcal{B}^*$ so that for all $a \in A$, $\underline{a}^{\mathcal{B}^*} = a$. Then the identity function $id : A \to \mathcal{B}$ is an elementary partial map from $\mathcal{A}$ to $\mathcal{B}$ and thus by Lemma 8.1 (i), we may assume in addition that $\mathcal{A} \preceq \mathcal{B}$. Letting $b_i = c_i^{\mathcal{B}^*}$ for $1 \leq i \leq n$, $b = (b_1,...,b_n)$ realizes p in $\mathcal{B}$. □

8.4 Remark.

(i) Suppose $A \subseteq \mathcal{A}$. Letting the sets $\{p \in S_n(A) |\ \phi(x,a) \in p\}$ for $\phi(x,a) \in L_{\omega\omega}(A)$, be a basis for a topology on $S_n(A)$, we get a Hausdorff topology which is by Lemma 8.3 also compact (exercise). This space is called a Stone space.

(ii) If T is complete and $\mathcal{A},\mathcal{B} \models T$, then $S_n(\emptyset;\mathcal{A}) = S_n(\emptyset;\mathcal{B})$ (by Lemma 8.3 (iii)) and thus when T is given, we can talk about $S_n(\emptyset)$ without need to specify the model.

8.5 Definition.

(i) We say that $\mathcal{A}$ is κ-saturated if if for all $n \in \mathbb{N} - \{0\}$, $A \subseteq \mathcal{A}$ of power $< \kappa$ and $p \in S_n(A)$, some $a \in \mathcal{A}^n$ realizes p. We say that $\mathcal{A}$ is saturated if it is $|\mathcal{A}|$-saturated.

(ii) We say that $\mathcal{A}$ is strongly κ-homogeneous if for all partial elementary maps $f : \mathcal{A} \to \mathcal{A}$ with $dom(f)$ of power $< \kappa$, there is an automorphism g of $\mathcal{A}$ such that $g \restriction dom(f) = f$.

(iii) We say that $\mathcal{A}$ is κ-universal if for all $\mathcal{B} \models Th(\mathcal{A})$ of power $< \kappa$, there is an elementary embedding $f : \mathcal{B} \to \mathcal{A}$.

8.6 Lemma. *Suppose $\kappa \geq \omega$. If for all $A \subseteq \mathcal{A}$ of power $< \kappa$ and $p \in S_1(A)$, some $a \in \mathcal{A}$ realizes p, then $\mathcal{A}$ is κ-saturated.*

Proof. Exercise. □

8.7 Theorem. *Suppose T is a complete theory and κ is a cardinal. Then there is $\mathcal{A} \models T$ which is κ-saturated, κ-universal and strongly κ-homogeneous.*

Proof. By Lemma 8.1 and Corollary 4.7, the proof is the same as that of Theorem 5.11, verbatim, except that one needs to replace $\subseteq$ by $\preceq$, partial isomorphisms by partial elementary maps and quantifier free formulas by complete types (exercise). □

8.8 Remark. *It is not an accident that the proofs of 8.7 and 5.11 are the same, see the work on abstract elementary classes.*

8.9 Lemma. *If T is a complete theory and $\mathcal{A}, \mathcal{B} \models T$ are infinite saturated models of the same size, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. Let us enumerate $\mathcal{A} = \{a_i |\ i < \kappa\}$ and $\mathcal{B} = \{b_i |\ i < \kappa\}$. By induction on $i \leq \kappa$, we construct partial elementary maps $f_i, g_i : \mathcal{A} \to \mathcal{B}$ so that

(i) for $i < j \leq \kappa$, $f_i \subseteq g_j \subseteq f_j$,

(ii) for all $i < \kappa$, $a_i \in dom(g_{i+1})$ and $b_i \in rng(f_{i+1})$,

(iii) for all $i < \kappa$, $|dom(f_i)|, |dom(g_i)| < |i|^+ + \omega$.

For $i = 0$, we let $f_i = g_i = \emptyset$ (these are elementary because T is complete) and for limit i we let $g_i = f_i = \bigcup_{j<i} f_j$ $(= \bigcup_{j<i} g_j$ by (i)).

Suppose $i = j + 1$. Let $A = dom(f_j)$ and $p = t(a_j/A; \mathcal{A})$. Let $f_j(p) = \{\phi(x, f_j(a'_1), ..., f_j(a'_n)) |\ \phi(x, a'_1, ..., a'_n) \in p\}$. Since f_j is elementary, by Lemma 8.3 (ii), $f_j(p)$ is consistent. Since $\mathcal{B}$ is saturated and f_j satisfies (iii), $f_j(p)$ is realized in $\mathcal{B}$ by some b. Let g_i be such that $dom(g_i) = dom(f_j) \cup \{a_j\}$, $g_i \restriction A = f_j$ and $g_i(a_j) = b$. Clearly (i)-(iii) are satisfied. f_i can be found similarly, only start from g_i and look the inverses.

Then f_κ is the isomorphism claimed to exist. □

8.10 Example. *Every uncountable model of T_{acf_0} is saturated and thus T_{acf_0} is κ-categorical for all $\kappa > \omega$.*

Proof. Exercise. □

8.11 Fact. *$T_{dlo} = Th((\mathbf{Q}, <))$ has a saturated model of power κ iff $\kappa^{<\kappa} = \kappa \geq \omega$, where $\kappa^{<\kappa}$ is the cardinality of the set $\{f : \alpha \to \kappa |\ \alpha < \kappa\}$. (So T_{dlo} has a saturated model of power $\aleph_1$ iff the continuum hypothesis holds and it has never a saturated model of power $\aleph_\omega$. In fact, it is consistent that upto isomorphism, $(\mathbf{Q}\ <)$ is the only saturated model of T_{dlo}.)*

8.12 Lemma.
(i) If $\mathcal{A}$ is κ-saturated, then it is κ^+-universal.
(ii) If $\mathcal{A}$ is saturated then it is strongly $|\mathcal{A}|$-homogeneous.

Proof. Exercise. $\square$

8.13 Exercise. *Show that if $\mathcal{A} \models T_{dlo}$ is saturated and $\kappa = |\mathcal{A}|$, then $\kappa^{<\kappa} = \kappa$. (Hint: See Exercise 6.6.)*

9. Omitting types and ω-categoricity

9.1 Definition.

(i) We say that a theory T locally omits an n-type p over $\emptyset$, if the following holds: For every formula $\phi(x)$, $x = (v_1, ..., v_n)$, if $T \not\models \forall v_1 ... \forall v_n \neg\phi(x)$, then there is $\theta(x) \in p$ such that $T \not\models \forall v_1 ... \forall v_n(\phi(x) \to \theta(x))$.

(ii) We say that $\mathcal{A}$ omits an n-type p over $\emptyset$ if no $a \in \mathcal{A}^n$ realize p.

In the following theorem, it is crucial that the vocabulary is countable (i.e. of size ω or finite) and that we claim only that $\mathcal{A}$ is countable, see Remark 9.3. In fact for uncountable vocabularies there are no known methods, anywhere as powerful as 9.2, to omit types.

9.2 Omitting types theorem. *Suppose L is countable, T is a theory and D is a countable collection of types over $\emptyset$. If T locally omits every $p \in D$ then T has a countable model $\mathcal{A}$ which omits every $p \in D$.*

Proof. This proof is a modification of the usual proof of the completeness theorem, see the course Matemaattinen logiikka. For simplicity, we assume that D is a singleton and that the only type p in D is a 1-type (exercise: what changes are needed to prove the general case?). Let c_i, $i < \omega$, be new constants and denote $L^* = L \cup \{c_i |\ i < \omega\}$. Let ϕ_i, $i < \omega$, enumerate all L^*-sentences so that if c_j appears in ϕ_i then $j < i$. By recursion on $k < \omega$, we construct an increasing sequence of consistent L^*-theories T_k so that

(i) $T_k - T$ is finite and if c_i appears in some $\theta \in T_k$, then $i \le k$,

(ii) $\phi_k \in T_{k+1}$ or $\neg\phi_k \in T_{k+1}$,

(iii) if $\phi_k = \exists x\theta(x) \in T_{k+1}$, then $\theta(c_{k+1}) \in T_{k+1}$,

(iv) there is $\theta(v_1) \in p$ such that $\neg\theta(c_k) \in T_{k+1}$.

We let $T_0 = T$.

For T_{k+1}, first we choose L-formula $\phi(v_1, x_0, ..., x_{k-1})$, so that

$$\models \phi(c_k, c_0, ..., c_{k-1}) \leftrightarrow \bigwedge\{\theta |\ \theta \in T_k - T\}$$

(if $k = 0$, we let $\phi = v_1 = v_1$). Now $T \not\models \phi(c_k, c_0, ..., c_{k-1}) \to \theta(c_k)$ for some $\theta(v_1) \in p$ because otherwise (exercise or see the course Matemaattinen logiikka)

$$T \models \forall v_1(\exists x_0 ... \exists x_{k-1}\phi(v_1, x_0, ..., x_{k-1}) \to \theta(v_1))$$

for all $\theta(v_1) \in p$ contradicting the assumption that T locally omits p. So there is $\theta(v_1) \in p$ such that $T^*_{k+1} = T_k \cup \{\neg\theta(c_k)\}$ is consistent. This takes care of (iv).

Clearly either $T^*_{k+1} \cup \{\phi_k\}$ or $T^*_{k+1} \cup \{\neg\phi_k\}$ is consistent and let T^{**}_{k+1} be the one of these that is consistent. This takes care of (ii).

Unless $\phi_k = \exists x\psi(x)$ for some ψ and $\phi_k \in T^{**}_k$, we let $T_{k+1} = T^{**}_{k+1}$. Otherwise, since c_{k+1} does not appear in T^{**}_{k+1}, $T_{k+1} = T^{**}_{k+1} \cup \{\psi(c_{k+1})\}$ is consistent (exercise or see the course Matemaattinen logiikka). This takes care of (iii). Clearly (i) holds.

Then $T^* = \cup_{k<\omega} T_k$ is consistent and it has a model, say $\mathcal{B}^*$. Let $\mathcal{B} = \mathcal{B}^* \restriction L$ and $A = \{c_i^{\mathcal{B}} |\ i < \omega\}$.

1 Claim. *For all constants $c \in L$, $c^{\mathcal{B}} \in A$ and for all $n-ary$ function symbols $f \in L$ and $a \in A^n$, $f^{\mathcal{B}}(a) \in A$.*

Proof. Exercise. □ Claim 1.

By Claim 1 we can let $\mathcal{A}$ be the L-model such that $dom(\mathcal{A}) = A$, for all $R \in L$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^{\#R}$, for all $f \in L$, $f^{\mathcal{A}} = f^{\mathcal{B}} \restriction A^{\#f}$ and for all $c \in L$, $c^{\mathcal{A}} = c^{\mathcal{B}}$. Then $\mathcal{A}$ is a substructure of $\mathcal{B}$ and by (iii) and Tarski-Vaught, $\mathcal{A} \preceq \mathcal{B}$ (if $\mathcal{B} \models \exists x\psi(x, c^{\mathcal{B}})$, $c = (c_{i_1}, ..., c_{i_n})$ and $c^{\mathcal{A}} = (c^{\mathcal{B}}_{i_1}, ..., c^{\mathcal{B}}_{i_n})$, then $\exists x\psi(x, c) \in T^*$ and so $\psi(c_k, c) \in T^*$ for some k i.e. $\mathcal{B} \models \psi(c^{\mathcal{B}}_k, c^{\mathcal{B}})$). So $\mathcal{A} \models T$ and by (iv), $\mathcal{A}$ omits p. □

9.3 Remark. *Let us look at the theory T_{f_0}. Let*

$$p = \{\neg(\Sigma^{i=n}_{i=0} a_i v^i_1 = 0) |\ n > 0,\ a_i \in \mathbf{Z}, a_n \neq 0\}.$$

Then every model of T_{f_0} which omits p is countable. Using this, one can find L', T' and a type p' such that T' locally omits p' but no model of T' omit p' (exercise).

9.4 Definition. *Assume T is complete.*

(i) We say that $p \in S_n(\emptyset)$ is isolated if T does not locally omit p i.e. there is $\phi(v_1, ..., v_n) \in p$ such that for all $\psi(v_1, ..., v_n) \in p$, $T \models \forall v_1 ... \forall v_n(\phi(v_1, ..., v_n) \rightarrow \psi(v_1, ..., v_n))$. When this happens, we say that ϕ isolates p.

(ii) $\mathcal{A} \models T$ is atomic if for all $n \in \mathbb{N}$ and $a \in \mathcal{A}^n$, $t(a/\emptyset)$ is isolated.

(iii) We say that $\phi(v_1, ..., v_n)$ is complete if it isolates some $p \in S_n(\emptyset)$.

(iv) We say that T is atomic if for all $\phi(v_1, ..., v_n)$ either $T \models \forall v_1 ... \forall v_n \neg\phi$ or there is complete $\psi(v_1, ..., v_n)$ such that $T \models \forall v_1 ... \forall v_n(\psi \rightarrow \phi)$.

9.5 Exercise. *Let T be a complete theory and suppose that for all $n \in \mathbb{N} - \{0\}$, $S_n(\emptyset)$ is countable. Show that T has an atomic model. Conclude that T_{acf_0} has an atomic model.*

9.6 Lemma. *Suppose T is a complete theory. Then the following are equivalent.*

(i) T is atomic.

(ii) T has an atomic model.

Proof. (ii)$\Rightarrow$(i): Exercise.

(i)$\Rightarrow$(i): For all $n \in \mathbb{N} - \{0\}$, let

$$p_n = \{\neg\phi(v_1, ..., v_n)|\ \phi \text{ is complete}\}.$$

Since T is atomic, T locally omits every p_n. Thus by Theorem 9.2, T has a model $\mathcal{A}$ which omits every p_n. Clearly $\mathcal{A}$ is atomic. □

9.7 Exercise. *Find a theory that does not have an atomic model.*

9.8 Lemma. *Assume T is complete (with infinite models) and $\mathcal{A}$ and $\mathcal{B}$ are countable atomic models of T. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. This proof is essentially the same as that of Lemma 8.9: Let $\{a_i|\ i < \omega\}$ and $\{b_i|\ i < \omega\}$ be enumerations of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then we construct an increasing sequences of finite partial elementary maps $f_i, g_i : \mathcal{A} \to \mathcal{B}$, $i < \omega$, as in the proof of Lemma 8.9. We let $f_0 = g_0 = \emptyset$ and g_{i+1} is found as follows: Let $\{a'_1, ..., a'_n\} = dom(f_i)$. Then $t((a'_1, ..., a'_n, a_i)/\emptyset)$ is isolated, say by $\phi(v_1, ..., v_{n+1})$. Since f_i is elementary, $\mathcal{B} \models \exists v_{n+1}\phi(f_i(a'_1), ..., f_i(a'_n), v_{n+1})$. So there is $b \in \mathcal{B}$ such that $\mathcal{B} \models \phi(f_i(a'_1), ..., f_i(a'_n), b)$. Since ϕ isolates $t((a'_1, ..., a'_n, a_i)/\emptyset)$, $t((f_i(a'_1), ..., f_i(a'_n), b)/\emptyset) = t((a'_1, ..., a'_n, a_i)/\emptyset)$. This means that g_{i+1} is elementary when $dom(g_{i+1}) = dom(f_i) \cup \{a_i\}$, $g_{i+1} \restriction dom(f_i) = f_i$ and $g_{i+1}(a_i) = b$. f_{i+1} is found similarly. $\cup_{i<\omega} f_i$ is the required isomorphism. □

9.9 Lemma. *Assume T is complete (with infinite models), $\mathcal{B} \models T$ and $\mathcal{A}$ is a countable atomic model of T. Then there is an elementary embedding $f : \mathcal{A} \to \mathcal{B}$.*

Proof. As the previous lemma (exercise). □

9.10 Theorem (Ryll-Nardzewski). *Assume L is countable and T is complete and has infinite models. Then the following are equivalent:*

(i) T is ω-categorical,

(ii) for all $n \in \mathbb{N} - \{0\}$, $S_n(\emptyset)$ is finite.

Proof. (ii)$\Rightarrow$(i): If $S_n(\emptyset)$ is finite, then every $p \in S_n(\emptyset)$ is isolated (if $S_n(\emptyset) = \{p_0, ..., p_n\}$, $p = p_0$, then $\bigwedge_{1 \le i \le n} \phi_i$ isolates p when the formulas ϕ_i are chosen so that $\phi_i \in p - p_i$). Thus every model of T is atomic and so (i) follows from Lemma 9.8.

(i)$\Rightarrow$(ii): Suppose $S_n(\emptyset)$ is infinite. We show that T is not ω-categorical.

1 Claim. *There is non-isolated $r \in S_n(\emptyset)$.*

Proof. Suppose not. For every $p \in S_n(\emptyset)$, let $\phi_p \in p$ be a formula that isolates p. Then $q = \{\neg\phi_p|\ p \in S_n(\emptyset)\}$ can be realized in a model of T by compactness since for all $p \in S_n(\emptyset)$ every realization of p realizes $q - \{\phi_p\}$. Let $\mathcal{B} \models T$ and $b \in \mathcal{B}^n$ be such that b realizes q. Then $r = t(b/\emptyset)$ is a complete consistent type but $r \notin S_n(\emptyset)$, a contradiction. □ Claim 1.

Let r be as in Claim 1. By omitting types theorem, T has a countable model $\mathcal{A}$ that omits r. Since T is complete and has infinite models, every model of T is infinite and so $\mathcal{A}$ has power ω.

On the other hand, since $r \in S(\emptyset)$, there is a model $\mathcal{B}$ of T that realizes r. As above $\mathcal{B}$ is infinite and so by Lemma 4.8 can be chosen to have cardinality ω. Clearly $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. ◻

9.11 Example. *T_{acf_0} is not ω-categorical.*

Proof. Exercise. ◻

10. Indiscernible sequences

In the next section indiscernible sequences will play an important role. In this section we make some general observations about them.

10.1 Definition. *Suppose $(I,<)$ is a linear ordering, for all $i \in I$, $a_i \in \mathcal{A}^n$ and $A \subseteq \mathcal{A}$. We say that $(a_i)_{i \in I}$ is m^*-indiscernible over A of for all $m \leq m^*$, $\phi(x^1, ..., x^m, a)$, $x^k = (x^k_1, ..., x^k_n)$ and $a \in A^{n^*}$, the following holds: If $i_1 < i_2 < ... < i_m$ and $j_1 < j_2 < ... < j_m$ are from I, then $\mathcal{A} \models \phi(a_{i_1}, ..., a_{i_m}, a)$ iff $\mathcal{A} \models \phi(a_{j_1}, ..., a_{j_m}, a)$. $(a_i)_{i \in I}$ is indiscernible over A if it is m^*-indiscernible over A for all $m^* \in \mathbb{N}$.*

If for all $i, j \in I$, $a_i = a_j$, $(a_i)_{i \in I}$ is called trivial (indiscernible sequence). When we talk about indiscernible sequences we mean non-trivial ones.

10.2 Definition. *Let κ, λ and ξ be cardinals and $X \subseteq \kappa$.*

(i) By $[X]^n$ we mean the set $\{(\alpha_1, ..., \alpha_n) \in X^n |\ \alpha_1 < \alpha_2 < ... < \alpha_n\}$.

(ii) We write $\kappa \to (\lambda)^n_\xi$ if the following holds: For all functions $f : [\kappa]^n \to \xi$ there is $X \subseteq \kappa$ of power λ such that $f \restriction [X]^n$ is constant (such X is called homogeneous).

10.3 Ramsay's theorem. *$\omega \to (\omega)^n_k$ for all $n, k \in \mathbb{N} - \{0\}$.*

Proof. By induction on n:

$n = 1$: This is just the pigeon hole principle.

$n = m+1$: By the induction assumption we can find by recursion on $i < \omega$, infinite sets $X_i \subseteq \omega$, $b_i \in X_i$, functions $f_i : [X_i - \{b_i\}]^m \to k$ and $c_{i+1} \in k$ as follows:

$i = 0$: $X_0 = \omega$, $b_0 = 0$ and $f_0(a_1, ..., a_m) = f(b_0, a_1, ..., a_m)$.

$i = j+1$: We let $X_i \subseteq X_j$ and $c_i \in k$ be such that X_i is infinite and for all $(a_1, ..., a_m) \in [X_i]^m$, $f_j(a_1, ..., a_m) = c_i$. We let b_i be the least element of X_i and $f_i(a_1, ..., a_m) = f(b_i, a_1, ..., a_m)$.

By the case $i = 1$, we can find infinite $I \subseteq \omega$ and $c \in k$ such that for all $i \in I$, $c_{i+1} = c$. Then $X = \{b_i |\ i \in I\}$ is as wanted (exercise). ◻

The following theorem is just one example of what kind of indiscernible sequences can be found by compactness.

10.4 Theorem. *Suppose $(I,<)$ is a linear ordering, $a_i = (a^i_1, ..., a^i_n) \in \mathcal{A}^n$, $i < \omega$, and $d \in \mathcal{A}^m$ are such that for $i \neq j$, $\mathcal{A} \models \phi(a_i, a_j, d)$ iff $i < j$, where $\phi(x, y, z)$ is a formula. Then there are $\mathcal{A} \preceq \mathcal{B}$ and $e_i \in \mathcal{B}^n$ such that $(e_i)_{i \in I}$ is indiscernible over $\mathcal{A}$ and for all $i, j \in I$, $i \neq j$, $\mathcal{B} \models \phi(e_i, e_j, d)$ iff $i < j$.*

Proof. Let c^i_j, $i \in I$ and $1 \leq j \leq n$, be constants not in $L(\mathcal{A})$. Denote $c_i = (c^i_1, ..., c^i_n)$. Clearly it is enough to show that the following theory T is consistent:

$$T = Th(\mathcal{A}, \mathcal{A}) \cup$$

$$\{\psi(c_{i_1}, ..., c_{i_k}, \underline{b}) \leftrightarrow \psi(c_{j_1}, ..., c_{j_k}, \underline{b}) |\ \psi(z_1, ..., z_k, z)\ L\text{-formula},\ b \in \mathcal{A}^{lg(z)},$$

$$i_1 < i_2 < ... < i_k,\ j_1 < j_2 < ... < j_k\} \cup$$

$$\{\phi(c_i, c_j, \underline{d}) \wedge \neg\phi(c_j, c_i, \underline{d}) |\ i, j \in I,\ i < j\}$$

where $lg(z)$ is the length of the sequence z. By compactness, it is enough to show that

$$T' = Th(\mathcal{A}, \mathcal{A}) \cup$$

$$\{\psi_s(c_{i^s_1}, ..., c_{i^s_{k_s}}, \underline{b}) \leftrightarrow \psi_s(c_{j^s_1}, ..., c_{j^s_{k_s}}, \underline{b}) |\ s \leq s^*\} \cup$$

$$\{\phi(c_{i^0_l}, c_{i^0_{l'}}, \underline{d}) \wedge \neg\phi(c_{i^0_{l'}}, c_{i^0_l}, \underline{d}) |\ 1 \leq l < l' \leq k_0\}$$

is consistent for arbitrary L-formulas $\psi_s(z_1, ..., z_{k_s}, z)$, $s \leq s^* \in \mathbb{N}$, and $i^s_1 < i^s_2 < ... < i^s_{k_s}$ and $j^s_1 < j^s_2 < ... < j^s_{k_s}$ from I.

For each $s \leq s^*$ define $f_s : [\omega]^{k_s} \to 2$ so that $f_s(n_1, ..., n_{k_s}) = 1$ if $\mathcal{A} \models \psi_s(a_{n_1}, ..., a_{n_{k_s}}, b)$ and otherwise $f_s(n_1, ..., n_{k_s}) = 0$. By Ramsey's theorem there is infinite $X \subseteq \omega$ such that it is homogeneous for every function f_s, $s \leq s^*$.

Let $\pi : \{i^s_l, j^s_l |\ s \leq s^*. 1 \leq l \leq k_s\} \to X$ be order preserving. Let $\mathcal{A}^*$ be a model we get from $(\mathcal{A}, \mathcal{A})$ by interpreting $c^{i^s_l}_p$ to $a^{\pi(i^s_l)}_p$ and $c^{j^s_l}_p$ to $a^{\pi(j^s_l)}_p$, where $1 \leq l \leq k_s$ and $1 \leq p \leq n$. Clearly $\mathcal{A}^* \models T'$. □

10.5 Theorem. *Suppose $(I,<) \subseteq (J,<)$ are infinite linear orderings, $A \subseteq \mathcal{A}$ and $a_i \in \mathcal{A}^n$, $i \in I$, are such that $(a_i)_{i \in I}$ is indiscernible over A. Then there are $\mathcal{A} \preceq \mathcal{B}$ and $b_i \in \mathcal{B}^n$ such that $(b_i)_{i \in J}$ is indiscernible over A and for all $i \in I$, $b_i = a_i$.*

Proof. As the proof of the previous theorem (exercise, Ramsey's theorem is not needed). □

10.6 Lemma. *Suppose $(I,<) \subseteq (J,<)$ are linear orderings such that I is an end segment of J (i.e. if $j > i \in I$, then $j \in I$). If $(a_i)_{i \in J}$ is indiscernible over $A \subseteq \mathcal{A}$, $a_i = (a^i_1, ..., a^i_n) \in \mathcal{A}^n$, then $(a_i)_{i \in I}$ is indiscernible over $A \cup \{a^j_k |\ j \in J - I,\ 1 \leq k \leq n\}$.*

Proof. Exercise. □

10.7 Example. *Suppose $\mathcal{A} \models T_{acf_0}$. For $A \subseteq \mathcal{A}$, by $acl(A)$ we mean the algebraic closure of A (i.e. the set of all roots from $\mathcal{A}$ of all non-zero polynomials $P(X)$ over the field generated by A). Then $(a_i)_{i<\omega}$, $a_i \in \mathcal{A}$, is indiscernible over $\emptyset$ (and non-trivial) iff for all $i < \omega$, $a_i \notin acl(\{a_j|\ j < i\})$.*

Proof. Exercise. ▫

11. Ehrenfeucht-Mostowski models

11.1 Definition. *Given a vocabulary L, a skolemization L^S of L is the vocabulary $L \cup \{f_{\phi(v_i,x)}|\ \phi(v_i,x)\ L$-formula, $x = (x_1, ..., x_n)\}$, where $f_{\phi(v_i,x)}$ are new n-ary function symbols (0-ary function symbols are constants). Skolem theory T^S is the set of all sentences*

$$\forall x_1 ... \forall x_n (\exists v_i \phi(v_i, x) \rightarrow \phi(f_{\phi(v_i,x)}(x), x)),$$

where $\phi(v_i, x)$ is an L-formula.

11.2 Lemma.
(i) For all L-structures $\mathcal{A}$ there is L^S-structure $\mathcal{A}^S$ such that $\mathcal{A}^S \models T^S$ and $\mathcal{A}^S \restriction L = \mathcal{A}$.
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are L^S-structures and $\mathcal{B} \subseteq \mathcal{A} \models T^S$, then $\mathcal{B} \restriction L \preceq \mathcal{A} \restriction L$.

Proof. (i) is trivial and (ii) follows immediately from Tarski-Vaught. ▫

11.3 Definition. *Let $\mathcal{A}$ be an L^S-structure and $A \subseteq \mathcal{A}$. By $SH(A)$ (Skolem hull) we mean the set*

$$\{t^{\mathcal{A}}(a)|\ t(x)\ L^S\text{-term},\ x = (x_1, ..., x_n),\ a \in A^n\}.$$

Then $SH(A)$ is closed under all $f^{\mathcal{A}}$, $f \in L^S$ a function symbol, and contains all $c^{\mathcal{A}}$, $c \in L^S$ a constant symbol, and thus it can be equipped with the structure induced from $\mathcal{A}$. This substructure of $\mathcal{A}$ is also called $SH(A)$.

Notice that by Lemma 11.2 (ii), $SH(A) \restriction L \preceq \mathcal{A} \restriction L$.

11.5 Definition. *Let κ be an infinite cardinal. For all ordinals α, a cardinal $\beth_\alpha(\kappa)$ is defined as follows: $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$ and for limit α, $\beth_\alpha(\kappa) = \bigcup_{\beta<\alpha} \beth_\beta(\kappa)$. Also we write $\beth_\alpha = \beth_\alpha(\omega)$.*

Notice that for all infinite cardinals κ, $\beth_{\kappa^+}(\kappa^+) = \beth_{\kappa^+}$.

11.6 Fact (Erdös-Rado). *For all infinite cardinals κ and $n \in \mathbb{N}$,*

$$(\beth_n(\kappa))^+ \rightarrow (\kappa^+)^{n+1}_\kappa.$$

Proof. See e.g. [Je]. ▫

Only for notational simplicity, in the following theorem we look at elements $a_i^\alpha \in \mathcal{A}_\alpha$ instead n-sequences $a_i^\alpha \in \mathcal{A}_\alpha^n$ for $n \in \mathbb{N}$.

11.7 Theorem. *Let $\kappa = |L_{\omega\omega}|$ and $\lambda = (2^\kappa)^+$. Suppose that for all $\alpha < \lambda$, we have $\mathcal{A}_\alpha$ and $a_i^\alpha \in \mathcal{A}_\alpha$, $i < (\beth_\alpha(\lambda))^+$, such that $a_i^\alpha \neq a_j^\alpha$ for $i \neq j$. For all $\alpha < \lambda$, let $\mathcal{A}_\alpha^S$ be as in Lemma 11.2 (i) for $\mathcal{A}_\alpha$. Then there is a collection Φ of L^S-formulas with the following properties:*

(i) for every L^S-formula $\phi(v_1, ..., v_n)$, $n \in \mathbb{N}$, either $\phi \in \Phi$ or $\neg\phi \in \Phi$,

(ii) for all linear orderings $(I, <)$, there are L^S-structure $\mathcal{B}$ and $b_i \in \mathcal{B}$, $i \in I$, such that

(a) for all $\phi(v_1, ..., v_n)$ and $i_1 < i_2 < ... < i_n$ from I, $\mathcal{B} \models \phi(b_{i_1}, ..., b_{i_n})$ iff $\phi(v_1, ..., v_n) \in \Phi$,

(b) for all $i_1 < i_2 < ... < i_n$ from I, there are $\alpha < \lambda$, $\gamma_1 < ... < \gamma_n < (\beth_\alpha(\lambda))^+$ and an isomorphism $\pi : SH(\{b_{i_1}, ..., b_{i_n}\}) \to SH(\{a_{\gamma_1}^\alpha, ..., a_{\gamma_n}^\alpha\})$ such that for all $1 \leq k \leq n$, $\pi(b_{i_k}) = a_{\gamma_k}^\alpha$.

Proof. By Lemma 10.6 it is enough to prove (ii) in the case $I = \omega$. Furthermore, by (i) and (a), it is enough to prove (b) in the case $i_k = k$ for all $1 \leq k \leq n \in \mathbb{N} - \{0\}$. We do this.

By recursion on $n \in \mathbb{N}$ we construct n-types Φ_n over $\emptyset$ in vocabulary L^S, and for $\alpha < \lambda$, $\alpha^n \in \lambda - \alpha$ and $X_n^\alpha \subseteq (\beth_{\alpha^n}(\kappa))^+$ of power $\geq (\beth_\alpha(\lambda))^+$ so that

(I) $\alpha^n < \beta^n$ for $\alpha < \beta$ and $\alpha^{n+1} = \beta^n$ for some $\beta \geq \alpha$ and then $X_{n+1}^\alpha \subseteq X_n^\beta$,

(II) for all $\phi(v_1, ..., v_n)$ and $i_1 < ... < i_n$ from X_n^α, $\phi(v_1, ..., v_n) \in \Phi_n$ iff $\mathcal{A}_{\alpha^n}^S \models \phi(a_{i_1}^{\alpha^n}, ..., a_{i_n}^{\alpha^n})$,

(III) if $(a_1, ..., a_n) \in \mathcal{A}^n$ realizes Φ_n, then $(a_1, ..., a_n)$ is n-indiscernible over $\emptyset$,

(IV) $\Phi_n \subseteq \Phi_{n+1}$.

$n = 0$: Since the number of possible L^S-theories is $< \lambda$, there is $X \subseteq \lambda$ of power λ such that for all $\alpha, \beta \in X$, $Th(\mathcal{A}_\alpha^S) = Th(\mathcal{A}_\beta^S)$ and we let Φ_0 be the common theory. $\alpha^0 = min(X - \{\beta^0 |\ \beta < \alpha\})$ and $X_0^\alpha = (\beth_{\alpha^0}(\kappa))^+$. Clearly (I)-(IV) hold.

$n = m + 1$: For all $\alpha < \lambda$, let $\alpha_*^n = (\alpha + n)^m$ and notice that

(*) $|X_m^{\alpha+n}| \geq (\beth_m(\beth_\alpha(\lambda)))^+$.

Define $f_n^\alpha : [X_m^{\alpha+n}]^n \to S_n(\emptyset)$ so that $f_n^\alpha(i_1, ..., i_n) = t((a_{i_1}^{\alpha_*^n}, ..., a_{i_n}^{\alpha_*^n})/\emptyset; \mathcal{A}_{\alpha_*^n}^S)$. Since

(**) $|S_n(\emptyset)| < \lambda$,

by Erdös-Rado and (*) above, there is homogeneous $X^{\alpha_*^n} \subseteq X_m^{\alpha+n}$ of power $\geq \beth_\alpha(\lambda)$. Let $p_{\alpha_*^n}$ be the constant value.

By (**), there is $X \subseteq \{\alpha_*^n |\ \alpha < \lambda\}$ of power λ and p such that for all $\gamma \in X$, $p_\gamma = p$. Now let $\alpha^n = min(X - \{\beta^n |\ \beta < \alpha\})$, $X_n^\alpha = X^{\alpha_n}$ and $\Phi_n = p$. Notice that by the assumptions (I) and (II) for m, $\Phi_m \subseteq \Phi_n$. So clearly (I)-(IV) hold.

Then we let $\Phi = \cup_{n<\omega}\Phi_n$. This is as wanted (exercise). ▫

11.8 Corollary. *Let $\kappa = |L_{\omega\omega}|$ and suppose that T is a theory and D is a collection of types over $\emptyset$. If for all $\lambda < \beth_{(2^\kappa)^+}$ there is $\mathcal{A} \models T$ of power $\geq \lambda$ such that it omits every $p \in D$, then for all $\theta \geq \kappa$ there is $\mathcal{A} \models T$ of power θ such that it omits every $p \in D$.*

Proof. For all $\alpha < (2^\kappa)^+$, since $(\beth_\alpha((2^\kappa)^+))^+ < \beth_{(2^\kappa)^+}$, we can find $\mathcal{A}_\alpha \models T$ and $a_i^\alpha \in \mathcal{A}_\alpha$, $i < (\beth_\alpha((2^\kappa)^+))^+$ such that

(i) $a_i^\alpha \neq a_j^\alpha$ for $i \neq j$,

(ii) $\mathcal{A}_\alpha$ omits every $p \in D$.

Let Φ be as in Theorem 11.7. Let $\mathcal{B}$ and b_i, $i < \theta$, be as in Theorem 11.7 (ii) for $(I, <) = (\theta, <)$. We claim that $\mathcal{C} = SH(\{b_i | \ i < \theta\}) \restriction L$ is as wanted.

Clearly $|\mathcal{C}| = \theta$. Also $\mathcal{C} \preceq \mathcal{B}$ and thus $\mathcal{C} \models T$. Finally, suppose $p \in D$. For a contradiction, assume that $c = (c_1, ..., c_m) \in \mathcal{C}^m$ realizes p. Then there are $i_1 < ... < i_n$ in θ such that $c_i \in SH(\{b_{i_1}, ..., b_{i_n}\}) = \mathcal{D}$ for all $1 \leq i \leq m$. Since $\mathcal{D} \restriction L \preceq \mathcal{C}$, c realizes p in $\mathcal{D}$. Let α, $\gamma_1, ..., \gamma_n$ and π be as in Theorem 11.7 (ii)(b). Then $d = (\pi(c_1), ..., \pi(c_m))$ realizes p is $\mathcal{D}' = SH(\{a^\alpha_{\gamma_1}, ..., a^\alpha_{\gamma_n}\}) \restriction L$. Since $\mathcal{D}' \preceq \mathcal{A}_\alpha$, d realizes p in $\mathcal{A}_\alpha$, a contradiction. $\square$

The model $\mathcal{C} = SH(\{b_i | \ i \in I\}) \restriction L$ from the proof of Corollary 11.8 is called an Ehrenfeucht-Mostowski model and is denoted by $EM(I, \Phi)$. Notice that I and Φ determine $EM(I, \Phi)$ upto isomorphism (and not more). It is also important to notice that although the easiest way to show that $EM(I, \Phi)$ exists (i.e. that the set Φ constructed in the proof of Theorem 11.7 satisfies (ii) from the theorem) is to use compactness, this can also be done without it and thus the construction works in many other context than the one above.

Recall that Remark 9.3 shows that in Corollary 11.8, one can not replace $\beth_{(2^\kappa)^+}$ by κ^+.

11.9 Exercise. *Show that in Corollary 11.8, one can not replace $\beth_{(2^\kappa)^+}$ by any cardinal $< \beth_{\kappa^+}$. (Hint: Look at models in which there are a countable set, codes for subsets of the set, codes for subsets of subsets of the set etc.)*

12. $L_{\kappa\omega}$ and omitting types

12.1 Definition. *$L_{\kappa\omega}$-formulas are defined as follows:*

(i) atomic formulas ϕ are $L_{\kappa\omega}$-formulas and v_i is free in ϕ is it appears in ϕ,

(ii) if ψ is $L_{\kappa\omega}$-formula, then so are $\neg\psi$ and $\exists v_k \psi$ and v_i is free in $\neg\psi$ if it is free in ψ and it is free in $\exists v_k \psi$ if it is free in ψ and $k \neq i$,

(iii) if $|I| < \kappa$, for all $i \in I$, ψ_i is $L_{\kappa\omega}$-formula and there is $n \in \mathbb{N}$ such that for all $i \in I$, if v_k is free in ψ_i, then $k < n$, then $\bigwedge_{i \in I} \psi_i$ is a formula and v_k is free in $\bigwedge_{i \in I} \psi_i$ if it is free in some ψ_i.

An $L_{\kappa\omega}$-formula ϕ is an $L_{\kappa\omega}$-sentence if no v_k is free in ϕ.

Notice that for all $L_{\kappa\omega}$-formulas ϕ, only finitely many v_k are free in ϕ. The notation $\phi(x)$ is used as in the case of first-order logic and also $\mathcal{A} \models \phi(a)$ is defined as in the case of first-order logic ($\mathcal{A} \models (\bigwedge_{i \in I} \psi_i)(a)$ if $\mathcal{A} \models \psi_i(a)$ for all $i \in I$). Symbols $\bigvee$, $\rightarrow$, $\leftrightarrow$ and $\forall$ are used as in the case of first-order logic ($\bigvee_{i \in I} \psi_i = \neg \bigwedge_{i \in I} \neg\psi_i$). Notice that $L_{\omega\omega}$ is still the first-order logic (i.e. the two definitions for $L_{\omega\omega}$ coincide) and we say that ϕ is $L_{\infty\omega}$-formula if it is $L_{\kappa\omega}$-formula for some κ.

12.2 Definition. *Let κ be a cardinal or ∞. We say that $\mathcal{A}$ and $\mathcal{B}$ are $L_{\kappa\omega}$ equivalent ($\mathcal{A} \equiv_{\kappa\omega} \mathcal{B}$) if for all $L_{\kappa\omega}$-sentences ϕ, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.*

12.3 Theorem. *The following are equivalent:*
(i) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$,
(ii) $II \uparrow EF_\omega(\mathcal{A},\mathcal{B})$.

Proof. (ii)$\Rightarrow$(i): Exactly as in the first-order case (just forget the quantifier ranks).

(i)$\Rightarrow$(ii): Clearly it is enough to prove the following claim:

1 Claim. *If* $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$, *then for all* $a \in \mathcal{A}$, *there is* $b \in \mathcal{B}$ *such that* $(\mathcal{A}, a) \equiv_{\infty\omega} (\mathcal{B}, b)$.

Proof. Suppose not. Then for all $b \in \mathcal{B}$ there is an $L_{\infty\omega}$-formula $\phi_b(v_1)$ such that $\mathcal{A} \models \phi_b(a)$ but $\mathcal{B} \not\models \phi_b(b)$. Then $\mathcal{A} \models \exists v_1 \bigwedge_{b\in\mathcal{B}} \phi_b(v_1)$ but $\mathcal{B} \not\models \exists v_1 \bigwedge_{b\in\mathcal{B}} \phi_b(v_1)$, a contradiction. $\square$ Claim 1.
$\square$

12.4 Definition. $L_{\kappa\omega}$*-formulas in negation normal form are defined as follows:* $L_{\kappa\omega}$*-formula* ϕ *is in negation normal form if it is atomic or negated atomic formula, or of the form* $\exists v_i \psi$ *or* $\forall v_i \psi$, *where* ψ *is in negation normal form or of the form* $\bigwedge_{i\in I} \psi_i$ *or* $\bigvee_{i\in I} \psi_i$, *where each* ψ_i *is in negation normal form.*

12.5 Lemma. *For all* $L_{\kappa\omega}$*-formulas* $\phi(x)$, $x = (x_1, ..., x_n)$, *there is an* $L_{\kappa\omega}$*-formula* $\psi(x)$ *in negation normal form such that for all* $\mathcal{A}$ *and* $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ *iff* $\mathcal{A} \models \psi(a)$.

Proof. Clearly it is enough to prove the following claim (exercise):

1 Claim. *For all* $L_{\kappa\omega}$*-formulas* $\phi(x)$ *in negation normal form,* $x = (x_1, ..., x_n)$, *there is an* $L_{\kappa\omega}$*-formula* $\psi(x)$ *in negation normal form such that for all* $\mathcal{A}$ *and* $a \in \mathcal{A}^n$, $\mathcal{A} \models \neg\phi(a)$ *iff* $\mathcal{A} \models \psi(a)$.

Proof. Easy induction on ϕ. E.g. if $\phi = \bigwedge_{i\in I} \theta_i$, then by the induction assumption there are $L_{\kappa\omega}$-formulas θ'_i in negation normal form such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \neg\theta_i(a)$ iff $\mathcal{A} \models \theta'_i(a)$ and we can choose $\psi = \bigvee_{i\in I} \theta'_i$. $\square$ Claim 1.
$\square$

12.6 Lemma. *Suppose* T *is a theory and* D *is a collection of types. Let* κ *be such that* $|L_{\omega\omega}|, |D| < \kappa$. *Then there is an* $L_{\kappa\omega}$*-sentence* ϕ *such that for all* $\mathcal{A}$, $\mathcal{A} \models \phi$ *iff* $\mathcal{A} \models T$ *and* $\mathcal{A}$ *omits every* $p \in D$.

Proof. Exercise. $\square$

In the following definition, we assume that v_0 does not appear in ϕ and when we write $\phi(x)$, $x = (x_1, ..., x_n)$, we assume that x is chosen so that $v_i \in \{x_1, ..., x_n\}$ iff v_i is free in ϕ. And in item (iv) our notation is even more sloppy than usually.

12.7 Definition. *Suppose* ϕ *is an* $L_{\kappa\omega}$*-sentence in negation normal form.*
(i) A fragment F_ϕ *of* ϕ *is defined as follows:*
(a) if ϕ *is atomic or negated atomic formula, then* $F_\phi = \{\phi\}$,
(b) if $\phi = \bigwedge_{i\in I} \psi_i$ *or* $\phi = \bigvee_{i\in I} \psi_i$, *then* $F_\phi = \{\phi\} \cup \bigcup_{i\in I} F_{\psi_i}$,

(c) if $\phi = \exists v_k \psi$ or $\phi = \forall v_k \psi$, then $F_\phi = \{\phi\} \cup F_\psi$.

(ii) $L^\phi = L \cup \{R_{\psi(x_1,...,x_n)} |\ \psi \in F_\phi\}$, where R_ψ are new $n+1$-ary relation symbols.

(iii) T^ϕ consists of the following formulas:

(a) if $\psi(x_1,...,x_n) \in F_\phi$ is atomic or negated atomic formula then $\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0,x_1,...,x_n) \leftrightarrow \psi(x_1,...,x_n)) \in T^\phi$,

(b) if $\psi(x_1,...,x_n) = \bigwedge_{i\in I} \psi_i \in F_\phi$, then for all $i \in I$, $\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0,x_1...x_n) \rightarrow R_{\psi_i}(v_0,x_1,...,x_n)) \in T^\phi$,

(c) if $\psi(x_1,...,x_n) = \bigvee_{i\in I} \psi_i \in F_\phi$, then for all $i \in I$, $\forall v_0 \forall x_1 ... \forall x_n (R_{\psi_i}(v_0,x_1...x_n) \rightarrow R_\psi(v_0,x_1,...,x_n)) \in T^\phi$,

(d) if $\psi(x_1,...,x_n) = \exists x \theta(x,x_1,...,x_n) \in F_\phi$, then $\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0,x_1,...,x_n) \leftrightarrow \exists x R_\theta(v_0,x,x_1,...,x_n)) \in T^\phi$,

(e) if $\psi(x_1,...,x_n) = \forall x \theta(x,x_1,...,x_n) \in F_\phi$, then $\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0,x_1,...,x_n) \leftrightarrow \forall x R_\theta(v_0,x,x_1,...,x_n)) \in T^\phi$,

(f) $\forall v_0 R_\phi(v_0)$.

(iv) D^ϕ consists of the following types:

(a) if $\psi(x_1,...,x_n) = \bigwedge_{i\in I} \psi_i \in F_\phi$, then $\{\neg R_\psi(v_0,v_1...v_n)\} \cup \{R_{\psi_i}(v_0,v_1,...,v_n) |\ i \in I\} \in D^\phi$,

(b) if $\psi(x_1,...,x_n) = \bigvee_{i\in I} \psi_i \in F_\phi$, then $\{R_\psi(v_0,v_1...v_n)\} \cup \{\neg R_{\psi_i}(v_0,v_1,...,v_n) |\ i \in I\} \in D^\phi$.

12.8 Lemma. *Suppose ϕ is an $L_{\kappa\omega}$-sentence in negation normal form.*

(i) For all L-structures $\mathcal{A}$, if $\mathcal{A} \models \phi$, then there is an L^ϕ-structure $\mathcal{B} \models T^\phi$ so that $\mathcal{B}$ omits every $p \in D^\phi$ and $\mathcal{B} \restriction L = \mathcal{A}$. Furthermore such $\mathcal{B}$ is unique.

(ii) If $\mathcal{A} \models T^\phi$ is an L^ϕ-structure and $\mathcal{A}$ omits every $p \in D^\phi$, then $\mathcal{A} \restriction L \models \phi$.

Proof. Just check the definitions (exercise). □

12.9 Theorem. *Suppose ϕ is an $L_{\kappa^+\omega}$-sentence such that for all $\lambda < \beth_{(2^\kappa)^+}$, there is $\mathcal{A} \models \phi$ of power $\geq \lambda$. Then for all $\lambda \geq \kappa$, there is $\mathcal{A} \models \phi$ of power λ.*

Proof. Clearly in ϕ at most κ symbols from the vocabulary can appear and thus we may assume that $|L_{\omega\omega}| \leq \kappa$. But then the claim is immediate by Lemmas 12.5 and 12.8 and Corollary 11.8. □

12.10 Remark. *Suppose F is a collection of $L_{\kappa^+\omega}$-formulas of power $\leq \kappa$ and $A \subseteq \mathcal{A}$. Then there is a substructure $\mathcal{B}$ of $\mathcal{A}$ of power $|A| + \kappa$ such that $A \subseteq \mathcal{B}$ and for all $\phi(x_1,...,x_n) \in F$ and $a \in \mathcal{B}^n$, $\mathcal{B} \models \phi(a)$ iff $\mathcal{A} \models \phi(a)$.*

Proof. Clearly we may assume that if $\wedge_{i\in I} \psi_i \in F$, then $\psi_i \in F$ for all $i \in I$ and that if $\neg\psi \in F$ or $\exists v_k \psi \in F$, then $\psi \in F$. Then we can proceed as in the first-order case (exercise). □

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