GEOMETRIES AND MODELS
- an Introduction

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Abstract

These lecture notes give a short introduction to ideas shared by classical theory of geometries and modern model theory. This course is not on stability theory and thus all the non-trivial results from stability theory are replaced by stronger assumptions, except three: Shelah's finite equivalence relation theorem, one consequence of splitting and the fact that ∞-definable groups in models of ω-stable theories are definable. These three are considered as facts i.e. the proofs are skipped. Also, we try to keep all the objects studied as concrete as possible and so although our main theorem (Conclusion 11.3) is a variant of a special case of a result from [Hr], the proof follows [HLS] and [Hy] most of the time. Additional benefit of this is that most of the proofs work also in many non-elementary cases. For the history of this topic, see [Pi] and [Po].

Contents

PART I: Geometries

In this part we look at various geometries, their properties and how to find such objects.

1 Basic definitions from model theory

In this section we recall the basic definitions of the first-order logic and some notions from model theory that are needed already when we study affine and projective planes.

1.1 Definition. *A vocabulary L is a collection of relation, function and constant symbols. Each relation symbol R and function symbol f come with the arity $\#R, \#f \in \mathbb{N} - \{0\}$.*

We let $L = \{R_i, f_j, c_k|\ i \in I^*,\ j \in J^*,\ k \in K^*\}$ be a fixed but arbitrary vocabulary (i.e. when we talk about arbitrary models, this is the vocabulary).

1.2 Definition. *The collection of (L-)terms is defined as follows:*
(i) variables v_i, $i \in \mathbb{N}$, are terms,
(ii) constant symbols c_k, $k \in K^*$, are terms,
(iii) if $n = \#f_j$, $j \in J^*$, and $t_1, ..., t_n$ are terms, then $f_j(t_1, ..., t_n)$ is a term.

1.3 Definition. *The collection of atomic (L-)formulas is defined as follows:*
(i) if t and u are terms, then $t = u$ is an atomic formula,
(ii) if $n = \#R_i$, $i \in I^*$, and $t_1, ..., t_n$ are terms, then $R_i(t_1, ..., t_n)$ is an atomic formula.

1.4 Definition. *The collection of (L-)formulas is defined as follows:*
(i) atomic formulas are formulas,
(ii) if ϕ is a formula, then $\neg\phi$ is a formula,
(iii) if ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula,
(iv) if ϕ is a formula and $i \in \mathbb{N}$, then $\exists v_i \phi$ is a formula.
By $L_{\omega\omega}$ we denote the set of all L-formulas. Notice that the cardinality of $L_{\omega\omega}$ is $max\{\omega, the\ cardinality\ of\ L\}$.

The following notation is used:

$$\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$$

$$\phi \rightarrow \psi = \neg\phi \vee \psi$$

$$\phi \leftrightarrow \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$$

$$\forall v_i \phi = \neg\exists v_i \neg\phi.$$

1.5 Definition. *The notion v_i is free in ϕ is defined as follows:*
(i) ϕ is atomic: v_i is free in ϕ if v_i appears in ϕ,
(ii) $\phi = \neg\psi$: v_i is free in ϕ if it is free in ψ,
(iii) $\phi = \psi \wedge \theta$: v_i is free in ϕ if it is free in ψ or θ,
(iv) $\phi = \exists v_j \psi$: v_i is free in ϕ if it is free in ψ and $i \neq j$.

A sentence is a formula in which no v_i is free.

If $x = (x_1, ..., x_n)$ is a sequence of variables (when we write like this we assume that for $k \neq m$, $x_k \neq x_m$), then the notation $\phi(x)$ means that if v_i is free in ϕ then $v_i \in \{x_1, ..., x_n\}$. Similarly for a term t, $t(x)$ means that if v_i appears in t, then $v_i \in \{x_1, ..., x_n\}$. Often we split x into two (or more) sequences y and z and write $\phi(y, z)$ in place of $\phi(x)$.

1.6 Definition. *A (L-)structure (i.e. model) is a sequence*

$$\mathcal{A} = (\mathcal{A}, R_i^{\mathcal{A}}, f_j^{\mathcal{A}}, c_k^{\mathcal{A}})_{i \in I^*, j \in J^*, k \in K^*}$$

where

(i) $\mathcal{A}$ is a non-empty set (the universe of $\mathcal{A}$, when we want to make a distinction between the model and its universe, we write $dom(\mathcal{A})$ for the universe),

(ii) $R^{\mathcal{A}_i} \subseteq \mathcal{A}^{\#R_i}$,

(iii) $f_j^{\mathcal{A}} : \mathcal{A}^{\#f_j} \to \mathcal{A}$,

(iv) $c_k^{\mathcal{A}} \in \mathcal{A}$.

When it does not risk confusion we write just $R_i = R_i^{\mathcal{A}}$ etc.

1.7 Definition. *For a term $t(x)$, $x = (x_1, ..., x_n)$, structure $\mathcal{A}$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, $t^{\mathcal{A}}(a)$ is defined as follows:*

(i) $t = v_i$: $t^{\mathcal{A}}(a) = a_m$, where m is such that $v_i = x_m$,

(ii) $t = c_k$: $t^{\mathcal{A}}(a) = c_k^{\mathcal{A}}$,

(iii) $t = f_j(t_1, ..., t_m)$: $t^{\mathcal{A}}(a) = f_j^{\mathcal{A}}(t_1^{\mathcal{A}}(a), ..., t_m^{\mathcal{A}}(a))$.

1.8 Definition. *For a formula $\phi(x)$, $x = (x_1, ..., x_n)$, structure $\mathcal{A}$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ is defined as follows:*

(i) $\phi = t = u$: $\mathcal{A} \models \phi(a)$ if $t^{\mathcal{A}}(a) = u^{\mathcal{A}}(a)$,

(ii) $\phi = R_i(t_1, ..., t_m)$: $\mathcal{A} \models \phi(a)$ if $(t_1^{\mathcal{A}}(a), ..., t_m^{\mathcal{A}}(a)) \in R_i^{\mathcal{A}}$,

(iii) $\phi = \neg\psi$: $\mathcal{A} \models \phi(a)$ if $\mathcal{A} \not\models \psi(a)$,

(iv) $\phi = \psi \wedge \theta$: $\mathcal{A} \models \phi(a)$ if $\mathcal{A} \models \psi(a)$ and $\mathcal{A} \models \theta(a)$,

(v) $\phi = \exists v_i \psi$: $\mathcal{A} \models \phi(a)$ if there is $b \in \mathcal{A}$ such that $\mathcal{A} \models \psi(b, a_1, ..., a_n)$ for $\psi = \psi(v_i, x_1, ..., x_n)$.

1.9 Remark. *In the Definition 1.8 (v) we assumed that $v_i \notin \{x_1, ..., x_n\}$. This can be done without loss of generality, see the course Matemaattinen logiikka. This kind sloppy notation will be used regularly in these notes.*

1.10 Fact. *For all $\phi(x)$, $x = (x_1, ..., x_n)$, and $y = (y_1, ..., y_n)$, there is $\psi(y)$ such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{A} \models \psi(a)$.*

Proof. See the course Matemaattinen logiikka. □

1.11 Definition. *For a structure $\mathcal{A}$, a relation $R \subseteq \mathcal{A}^n$ is definable over $A \subseteq \mathcal{A}$, if there are a formula $\phi(x, y)$, $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$, and $b \in A^m$ such that for all $a \in \mathcal{A}^n$, $a \in R$ iff $\mathcal{A} \models \phi(a, b)$. The elements of b are called the parameters of the definition. We say that R is definable if it is definable over $\mathcal{A}$. If parameters are not needed i.e. it is definable over $\emptyset$, we say that R is definable without parameters. A (possibly partial) function $f : \mathcal{A}^n \to \mathcal{A}$ is definable if its graph i.e. the relation $\{(a_1, ..., a_{n+1}) \in \mathcal{A}^{n+1} | \, f(a_1, ..., a_n) = a_{n+1}\}$ is definable. And an element $a \in \mathcal{A}$ is definable if $\{a\}$ is definable.*

1.12 Exercise. *Show that the set of integers is definable without parameters in* $(\mathbf{C}, +, \times, exp, 0, 1)$, *where* $+$ *and* $\times$ *are the addition and multiplication of complex numbers and* $exp(x) = e^x$. *Conclude that* $\{\lceil\phi\rceil | \ (\mathbb{C}, +, \times, exp, 0, 1) \models \phi\}$ *is not recursive, where* $\lceil\phi\rceil$ *is the Gödel number of* ϕ. *Hint: See Exercise 1.15 and use the fact that the answer to Hilbert's tenth problem is: there is no such algorithm.*

1.13 Exercise. *Suppose* $L^* \supseteq L$ *is a vocabulary,* $\mathcal{A}$ *is an* L^**-structure and for all* $R, f, c \in L^*$, $R^{\mathcal{A}}$, $f^{\mathcal{A}}$ *and* $c^{\mathcal{A}}$ *are definable in* $\mathcal{A} \restriction L$. *Show that if* $X \subseteq \mathcal{A}^n$ *is definable in* $\mathcal{A}$ *then it is definable already in* $\mathcal{A} \restriction L$.

If $g : A \to B$ and $f : B^n \to C$ then $f \circ g$ is a function from A^n to C such that for all $a = (a_1, ..., a_n) \in A^n$, $(f \circ g)(a) = f(g(a_1), ..., g(a_n))$. Also if $g : A \to B^n$ and $f : B \to C$, then $f \circ g$ is a function from A to C^n such that for all $a \in A$, $(f \circ g)(a) = (f(b_1), ..., f(b_n))$, where $(b_1, ..., b_n) = g(a)$.

1.14 Definition. *Let* L^* *be a vocabulary and* $\mathcal{B}$ *an* L^**-structure. We say that* $\mathcal{B}$ *is interpretable in* $\mathcal{A}$ *over* $A \subseteq \mathcal{A}$ *if there are a natural number* n *and a one-to-one function* $F : \mathcal{B} \to \mathcal{A}^n$ *such that* $F(\mathcal{B})$, $F(R^{\mathcal{B}})$, $F \circ (f^{\mathcal{B}} \circ F^{-1})$ *and* $F(c^{\mathcal{B}})$ *are definable over* A *for all* $R, f, c \in L^*$. *If* $A = \mathcal{A}$, *we say just interpretable in* $\mathcal{A}$ *and if* $A = \emptyset$, *we say that* $\mathcal{B}$ *is interpretable in* $\mathcal{A}$ *without parameters.*

Notice that $F \circ f^{\mathcal{B}} \circ F^{-1}$ being definable means that the set

$$\{(F(x_1), ..., F(x_{m+1})) | \ f^{\mathcal{B}}(x_1, ..., x_m) = x_{m+1}\}$$

is definable (the graph of $F \circ f^{\mathcal{B}} \circ F^{-1}$ is this set). And so $\mathcal{B}$ is interpretable in $\mathcal{A}$ if 'an isomorphic copy of $\mathcal{B}$ is definable in $\mathcal{A}$'. In fact we say that $\mathcal{B}$ is definable in $\mathcal{A}$ if the identity function id witnesses that $\mathcal{B}$ is interpretable in $\mathcal{A}$.

1.15 Exercise. *Suppose that* $F : \mathcal{B} \to \mathcal{A}^n$ *is a one-to-one function which witnesses that* $\mathcal{B}$ *is interpretable in* $\mathcal{A}$ *over* $A \subseteq \mathcal{A}$.

(i) Show that if G *is an automorphism of* $\mathcal{A}$, *then* $G \circ F$ *is a one-to-one function which also witnesses that* $\mathcal{B}$ *is interpretable in* $\mathcal{A}$. *Furthermore, if (e.g.)*

$$\phi(x_1, ..., x_n, a_1, ..., a_m)$$

defines $F(R^{\mathcal{B}})$, *then*

$$\phi(x_1, ..., x_n, G(a_1), ..., G(a_m))$$

defines $(G \circ F)(R^{\mathcal{B}})$. *Conclude that if* $G \restriction A = id$, *then* $F^{-1} \circ G \circ F$ *is an automorphism of* $\mathcal{B}$.

(ii) Show that for all L^**-formulas* $\phi(x)$, $x = (x_1, .., x_m)$, *there is an* L*-formula* $\phi^*(y^1, ..., y^m)$, $y^i = (y^i_1, ..., y^i_n)$, *such that for all* $a = (a_1, ..., a_m) \in \mathcal{B}^m$, $\mathcal{B} \models \phi(a)$ *iff* $\mathcal{A} \models \phi^*(F(a_1), ..., F(a_m))$. *Hint: See Lemma 7.4 in the lecture notes of the course Model theory.*

We finish this section by fixing an L-structure M (for those familiar with stability theory, one can choose M to be the monster model).

We say that p is a type over $A \subseteq M$ if for some finite sequence x of variables, it is a collection of formulas of the form $\phi(x, a)$, $a \in A^n$ and $n \in \mathbb{N}$. If m is the length of the sequence x, then we say also that p is an m-type. We say that p is realized in M if some sequence b of elements of M realizes it i.e. $M \models \phi(b, a)$ for all $\phi(x, a) \in p$. We say that p complete (over A) if every finite subtype (i.e. subset) of p is realized in M and for all $\phi(x, a)$, $a \in A^n$ for some $n \in \mathbb{N}$, either $\phi(x, a) \in p$ or $\neg\phi(x, a) \in p$. For a finite sequence b of elements of M, the type of b over A (in M), $t(b/A)$, is the unique complete type p over A realized by b (i.e. $t(b/A) = \{\phi(x, a) |\ M \models \phi(b, a),\ a \in M^n,\ n \in \mathbb{N}\}$).

1.16 Assumptions. *Outside definitions, when we talk about L-structure M, we mean an arbitrary but fixed L-structure such that the following holds:*

(i) The cardinality of M is $> |L| + \omega$.

(ii) M is saturated: For all $A \subseteq M$, if $|A| < |M|$, p is a type over A and every finite subtype of p is realized in M, then p is realized in M.

(iii) M is stable i.e. for all formulas $\phi(x, y, b)$, $b \in M^m$, $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, there is $N \in \mathbb{N}$ such that there are no sequences $a_i \in M^n$, $i < N$, such that $M \models \phi(a_i, a_j, b)$ iff $i < j$.

(iv) From Section 7 on, we will assume that L is countable.

(v) From Section 8 on, we will assume that M is ω-stable i.e. for all elementary extensions (see below) M' of M and countable $A \subseteq M'$, the set $\{t(a/A) |\ a \in M'\}$ is countable. Under the assumptions (i) and (ii), this is equivalent to: for all countable $A \subseteq M$, the set $\{t(a/A) |\ a \in M\}$ is countable (exercise).

1.17 Fact. *ω-stability implies stability.*

In our main theorems, assumptions (i) and (ii) are essentially without loss of generality (we will return to this later). On the way to prove the main main results, (i) and (ii) are vital assumption.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures, $A \subseteq \mathcal{A}$ and $f : A \to \mathcal{B}$. Then f is elementary if for all formulas $\phi(x)$, $x = (x_1, ..., x_n)$ and $a \in A^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$ (where $f(a) = (f(a_1, ..., f(a_n)))$. We say that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}$, if the identity function is an elementary function from $\mathcal{A}$ to $\mathcal{B}$

1.18 Exercise.

(i) M is strongly homogeneous i.e. if $f : A \to M$, $A \subseteq M$ and $|A| < |M|$, is elementary, then there is an automorphism g of $\mathbf{M}$ such that $f \subseteq g$.

(ii) If $\mathcal{A}$ is interpretable in M, then $\mathcal{A}$ is stable.

(iii) If $\mathcal{A}$ is interpretable in M, then $\mathcal{A}$ is saturated.

(iv) Show that if M is ω-stable, then for all countable $A \subseteq M$ and $n \in \mathbb{N}$, the set

$$\{t(a, A) |\ a \in M^n\}$$

is countable.

2 Affine planes

We call a structure of the form $\mathcal{A} = (A, P, L, I)$ a quasigeometry if $A = P \cup L$, $P \cap L = \emptyset$ and $I \subseteq P \times L$. Then the elements of P are called points and the elements of L are called lines. I is an incidence relation. However, we will make no difference between $l \in L$ and the set $\{p \in P | \ (p, l) \in I\}$ and thus we write e.g. $p \in l$ instead of $(p, l) \in I$ and $l \cap l' = \emptyset$ instead of saying that for no $p \in P$, both $(p, l) \in I$ and $(p, l') \in I$.

2.1 Definition. *A quasigeometry $\mathcal{A} = (A, P, L, I)$ is an affine plane if the following holds:*

(i) there are four points such that no three of them are collinear i.e. not contained in a one line,

(ii) for any two points there is a line containing them both,

(iii) any two distinct lines contain at most one common point,

(iv) for any point p and line l there is a unique line l' such that $p \in l'$ and l' and l are parallel (i.e. $l' \cap l = \emptyset$ or $l' = l$).

2.2 Exercise. *Suppose $\mathcal{A} = (A, P, L, I)$ is an affine plane.*

(i) Show that being parallel is an equivalence relation on the set of lines.

(ii) For a point $p \in P$, let $L(p)$ be the number of lines that contain p and for a line $l \in L$, let $P(l)$ be the number of points on l. Show that $L(p) = P(l) + 1$. Hint: Prove this first under the assumption that $p \notin l$.

In these notes, multiplication in a field $F = (F, +, \times, 0, 1)$ is commutative (we have 0 and 1 in the language of fields because they are needed in the elimination of quantifiers). If this requirement is dropped (but we still require that both $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$), the resulting object is called a division ring (a.k.a. skew field a.k.a. sfield a.k.a. field). Most of the time we work with fields although often the results hold also for division rings and even without any changes in the proof. Also we write F^+ for $(F, +)$, the additive group of F, and $F^\times$ for $(F - \{0\}, \times)$, the multiplicative group of F.

Sometimes we think vector spaces over a field F as structures in the sense of Section 1 and then the vocabulary is $\{+\} \cup \{f_a | \ a \in F\}$ and e.g. n-dimensional vector space $V_n(F)$ over a field F is $(F^n, +, f_a)_{a \in F}$, where $(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$ and $f_a(x_1, ..., x_n) = (ax_1, ..., ax_n)$. However, still, we will write just ax for $f_a(x)$.

2.3 Example. *Let F be a field and $V_2(F)$ a two dimensional vector space over F. Then we define a quasigeometry $\mathcal{A}_2(F) = (A, P, L, I)$ so that*

(i) $P = V_2(F)$ (i.e. the universe of $V_2(F)$ i.e. F^2),

(ii) L is the set of all lines $l_{abc} = \{(x, y) \in V_2(F) | \ ax + by + c = 0\}$, where $a, b, c \in F$ and either $a \neq 0$ or $b \neq 0$ (or both),

(iii) $(p, l) \in I$ if $p \in I$.

Then $\mathcal{A}_2(F)$ is an affine plane (check).

2.4 Theorem. *Let $\mathbb{R}$ be the field of real numbers. Then $\mathbb{R}$ is interpretable in $\mathcal{A}_2(\mathbb{R})$.*

Proof. Let k be a line in $A_2(\mathbb{R})$ and 0 and 1 be two distinct points of this line. Let d be the distance between the two points (in $V_2(\mathbb{R})$ with the usual metric). We define $F : \mathbb{R} \to k$ so that for all $x \in \mathbb{R}$, $F(x)$ is the unique point of k such that the distance between 0 and $F(x)$ is $d|x|$ and the distance between 1 and $F(x)$ is $d|x-1|$. Then it is enough to show that the sets

$$Y = \{(F(x), F(y), F(z)) |\ x + y = z\}$$

and

$$K = \{(F(x), F(y), F(z)) |\ xy = z\}$$

are definable in $A_2(\mathbb{R})$.

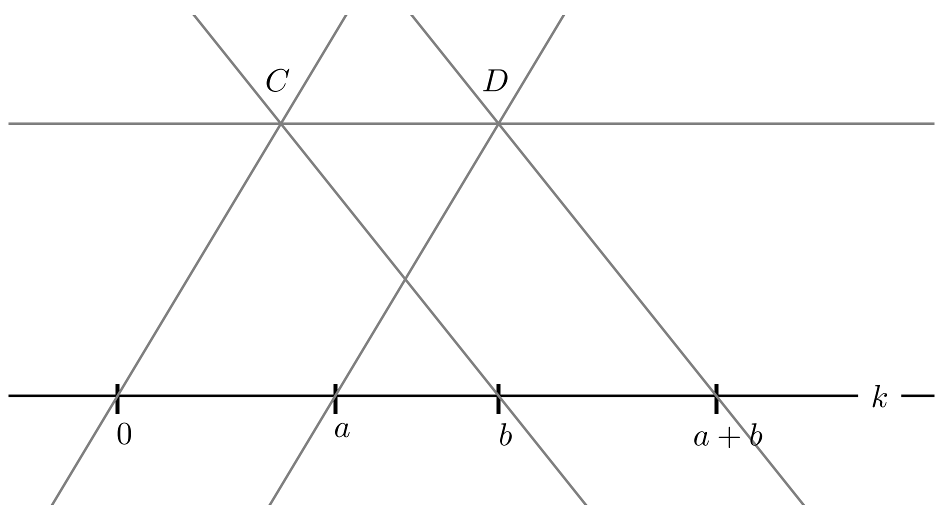
Y is defined by the formula $\phi(x, y, z, k, 0)$, where

$$\phi(x, y, z, z_1, z_2) = \exists x_1 \exists x_2 \exists y_1 \exists y_2 \exists y_3 \exists y_4 \exists y_5 ((y_1 \cap z_1 = \emptyset) \wedge (z_2 \in y_2)$$

$$\wedge (x_1 \in y_1 \cap y_2 \cap y_3) \wedge (y \in z_1 \cap y_3) \wedge (x \in z_1 \cap y_4) \wedge (x_2 \in y_1 \cap y_4 \cap y_5)$$

$$\wedge (y_2 \cap y_4 = \emptyset) \wedge (y_3 \cap y_5 = \emptyset) \wedge z \in z_1 \cap y_5))$$

(where e.g. $y_1 \cap z_1 = \emptyset$ is a shorthand for $\neg \exists x_3 (I(x_3, y_1) \wedge I(x_3, z_1))$. The fact that $\phi(x, y, z, k, 0)$ defines Y can be seen from the picture below (notice that the line segments $b(a+b)$, CD and $0a$ have the same length).



K is defined by $\psi(x, y, z, k, 0, 1)$, where

$$\psi(x, y, z, z_1, z_2, z_3) = x \in z_1 \wedge y \in z_1 \wedge z \in z_1 \wedge$$

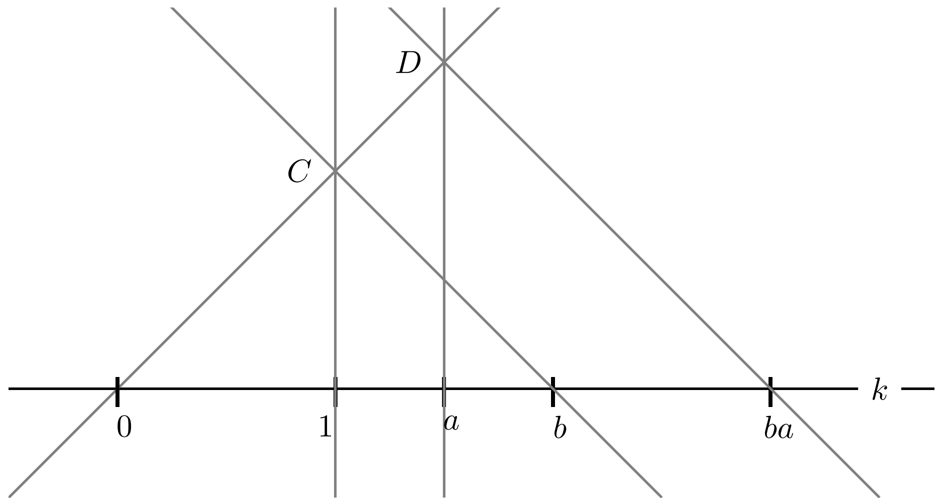
$$[((x = 0 \vee y = 0) \to z = 0) \wedge ((x \neq 0 \wedge y \neq 0) \to \psi'(x, y, z, z_1, z_2, z_3))]$$

and

$$\psi'(x, y, z, z_1, z_2, z_3) = \exists x_1 \exists x_2 \exists y_1 \exists y_2 \exists y_3 \exists y_4 \exists y_5 ((z_2 \in y_1) \wedge (x_1 \in y_1 \cap y_2 \cap y_4)$$

$$\wedge(x_1 \notin z_1) \wedge (z_3 \in y_2) \wedge (x \in y_3) \wedge (y \in y_4) \wedge (z \in y_5) \wedge (x_2 \in y_1 \cap y_3 \cap y_5)$$
$$\wedge(y_2 \cap y_3 = \emptyset) \wedge (y_4 \cap y_5 = \emptyset)).$$

The fact that $\psi(x, y, z, k, 0)$ defines K can be seen from the picture below (notice that $|0(ba)|/|0b| = |0D|/|0C| = |0a|/|01|$, where e.g $|0a|$ denotes the length of the line segment $0a$).



□

From the work we do in Section 4, it follows that in $A_2(F)$ for any field F, the formulas $\phi(x, y, z, k, 0)$ and $\psi(x, y, z, k, 0, 1)$ define addition and multiplication to k so that one gets a field that is isomorphic with F, see the discussion after Exercise 4.12. So

2.5 Fact. *For every field F, F is interpretable in $A_2(F)$.*

Alternatively one can directly show that the formulas $\phi(x, y, z, k, 0)$ and $\psi(x, y, z, k, 0, 1)$ define addition and multiplication to k so that one gets some field F' and then argue further that the field F' must be isomorphic with F. In fact more is true, see Fact 4.13.

2.6 Exercise. *(A. Kuusisto, J. Meyers and J. Virtema) Show that the monadic Π^1_1-theory of $A_2(\mathbb{R})$ is Π^1_1-hard i.e. there is a recursive $f : \mathbb{N} \to \mathbb{N}$ such that for all Π^1_1-sentences ϕ in the language $\{+, \times, 0, 1\}$, $f(\ulcorner\phi\urcorner)$ is the Gödel number of a monadic Π^1_1-sentences ψ in the language $\{P, L, I\}$ such that $(\mathbb{N}, +, \times, 0, 1) \models \phi$ iff $A_2(\mathbb{R}) \models \psi$. (For Π^1_1-sentences, see the literature.)*

3 On group actions and automorphism groups

Let X be a set. By $Sym(X)$ we denote the group of all permutations of X (i.e. bijections from X to X with composition as the group operation).

3.1 Definition. *Let $M = (M, \times, 1)$ be a monoid. A homomorphism $\pi : M \to Sym(X)$ is called an action of M on X. For all $g \in M$ and $x \in X$, we write gx for $(\pi(g))(x)$ (i.e. the group acts on left). For all $x \in X$, $O(x) = \{gx|\ g \in M\}$ is called the orbit of x and $M_x = \{g \in \mathbf{M}|\ gx = x\}$ is the stabilizer of x. Similarly, for $Y \subseteq X$ we write M_Y for $\{g \in \mathbf{M}|\ gx = x \text{ for all } x \in Y\}$.*

In these notes almost always when we study an action it will be an action of a group. And for an action $\pi : G \to Sym(X)$ of a group G on X the following observations hold:

(i) The orbits form a partition of X (i.e. for all $x, y \in X$, $O(x) = O(y)$ or $O(x) \cap O(y) = \emptyset$) and thus $x \in O(y)$ is an equivalence relation.

(ii) G_x is a subgroup of G.

(iii) The kernel $K = Ker(\pi)$ of the homomorphism π (this will be called the kernel of the action) is a normal subgroup of G and G/K acts on X via $\pi'(gK) = \pi(g)$. Then π' is an isomorphism from G/K to the subgroup $\pi(G)$ of $Sym(X)$.

3.2 Exercise.

(i) Prove the three observations above.

(ii) Show that every group is isomorphic to a subgroup of $Sym(X)$ for some X.

(iii) Suppose $f, g, h \in Sym(X)$. Show that

(a) f is an isomorphism from (X, g) onto (X, h) iff $fgf^{-1} = h$.

(b) f is an automorphism of (X, g) iff $fg = gf$.

(c) f is an automorphism of (X, g) iff g is an automorphism of (X, f).

3.3 Examples.

(i) Let G be a group. Then

$$g \mapsto (x \mapsto gx)$$

and

$$g \mapsto (x \mapsto gxg^{-1})$$

are actions of G on G (gxg^{-1} is often denoted by x^g). If G is a subgroup of $Sym(X)$ then identity is an action of G on X. In particular, if $\mathcal{A}$ is a structure and G is a subgroup of $Aut(\mathcal{A})$, then identity is an action of G on $\mathcal{A}$, where $Aut(\mathcal{A})$ is the automorphism group of $\mathcal{A}$. (With $Aut(\mathcal{A})$ and $A \subseteq \mathcal{A}$ it is common to write $Aut(\mathcal{A}/A)$ instead of $Aut(\mathcal{A})_A$.)

(ii) Let F be a field. By $GL_n(F)$ we denote the group of invertible $n \times n$ matrices (general linear group). Then

$$(*) \qquad A \mapsto (x \mapsto Ax)$$

is an action of $GL_n(F)$ on $V_n(F)$, where $((a_{ij})_{i,j<n})((x_i)_{i<n}) = (y_i)_{i<n}$ if $y_i = \Sigma_{k=0}^{n-1} a_{ik}x_k$. In fact, (*) above determines an action of the monoid of all $n \times n$ matrices on $V_n(F)$. $GL_n(F)$ acts also on $A_n(F)$: on points the action is as in (*) above and lines l are mapped to $Al = \{Ax |\ x \in l\}$.

When we wrote $((a_{ij})_{i,j<n})((x_i)_{i<n})$, we thought $(x_i)_{i<n}$ as a column vector but for the obvious reason we write also e.g. $((a_{ij})_{i,j<3})(x, y, z)$.

3.4 Definition. *We say that actions $\pi : G \to Sym(X)$ and $\pi' : G' \to Sym(X')$ are isomorphic if there are an isomorphism $F : G \to G'$ and a bijection $H : X \to X'$ such that for all $x \in X$ and $g \in M$, $F(g)H(x) = H(gx)$. Then we also say that the pair (F, H) is an isomorphism between the two actions. If the actions are the same (F, H) is called an automorphism of the action.*

In the exercises, if two actions $\pi : G \to Sym(X)$ and $\pi' : G' \to Sym(X)$ act on the same set X and it is claimed that they are isomorphic, then often, but not always, the isomorphism (F, H) can be chosen so that $H = id$.

3.5 Exercise.
(i) Show that actions $\pi : G \to Sym(X)$ *and* $\pi' : G' \to Sym(X')$ *are isomorphic if there are one-to-one and onto functions* $F : G \to G'$ *and* $H : X \to X'$ *such that for all* $g \in G$ *and* $x \in X$, $H(gx) = F(g)H(x)$.
(ii) Let us look at the action $g \mapsto (x \mapsto x^g)$ *of* G *on* G *from Example 3.3 (i). Let* K *be the kernel of this action. Show that the action of* G/K *is isomorphic with an action of a subgroup of* $Aut(G)$ *on* G.
(iii) Show that the actions of $GL_n(F)$ *and* $Aut(V_n(F))$ *on* $V_n(F)$ *are isomorphic.*
(iv) Show that the action of $GL_2(F)$ *on* $A_2(F)$ *is isomorphic with an action of a subgroup of* $Aut(A_2(F))$ *on* $A_2(F)$.
(v) Show that $Aut((\mathbb{R}, +, \times, 0, 1))$ *contains just one element.*
(vi) Suppose that x_1, x_2, x_3 *are points in* $A_2(F)$ *which are not collinear (recall: not contained in a one line) and similarly* y_1, y_2, y_3 *are points in* $A_2(F)$, *which are not collinear. Show that there is* $f \in Aut(A_2(F))$ *such that* $f(x_i) = y_i$ *for all* $1 \leq i \leq 3$. *Show that if in addition* $F = \mathbb{R}$, *then there is only one such* f.
(vii) Show that there is a point $x \in A_2(\mathbb{R})$, *such that the actions of* $GL_2(\mathbb{R})$ *and*

$$Aut(A_2(\mathbb{R}))_x$$

are isomorphic on $A_2(\mathbb{R})$.
(viii) Show that for no point $x \in A_2(\mathbb{C})$, *the actions of* $GL_2(\mathbb{C})$ *and*

$$Aut(A_2(\mathbb{C}))_x$$

on $A_2(\mathbb{C})$ *are isomorphic, where* $\mathbb{C}$ *is the field of complex numbers (Exercise 5.21 (ii) may help here).*

3.6 Definition. *Let* $(G, \times)$ *and* $(H, +)$ *be groups and suppose that there is an action* $g \mapsto (h \mapsto h^g)$ *of* G *on* H *so that for all* $g \in G$, $0^g = 0$ *and* $(x + y)^g = (x^g) + (y^g)$ *for all* $x, y \in H$ *(i.e. letting* K *be the kernel of the action, the action of* G/K *on* H *is isomorphic with the action of a subgroup of* $Aut(H)$*). Then we define a group* $G \rtimes H$ *(semidirect product) as follows: The elements of the group are pairs* $(g, h) \in G \times H$ *and* $(g, h)(a, b) = (ga, h + b^g)$.

3.7 Exercise. *Show that* $G \rtimes H$ *is a group and that for* $g \in G$ *and* $h \in H$, $(g, 0)(1, h)(g, 0)^{-1} = (1, h^g)$ *(and so* H *i.e.* $\{(1, h) | \ h \in H\}$ *is a normal subgroup of* $G \rtimes H$*).*

When we think $V_2(\mathbb{R})$ as a group, then it acts on $A_2(\mathbb{R})$ via translations i.e. for points p, $xp = p + x$ and for lines l, $xl = \{p + x | \ p \in l\}$. Also using the action of $GL_2(\mathbb{R})$ on $V_2(\mathbb{R})$ from Example 3.3 (ii), we can form $GL_2(\mathbb{R}) \rtimes V_2(\mathbb{R})$, which acts on $A_2(\mathbb{R})$ as follows: For points p, $(A, x)p = Ap + x$ and for lines l, $(A, p)l = \{Ap + x | \ p \in l\}$.

3.8 Exercise.

(i) Show that the two actions above are indeed actions.

(ii) Show that the actions of $Aut(A_2(\mathbb{R}))$ and $GL_2(\mathbb{R}) \rtimes V_2(\mathbb{R})$ on $A_2(\mathbb{R})$ are isomorphic.

We finish this section with two notions needed later.

3.9 Definition. *Suppose a group G acts on X.*

(i) We say that the action is n-transitive (a.k.a. n-fold transitive) if for all distinct $x_i \in X$, $i < n$, and distinct $y_i \in X$, $i < n$, there is $a \in G$ such that for all $i < n$, $ax_i = y_i$. Action is transitive if it is 1-transitive.

(ii) We say that the action is n-regular if it is n-transitive and for all distinct $x_i \in X$, $i < n$, and $a, b \in G$ the following holds: If for all $i < n$, $ax_i = bx_i$, then $a = b$. The action is regular if it is 1-regular.

4 Projective planes

4.1 Definition. *A quasigeometry $S = (S, P, L, I)$ is a projective plane if the following holds:*

(i) there are four points such that no three of them are collinear,

(ii) for two distinct points, there is exactly one line containing both of them,

(iii) for any two distinct lines there is exactly one point that is contained in both of then.

4.2 Exercise.

(i) Show that every projective plane contains at least 7 points.

(ii) Show that if $S = (S, P, L, I)$ is a projective plane then also

$$S' = (S, P', L', I')$$

is a projective plane when $P' = L$, $L' = P$ and $(l, p) \in I'$ if $(p, l) \in I$.

Suppose $A = (A, P, L, I)$ is an affine plane. We define $\overline{A} = (S, P', L', I')$ as follows: For each $l \in L$, by l_p we denote the equivalence class of l in the equivalence relation 'being parallel' on L. Then $P' = P \cup \{l_p |\ l \in L\}$ $L' = L \cup \{l_\infty\}$, where $l_\infty = \{l_p |\ l \in L\}$, $I' = I \cup \{(l_p, l_\infty), (l_p, l) |\ l \in L\}$ and $S = P' \cup L'$.

4.3 Exercise.

(i) Show that $\overline{A}$ is a projective plane.

(ii) Suppose $S = (S, P, L, I)$ is a projective plane and $l \in L$. Show that $S_l = (S', P', L', I')$ is an affine plane when $P' = P - l$, $L' = L - \{l\}$, $I' = I \cap (P' \times L')$ and $S' = P' \cup L'$.

4.4 Definition. *Let F be a field and $n > 0$. For all $(x_i)_{i \le n} \in V_{n+1}(F) - \{0\}$, let $[x_i]_{i \le n}$ be the equivalence class of $(x)_{i \le n}$ in the equivalence relation $\sim$ on $V_{n+1}(F) - \{0\}$, where $(x_i)_{i \le n} \sim (x'_i)_{i \le n}$ if for some $\lambda \in F^\times$, $\lambda (x_i)_{i \le n} = (x'_i)_{i \le n}$ (i.e. $[x_i]_{i \le n} \cup \{0\}$ is the 1-dimensional subspace of $V_{n+1}(F)$ containing $(x_i)_{i \le n}$).*

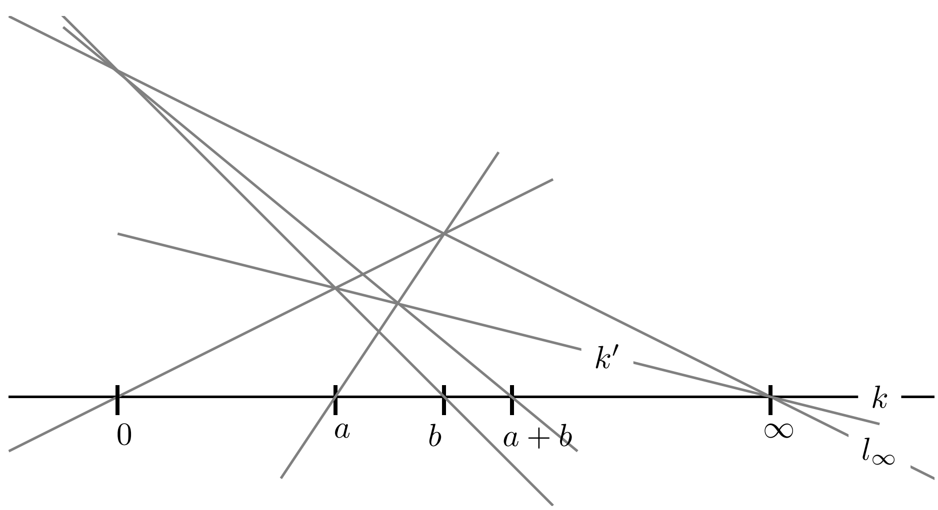
Then $P_n(F) = (S, P, L, I)$ is called a projective space over F, if $P = \{[x_i]_{i \le n} |\ (x_i)_{i \le n} \in V_{n+1}(F) - \{0\}\}$, L is the set of all 2-dimensional subspaces of $V_{n+1}(F)$, $(p, l) \in I$ if $p \subseteq l$ and $S = P \cup L$.

In the literature, projective space $P_n(F)$ often means just the set of points of the projective space $P_n(F)$ as defined above. Similarly affine space often means just the universe of $V_n(F)$. Also e.g. $[x, y, z]$ is usually denoted by $(x : y : z)$ but this notation is most inconvenient for our purposes.

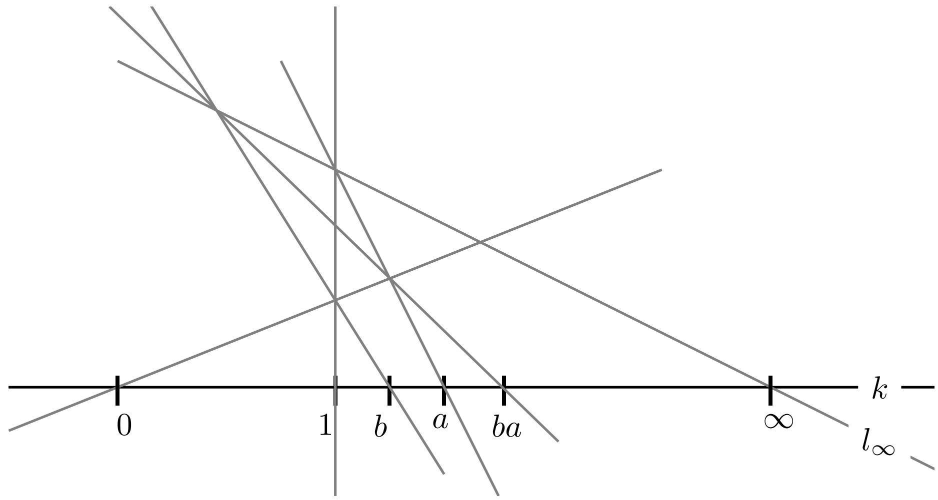
4.5 Exercise. *Show that $\overline{A_2(F)}$ is isomorphic with $P_2(F)$ (in particular, $P_2(F)$ is a projective plane).*

4.6 Theorem. *Let k and l_∞ be two distinct lines in $P_2(\mathbb{R})$ and $0, 1 \in k - l_\infty$ be two distinct points. Denote by ∞ the point in the intersection of k and l_∞. Then with the parameters $k, l_\infty, 0$ and 1 one can define addition and multiplication to $k - \{\infty\}$ so that with these, $k - \{\infty\}$ is isomorphic with F. In particular, $\mathbb{R}$ is interpretable in $P_2(\mathbb{R})$.*

Proof. Just copy the proof of Theorem 2.4. Then the picture from the proof of Theorem 2.4 for addition transforms to the picture (by Exercise 4.5 and the fact that any two distinct lines can be mapped to any other two distinct lines, see the discussion after Definition 4.8, we may think l_∞ as the line of points in the infinity i.e. lines are 'parallel' if they intersect at a point in l_∞):



And for multiplication it transforms as follows (here we have changed the order of a and b in an attempt to make the picture clearer):



□

Notice that for all $a \in k - l_\infty$ and $b \in k-(l_\infty \cup \{a,0\})$, if there are (and there are, see below) $\delta_a \in Aut(P_2(\mathbb{R})/l_\infty \cup \{k,k'\})$ such that $\delta_a(0) = a$ (for k' see the picture for addition) and $\rho_b \in Aut(P_2(\mathbb{R})/l_\infty \cup \{k,0\})$ such that $\rho_b(1) = b$, then $a+b = \delta_a(b)$ and $ab = \rho_b(a)$ (think how such automorphisms move the lines in the related pictures above).

Let F be a field and $n > 0$. Clearly $GL_{n+1}(F)$ acts on $P_n(F)$ by

$$A([x_i]_{i\leq n}) = [x']_{i\leq n},$$

where $(x')_{i\leq n} = A((x)_{i\leq n})$ and for lines l, $Al = \{A([x]_{i\leq n})|\ [x_i]_{i\leq n} \in l\}$. Notice that the action of every $A \in GL_{n+1}(F)$ is an automorphism of $P_n(F)$. Let K be the kernel of this action.

For all $\lambda \in F^\times$, let $A_\lambda \in GL_{n+1}(F)$ be the unique matrix such that

$$A_\lambda((x)_{i\leq n}) = (\lambda x_i)_{i\leq n}.$$

So A_1 is the neutral element of $GL_{n+1}(F)$ and we refer to it also as *id* (since this is the action of A_1 on $V_{n+1}(F)$). Similarly by *id* we refer also to $A_1K \in PGL_{n+1}(F)$.

4.7 Exercise. *Show that* $K = \{A_\lambda|\ \lambda \in F\}$.

4.8 Definition. *We let* $PGL_{n+1}(F) = GL_{n+1}(F)/K$ *(projective general linear group).*

Fix two distinct lines $k, l_\infty \in P_2(F)$.

Notice that if l_0, l_1 are distinct lines and also l'_0, l'_1 are distinct lines, then there is $A \in GL_3(F)$ such that for all $i < 2$, $Al_i = l'_i$ (exercise). Also if we denote by π the action of $GL_3(F)$ on $P_2(F)$, then the action of A on $P_2(F)$ together with the map $f \mapsto \pi(A)\circ f\circ \pi(A^{-1})$ on $Aut(P_2(F))$ form an automorphism of the action of $Aut(P_2(F))$ on $P_2(F)$ (exercise). (Similarly, the action of A on $P_2(F)$

together with the map $BK \mapsto ABA^{-1}K$ on $PGL_3(F)$ form an automorphism of the action of $PGL_3(F)$ on $P_2(F)$, exercise.) Thus keeping in mind that the action of A on $P_2(F)$ is an automorphism, essentially by Exercise 1.15, if one can solve the exercises for some choice of k and l_∞ if follows that the claims hold for all choices. So it is enough to prove the claims for some choice which makes the calculations easy. This same general principle holds when it comes time to deal with 0 and 1 below.

4.9 Exercise.

(i) Show that if $f, g \in Aut(P_2(F)/l_\infty \cup \{k\})$, $x, y \in k - l_\infty$ are distinct points, $f(x) = g(x)$ and $f(y) = g(y)$, then $f = g$ (notice that the easiest proof works in every projective plane).

(ii) Show that for all distinct $x_0, x_2 \in k - l_\infty$ and distinct $y_0, y_1 \in k - l_\infty$, there is $a \in PGL_3(F)_{l_\infty \cup \{k\}}$ such that for $i < 2$, $ax_i = y_i$. (So the action of $PGL_3(F)_{l_\infty \cup \{k\}}$ on $k - l_\infty$ is 2-regular.)

(iii) Show that the actions of $PGL_3(F)_{l_\infty \cup \{k\}}$ and $Aut(P_2(F)/l_\infty \cup \{k\})$ on $P_2(F)$ are isomorphic.

Denote $G = PGL_3(F)_{l_\infty \cup \{k\}}$. Since the action of every $g \in G$ fixes $k - l_\infty$ as a set, we can restrict the action to $k - l_\infty$ and so this way G acts on $k - l_\infty$ (i.e. if π is the action of G on $P_2(F)$, then $\pi'(g) = \pi(g) \restriction (k - l_\infty)$ is the action of G on $k - l_\infty$).

Also $F^\times$ acts on F^+ by multiplication and since $x(y+z) = xy + xz$ and $x0 = 0$, we can form $H = F^\times \rtimes F^+$. This group acts on F by $(x, y)a = xa + y$, where $(x, y) \in H$ and $a \in F$.

4.10 Exercise. *Show that the action of G on $k - l_\infty$ and the action of H on F are isomorphic.*

So G carries all the information needed to find the field F in $P_2(F)$ but is F interpretable in $P_2(F)$? For this a bit more work is needed.

We say that an element x of a group is an involution if $x^2 = 1$. We let G^+ be the set of all $g \in G$ such that the following holds: for all involutions $h \in G$, if $h \neq id$, then gh is an involution.

4.11 Exercise. *Show that G^+ is a normal subgroup of G and the action of G^+ on $k - l_\infty$ (as induced from G) is isomorphic with the action of F^+ on F (i.e. $xa = a + x$ for $x \in F^+$ and $a \in F$). Keep in mind that F may contain only two or three elements.*

Now fix some point $0 \in k - l_\infty$ and let $G^\times = G_0$. Since G^+ is a normal subgroup of G, $G^\times$ acts on G^+ via conjugation and since conjugation is an automorphism of G^+, we can form $G^\times \rtimes G^+$.

For all $x \in k - l_\infty$ let g_x be the unique element of G^+ such that $g_x(0) = x$ (by Exercise 4.11). Fix some $1 \in k - l_\infty$ so that $1 \neq 0$. For all $x \in k - l_\infty$, $x \neq 0$, let f_x be the unique element of $G^\times$ such that $g_1^{f_x} = f_x g_1 f_x^{-1} = g_x$.

4.12 Exercise.

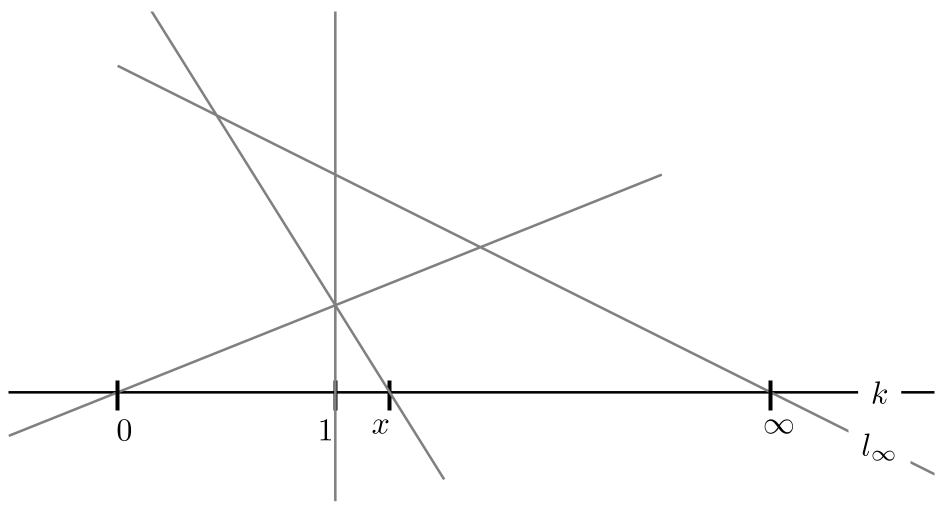
(i) Show that f_x exists and is unique.

(ii) *Show that* $G^{\times} = \{f_x |\ x \in k - l_\infty, x \neq 0\}$.

(iii) *Show that* $(f_x, g_y) \mapsto g_y f_x$ *is an isomorphism from* $G^{\times} \rtimes G^{+}$ *to* G.

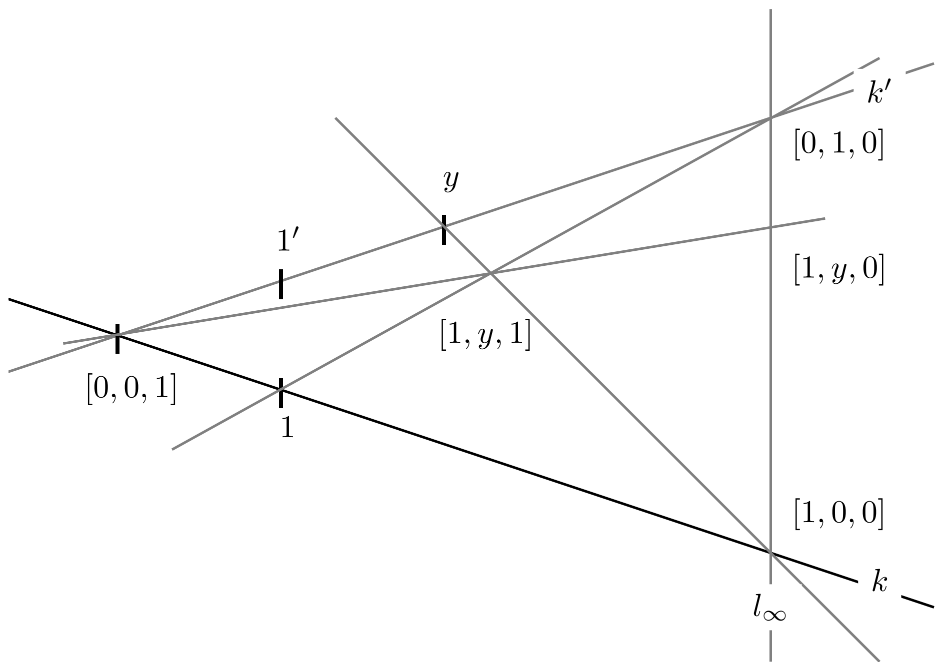
(iv) *Define addition and multiplication on* $k - l_\infty$ *as follows:* $x + y = z$ *if* $g_x g_y = g_x$ and $xy = z$ *if* $g_y^{f_x} = g_z$. *Show that* $(k - l_\infty, +, \times, 0, 1)$ *is isomorphic with* F.

(v) *Conclude that* F *is interpretable in* $P_2(F)$. *Hint: Use pairs* $(a0, a1)$ *to code elements* $a \in G$ *and use* $+$ *and* $\times$ *from (iv). Notice that for all* $x \in k - l_\infty$, ax *can be defined from* $(a0, a1)$ *as shown in the picture below (think what happens in the picture if* 0 *and* 1 *are moved by some* $f \in Aut(P_2(F)/l_\infty \cup \{k\})$*). See also Section 8.*



One can prove also Exercise 4.12 (v) from (iv) by showing that the formulas from the proof of Theorem 4.6 work, see Section 10 (and Exercise 3.2). However, this method is a bit ‘accidental’. The method in the hint is the one used in Section 8 and in many other places outside the scope of these notes.

4.13 Fact. *One can add a geometric requirement to the definition of projective plane (to get a notion of a Pappian plane), satisfied by all planes* $P_2(F)$ *so that under the requirement the construction above goes through and one can interpret some field* F *in the plane. Further more, one can coordinatize the plane (see the picture below) so that the coordinatization gives an isomorphism between the plane and* $P_2(F)$. *A weaker requirement gives a notion of a Desarguesian plane and in these the construction still works but it does not necessarily give a field but just a division ring. The coordinatization works as in Pappian planes.*



We finish this section with an exercise in which we describe the action of $PGL_2(F)$ on the projective line $P_1(F)$. This description will be needed in the end of these lecture notes.

4.14 Exercise. *Let F be a field.*

(i) Show that the action of $PGL_2(F)$ on $P_1(F)$ is 3-regular.

(ii) Pick some point $\infty \in P_1(F)$. Show that the action of $PGL_2(F)_\infty$ on $P_1(F) - \{\infty\}$ is isomorphic to the action of $F^\times \rtimes F^+$ on F.

(iii) Let $G : PGL_2(F)_\infty \to F^\times \rtimes F^+$ and $H : P_1(F) - \{\infty\} \to F$ witness the claim in (ii). For notational simplicity, we identify $x \in P_1(F) - \{\infty\}$ and $H(x)$. Then let $\alpha \in PGL_2(F)$ be such that $\alpha 0 = \infty$, $\alpha\infty = 0$ and $\alpha 1 = 1$ (by (i) there is exactly one such element and it is an involution). Show that for all $x \in P_1(F) - \{\infty, 0\}$, $\alpha x = x^{-1}$.

(iv) Show that for all $a \in PGL_2(F) - PGL_2(F)_\infty$, there are $c, b \in PGL_2(F)_\infty$ such that $a = c\alpha b$. Hint: c and b can be found using just the properties (ii) and (iii) above keeping in mind that the action is 3-regular.

(v) Show that if a group G acts on a set X so that (i)-(iii) above hold for G in place of $PGL_2(F)$ and X in place of $P_1(F)$, then the action of G on X is isomorphic with the action of $PGL_2(F)$ on $P_1(F)$.

5. Pregeometries and strongly minimal sets

For a set A and an element a, by Aa we denote the set $A \cup \{a\}$ (it will be clear from the context whether we mean a set Aa or the value of a in the action of a matrix A). Similarly if $a = (a_1, ..., a_n)$ is a sequence of elements, Aa means $A \cup \{a_1, ..., a_n\}$.

5.1 Definition. *We say that* (X, cl) *is a pregeometry (a.k.a. matroid) if the following hold:*

(i) X *is a non-empty set and* $cl : \mathcal{P}(X) \to \mathcal{P}(X)$,

(ii) if $A \subseteq B$, *then* $A \subseteq cl(A) \subseteq cl(B) = cl(cl(B))$,

(iii) if $a \in cl(A)$, *then* $a \in cl(B)$ *for some finite* $B \subseteq A$,

(iv) if $a \in cl(Ab) - cl(A)$, *then* $b \in cl(Aa)$.

(X, cl) *is a geometry if in addition* $cl(\emptyset) = \emptyset$ *and* $cl(\{a\}) = \{a\}$ *for all* $a \in X$.

5.2 Exercise. *Let* F *be a field,* V *a vector space over* F *and for all* $A \subseteq V$, $span(A)$ *be the subspace of* V *generated by* A. *Show that* $(V, span)$ *is a pregeometry.*

Suppose (X, cl) is a pregeometry and $Y \subseteq X$. We define $cl_Y : \mathcal{P}(X) \to \mathcal{P}(X)$ so that $cl_Y(A) = cl(A \cup Y)$. (X, cl_Y) is called a localization of X.

5.3 Exercise. *Let* (X, cl) *be a pregeometry.*

(i) Show that (X, cl_Y) *is a pregeometry for all* $Y \subseteq X$.

(ii) Suppose $Y \subseteq X$ *is not empty. Show that* $(X, cl) \restriction Y = (Y, cl^Y)$ *is a pregeometry, where* $cl^Y(A) = cl(A) \cap Y$.

(iii) Show that $a \sim b$ *if* $cl(\{a\}) = cl(\{b\})$ *is an equivalence relation on* $X^* = X - cl(\emptyset)$ *and that for all* $a, b \in X^*$, *either* $cl(\{a\}) = cl(\{b\})$ *or* $cl(\{a\}) \cap cl(\{b\}) = \emptyset$ *(and thus* $a/\sim = cl(\{a\}) - cl(\emptyset)$*).*

(iv) Let $\sim$ *and* X^* *be as above and assume that* $cl(\emptyset) \neq X$. *For* $A \subseteq X^*/\sim = \{a/\sim \mid a \in X^*\}$, *let* $cl^*(A) = \{a/\sim \mid a \in cl(\cup A) - cl(\emptyset)\}$. *Show that* $(X^*/\sim, cl^*)$ *is a geometry.*

We say that a set A of elements of a pregeometry (X, cl) is independent if for all $a \in A$, $a \notin cl(A - \{a\})$. A is a basis of X if in addition $cl(A) = X$.

5.4 Exercise. *Let* (X, cl) *be a pregeometry.*

(i) Suppose $A \subseteq X$ *is independent in* (X, cl) *and* B *is independent in* (X, cl_A). *Show that* $A \cup B$ *is independent in* (X, cl). *If in addition* B *is a basis of* (X, cl_A), *show that* $A \cup B$ *is a basis of* (X, cl).

(ii) Suppose $(I, <)$ *is a linear ordering. Show that* $\{a_i \mid i \in I\} \subseteq X$ *is independent if for all* $i \in I$, $a_i \notin cl(\{a_j \mid j < i\}$. *Hint: Show first that if there is a counter example, then there is a finite one.*

5.5 Theorem. *If* A *and* B *are bases of a pregeometry* (X, cl), *then* $|A| = |B|$ *(i.e. they have the same cardinality i.e. there is a bijection from* A *onto* B*).*

Proof. We prove first the claim in the case A is finite. The general case follows from this easily. Since $A = \emptyset$ iff $cl(\emptyset) = X$ iff $B = \emptyset$, we may assume that $A \neq \emptyset \neq B$. By symmetry, it is enough to show that $|B| \leq |A|$. We start with a claim:

5.5.1 Claim. *Suppose* $f : B' \to A$, $B' \subsetneq B$, *is a one-to-one function such that*

(*) $\quad (A - rng(f)) \cup dom(f)$ *is a basis of* (X, cl).

Then for any $b \in B - B'$ *there is a one-to-one function* $g : B'b \to A$, *such that* $f \subseteq g$ *and (*) holds for* g.

Proof. We prove the claim first in the special case $B' = \emptyset$ (for all (X, cl), A and B):

Let $b \in B$ be arbitrary. Then $b \in cl(A) - cl(\emptyset)$ and let $A' \subseteq A$ be a minimal set such that $b \in cl(A')$. Then $A' \neq \emptyset$ and we can choose some $a \in A'$. Let $A'' = A' - \{a\}$ and notice that by the minimality of A', $b \notin cl(A'')$ and thus $a \in cl(A''b)$ i.e. $cl(A''b) = cl(A''a)$ and in particular,

$$(I) \qquad b \notin cl(A - \{a\}).$$

We claim that $g = \{(b, a)\}$ is as wanted. Since g is one-to-one, it is enough to prove (*) for g (i.e. that $A^* = (A - \{a\})b$ is a basis of X):

For the independence, let $c \in A^*$. We need to show that $c \notin cl(A^* - \{c\})$. The case when $c = b$ follows from (I) above and so we may assume that $c \neq b$. But now if $c \in cl((A - \{a, c\})b)$, since $c \notin cl(A - \{a, c\})$, we get that $b \in cl((A - \{a, c\})c)$, which is a contradiction with (I) above.

Also $X \subseteq cl(A) \subseteq cl((A - A') \cup cl(A''b)) \subseteq cl(cl(A^*)) = cl(A^*)$ and thus A^* is a basis of X.

Now we get the general case $B \neq \emptyset$ as follows: It is easy to see that $A - rng(f)$ and $B - B'$ are bases of $(X, cl_{B'})$ and so the claim follows from the case $|B'| = 0$ and Exercise 5.4. $\square$ Claim 5.5.1.

Now the claim that $|B| \leq |A|$ follows immediately (for finite A) from Claim 5.5.1 since by applying it recursively (at most $|A|$ many times), one can construct a one-to-one function from B to A.

For infinite basis A and B (we showed above that if one of them is infinite then so is the other one) the claim can be see as follows: It is enough to show that $|A| \leq |B|$. For all $b \in B$, we can find finite $A_b \subseteq A$ such that $b \in cl(A_b)$. Let $A^* = \cup_{b \in B} A_b$. Since B is infinite, $|A^*| \leq |B|$ and so it is enough to show that $A^* = A$. For this it is enough to show that $cl(A^*) = X$. But this is clear since $X \subseteq cl(B) \subseteq cl(cl(A^*)) \subseteq cl(A^*)$. $\square$

5.6 Definition. *For a pregeometry (X, cl) and $Y \subseteq X$, by $dim(Y) = dim^X_{cl}(Y)$ (dimension of Y) we mean the size of a basis of $(X, cl) \restriction Y$. If in addition, $Z \subseteq X$, then by $dim(Y/Z)$ we mean the dimension of Y in the pregeometry (X, cl_Z) i.e. $dim^X_{cl}(Y/Z) = dim^X_{cl_Z}(Y)$. When we talk about dimensions of sequences, we think the sequences as sets, so for $a = (a_1, ..., a_n) \in X^n$, e.g. $dim(a/Z) = dim(\{a_1, ..., a_n\}/Z)$.*

For $Y \subseteq X^n$ and $Z \subseteq X$, $max\{dim(a/Z) | \; a \in Y\}$ is often called a rank (or dimension or something) of Y over Z with various names in front of the word rank depending on the situation.

5.7 Exercise. *Suppose (X, cl) is a pregeometry and $Y, Z \subseteq X$.*

(i) Show that $dim(Y/Z)$ is the size of a maximal independent subset of Y in the pregeometry (X, cl_Z).

(ii) Show that $dim(YZ) = dim(Z) + dim(Y/Z)$.

Let us now look one way of finding pregeometries inside structures. The pregeometries that we will find, are a special case of pregeometries that arise from regular types and non-forking.

Recall that in the end of Section 1 we fixed a structure M.

5.8 Definition. *For $A \subseteq M$, let $acl(A)$ be the union of all finite sets definable over A. If A is a collection of finite sequences of elements of M, we still write $acl(A)$ and mean $acl(\overline{A})$, where $\overline{A}$ is the set of all elements of M that appear in the sequences from A.*

5.9 Exercise.
(i) Show that if $A \subseteq B$, then $A \subseteq acl(A) \subseteq acl(B) = acl(acl(B))$.
(ii) If $a \in acl(A)$, then $a \in acl(B)$ for some finite $B \subseteq A$.
(iii) If $X \subseteq M^n$ is finite and definable over $A \subseteq M$, then $X \subseteq acl(A)^n$.

Notice also that if X is definable over A, then so is X^n.

5.10 Definition. *Let $\mathcal{A}$ be a structure and $A \subseteq \mathcal{A}$.*
(i) We say that $P \subseteq \mathcal{A}^n$ is minimal over A if it is definable over A, infinite and for all definable $X \subseteq \mathcal{A}^n$, either $P \cap X$ or $P - X$ is finite.
(ii) We say that $P \subseteq \mathcal{A}$ is strongly minimal over A if for all elementary extensions $\mathcal{B}$ of $\mathcal{A}$ the following holds: If $\phi(x, a)$, $a \in \mathcal{A}^m$, defines P, then $\phi(\mathcal{B}, a)$ is minimal in $\mathcal{B}$.
(iii) We say that $\mathcal{A}$ is strongly minimal if the universe of $\mathcal{A}$ is strongly minimal over $\emptyset$.

5.11 Exercise.
(i) Show that Definition 5.10 (ii) does not depend on the choice of $\phi(x, a)$.
(ii) Let $A \subseteq M$. Show that $P \subseteq M^n$ is minimal over A iff it is strongly minimal over A. Hint: M is saturated.

Recall that for $a \in M^n$ and $A \subseteq M$ we write $t(a/A)$ for the set

$$\{\phi(x, b) |\ \mathbf{M} \models \phi(a, b),\ b \in A^m\},$$

where $x = (v_1, ..., v_n)$.

5.12 Exercise. *Suppose $P \subseteq M^n$ is strongly minimal over A and $A \subseteq B \subseteq M$. Show that if $a, b \in P - acl(B)^n$, then $t(a/acl(B)) = t(b/acl(B))$. Conclude that if the cardinality of B is less than the cardinality of M, then there is $f \in Aut(M/acl(B))$ such that $f(a) = b$. Hint: Exercise 1.18.*

In order to simplify the proof of the following lemma, in the proof we use the assumption that M is stable although it is not necessary (exercise).

5.13 Lemma. *Suppose $P \subseteq M^n$ is strongly minimal over finite $A \subseteq M$. For all $X \subseteq P$, let $cl(X) = acl^P_A(X) = P \cap acl(X \cup A)^n$. Then (P, cl) is a pregeometry.*

Proof. The first requirement is clear and the next two requirements follow immediately from Exercise 5.9. So it suffices to prove the exchange property. For a contradiction suppose that $a_0 \in cl(Xa_1) - cl(X)$ but $a_1 \notin cl(Xa_0)$ for some finite X (if there is a counter example, there is one in which X is finite). So there is a formula $\phi(x,y,z)$ and $b \in A \cup X$, such that the set $\phi(M,a_1,b)$ of realizations of $\phi(x,a_1,b)$ is finite and $M \models \phi(a_0,a_1,b)$.

For all $i \in \mathbb{N}$, $i > 1$, choose a_i so that $a_i \notin cl(Xa_0...a_{i-1})$. By applying Exercise 5.12 twice, for all $i < j$ and $i' < j'$ there is $f \in Aut(M/A \cup \overline{X})$ such that $f(a_i) = a_{i'}$ and $f(a_j) = a_{j'}$. (By Exercise 5.12, there is $g \in Aut(M/A \cup \overline{X})$, such that $g(a_i) = a_{i'}$. Since $a_j \notin cl(Xa_i)$, $g(a_j) \notin acl^P_A(AXg(a_i)) = acl^P_A(AXa_{i'})$. Since also $a_{j'} \notin acl^P_A(AXa_{i'})$, by Exercise 5.12 again, there is $h \in Aut(M/A \cup \overline{Xa_{i'}})$ such that $h(g(a_j)) = a_{j'}$. Now $f = h \circ g$ is as wanted.) But then for all $i < j$, $M \models \phi(a_i,a_j,b)$ and for all $i \in \mathbb{N}$, $\phi(M,a_i,b)$ is finite. In particular, for $i < j$, $M \not\models \phi(a_j,a_i,b)$. But then $\phi(x,y,b) \wedge$ "$x \neq y$" defines an infinite ordering contradicting our assumption that M is stable. □

Following Definition 5.6, we write $dim^P_{acl^P_A}(a/B)$ for $dim^P_{acl^P_{A \cup B}}(a)$ for all $B \subseteq M$ (and not just for $B \subseteq P$).

5.14 Exercise. *Suppose $P \subseteq M^n$ is strongly minimal over A and $A \subseteq B \subseteq M$. Show that if $a_i, b_i \in P$, $1 \leq i \leq n$, and*

$$dim^P_{acl^P_A}((a_1,...,a_n)/B) = dim^P_{acl^P_A}((b_1,...,b_n)/B) = n,$$

then $t((a_1,...,a_n)/B) = t((b_1,...,b_n)/B)$. Make the same conclusion as in the Exercise 5.12.

5.15 Definition. *Let P be strongly minimal over A.*

(i) For $X \subseteq P^n$ and $B \subseteq M$, we say that $a \in X$ is generic in X over B if for all $b \in X$, $dim^P_{acl^P_A}(b/B) \leq dim^P_{acl^P_A}(a/B)$. If $X = P^n$, we say just generic over B and if in addition $B = \emptyset$, we say just generic.

(ii) For $a \in P^n$ and $A \subseteq B \subseteq C \subseteq M$, we write $a \downarrow_B C$ if $dim^P_{acl^P_A}(a/B) = dim^P_{acl^P_A}(a/C)$.

5.16 Exercise. *Suppose $A \subseteq B \subseteq M$ and P is strongly minimal over A. Show that if $X \subseteq P^n$ is definable over A, then $a \in X$ is generic over B iff*

$$dim^P_{acl^P_A}(a/B) = max\{dim^P_{acl^P_A}(b/A) | \ b \in X\}.$$

Conclude that $a \in P^n$ is generic over B if it is generic and $a \downarrow_A B$. Hint: Exercise 5.14.

The following lemma is a very special case of a general fact of 'definability of free extension' in stable structures.

5.17 Lemma. *If $P \subseteq M^n$ is strongly minimal over A and $\phi(x,y)$ is a formula, $x = (x_1,...,x_n)$ and $y = (y_1,...,y_k)$, then there is a formula $d_P\phi(y,e)$, such that e is a sequence of elements of A and for all $b \in M^k$ and $a \in P$ generic over Ab, $M \models \phi(a,b)$ iff $M \models d_P\phi(b)$.*

Proof. For all $N \in \mathbb{N}$, let $\phi_N(y,e)$ be a formula such that for all $b \in M^k$, $M \models \phi_N(b,e)$ if there are $\geq N$ many $a \in P$ such that $M \models \phi(a,b)$ (clearly there is such a formula with parameters from A). Similarly, let $\phi^-_N(y,e)$ be a formula such that for all $b \in M^k$, $M \models \phi^-_N(b,e)$ if there are $\geq N$ many $a \in P$ such that $M \models \neg\phi(a,b)$. We claim that for some $N \in \mathbb{N}$, $d_P\phi(y,e) = \phi_N(y,e)$ is as wanted.

Since there are infinitely many elements in P which are generic over Ab, by Exercise 5.12, the claim from left to right holds no matter how N is chosen i.e. if $\phi(a,b)$ holds for some $a \in P$ generic over Ab, then $\phi_N(b,e)$ holds for all $N \in \mathbb{N}$. Since the same holds also for $\neg\phi$ and ϕ^-_N in place of ϕ and ϕ_N, if there is no such N, then every finite subtype of the type

$$q = \{\phi_N(y,e), \phi^-_N(y,e) |\ N \in \mathbb{N}\}$$

is realized in M and thus by the saturation of M, q is realized in M by some b. But then $\phi(x,b)$ defines a subset X of M^n such that both $P \cap X$ and $P - X$ are infinite, a contradiction. □

5.18 Exercise.

(i) If $P \subseteq M^n$ is strongly minimal over A and $\phi(x,y)$ is a formula, $x = (x_1, ..., x_{mn})$ and $y = (y_1, ..., y_k)$, then there is a formula $d_P\phi(y)$ such that for all $b \in M^k$ and $a \in P^m$ generic over Ab, $M \models \phi(a,b)$ iff $M \models d_P\phi(b)$. Hint: Use '$d_P(...(d_P\phi(x,y))...)$'.

(ii) If M satisfies Assumptions 1.16 (i), (ii) and (iv) and is strongly minimal, then it is ω-stable.

Before looking examples, we need one fact.

5.19 Definition.

(i) We say that a structure $\mathcal{A}$ has elimination of quantifiers if for every set $X \subseteq \mathcal{A}^n$ the following holds: If X is definable over $A \subseteq \mathcal{A}$ then it is definable over A with a formula that does not contain any quantifiers (i.e. is a boolean combination of atomic formulas).

(ii) We say that a field F is algebraically closed if for all polynomials $P(X) \in F[X]$ the following holds: if P is not constant, then it has a root in F (i.e. there is $a \in F$ such that $P(a) = 0$).

5.20 Facts.

(i) If V is an infinite vector space, then it has elimination of quantifiers.

(ii) If F is an algebraically closed field, then it has elimination of quantifiers.

5.21 Exercise. *Prove the following claims:*

(i) If V is an infinite vector space over a field F, then it is strongly minimal, $acl = span$, and if in addition $dim^V_{acl}(V)$ is at least the cardinality of F, then V is saturated.

(ii) If F is an algebraically closed field, then it is strongly minimal, and if in addition is $dim^F_{acl}(F)$ is infinite, then it is saturated. Hint: see the remark below.

(iii) If F is an uncountable algebraically closed field and $k, l_\infty \in P_2(F)$ are distinct lines, then $k - l_\infty$ is strongly minimal over $\{k, l_\infty\}$ and $(k - l_\infty, acl^{k-l_\infty}_{\{k,l_\infty\}})$ is a geometry. Hint: Start by noticing that every automorphism of F extends

naturally to an automorphism of $P_2(F)$ fixing lines k and l_∞ (easier if k and l_∞ are chosen properly). Then use this to show that for every definable X, either $k \cap X$ or $k - X$ is countable. Finally, show that $P_2(F)$ is saturated, see Exercise 7.2, 7.3.3 and 1.18.

5.22 Remark.

(i) The claim in Exercise 5.21 (ii) can be proved so that from the proof it follows that for an algebraically closed field F and $A \subseteq F$, $a \in acl(A)$ if a is a root of some non-zero polynomial over the subfield of F generated by A. So in the case of algebraically closed fields, out notion of algebraic closure coincides with the notion of algebraic closure from the theory of fields.

(ii) In Exercise 5.21 (iii) the assumption that F is uncountable is made just to make the proof easier.

In this part we look at examples of what geometries tell about models and their automorphism groups.

6. Groups, fields and pregeometries

6.1 Definition.

(i) For an infinite structure $\mathcal{A}$ and $cl : \mathcal{P}(dom(\mathcal{A})) \to \mathcal{P}(dom(\mathcal{A}))$, we say that $(\mathcal{A}, cl)$ is a structure carrying an ω-homogeneous pregeometry if the following holds:

(a) $(\mathcal{A}, cl)$ is a pregeometry,

(b) $dim(\mathcal{A})$ is the same as the cardinality of $\mathcal{A}$,

(c) if $A \subseteq \mathcal{A}$ is finite and $a, b \in \mathcal{A} - cl(A)$, then there is $f \in Aut(\mathcal{A}/A)$ such that $f(a) = b$ and for all $B \subseteq \mathcal{A}$, $f(cl(B)) = cl(f(B))$.

(ii) Following Section 5, for a structure $(\mathcal{A}, cl)$ carrying an ω-homogeneous pregeometry and $A \subseteq \mathcal{A}$, we say that $a \in \mathcal{A}^n$ is generic over A if $dim^{\mathcal{A}}_{cl}(a/A) = n$.

(iii) We say that $X \subseteq \mathcal{A}^n$ is A-invariant, $A \subseteq \mathcal{A}$, if

$$f(X) = \{(f(x_1), ..., f(x_n)) |\ (x_1, ..., x_n) \in X\} = X$$

for all $f \in Aut(\mathcal{A}/A)$.

Notice that if $H : \mathcal{A} \to M$ witnesses that $\mathcal{A}$ is interpretable in M over $A \subseteq M$ and $H(\mathcal{A})$ is strongly minimal over A, then $\mathcal{A}$ is a structure carrying an ω-homogeneous pregeometry (exercise). Notice also that if $X \subseteq M^n$ is definable over $A \subseteq M$, then it is A-invariant.

6.2 Exercise. *Suppose that (G, cl) is a group carrying an ω-homogeneous pregeometry and $A \subseteq G$ is finite. Prove the following claims:*

(i) If $X \subseteq G$ is A-invariant, then either $X \subseteq cl(A)$ or $G - cl(A) \subseteq X$.

(ii) If $a \in G - cl(A)$, $b \in cl(A)$ and $c \in G - cl(Aa)$, then $a^{-1}, ab, ac \in G - cl(A)$ and $a^{-1} \in cl(\{a\})$.

(iii) If H is an A-invariant subgroup of G, then either $H \subseteq cl(A)$ or $H = G$.

The order of an element a of a group G is the cardinality of the subgroup generated by $\{a\}$ if finite and otherwise it is ∞. The following easy exercise is needed in the proof of Theorem 6.4.

6.3 Exercise. *If the order of every element of a group G is ≤ 2, then G is commutative.*

6.4 Theorem. *Suppose that (G, cl) is a group carrying an ω-homogeneous pregeometry. Then either G is commutative or unstable (in which case it is not interpretable in M^{eq} by Exercise 1.18 (ii), see Section 7).*

Proof. We assume that G is not commutative and we show that it is unstable. Let Z be the center of G (i.e. the subgroup of all elements of G that commute with every element of G). Since G is $\emptyset$-invariant and G is not commutative, by Exercise 6.2, $Z \subseteq cl(\emptyset)$. Let $G^* = G/Z$.

Our first goal is to show that G^* is centerless (i.e. the center of G^* contains just one element, the neutral element). For a contradiction, suppose $a \in G - Z$ is such that aZ commutes with every element in G^*. Then for all $x \in G$, $a^x = xax^{-1} \in aZ$. For all $z \in Z$, let $X_{az} = \{x \in G | a^x = az\}$. Since X_{az} is $\{a, z\}$-invariant, by Exercise 6.2 (i), for some $z_0 \in Z$,

(*) $G - X_{az_0} \subseteq cl(\{a, z_0\}) = cl(\{a\})$.

Choose $c \in G - cl(\{a\})$ and $d \in G - cl(\{a, c\})$. Then by (*) and Exercise 6.2 (ii), $c, d, cd \in X_{az_0}$. Now (since $z_0 \in Z$) $az_0 = a^{cd} = (a^d)^c = (az_0)^c = a^c z_0^c = a^c z_0 = (az_0)z_0$ and so $z_0 = 1$ (the neutral element of G). But then X_{az_0} is the centralizer C_a of a (i.e. the set of all elements of G that commute with a), in particular it is a subgroup of G and thus by Exercise 6.2 (iii), $C_a = G$ i.e. $a \in Z$, a contradiction.

Our next goal is to show that any two $a, b \in G^* - \{1Z\}$ are conjugates (i.e. for some $x \in G^*$, $a^x = b$). Let $c \in G$ be such that $a = cZ$. Since G^* is centerless, $C^*_c = \{x \in G | c^x \in cZ\}$ ($x \in C^*_c$ if xZ and $a = cZ$ commute) is a proper subgroup of G and since it is $\{c\}$-invariant, it is a subset of $cl(\{c\})$ by Exercise 6.2. Let $x \in G - cl(\{c\})$.

6.4.1 Claim. $c^x \notin cl(\{c\})$.

Proof. For a contradiction, suppose $c^x \in cl(\{c\})$. Then for all $y \in G - cl(\{c\})$, $c^y = c^x$ (there is an automorphism fixing c^x and mapping x to y). This means that for all $y \in G - cl(\{c\})$, $y^{-1}x \in C^*_c$, which contradicts Exercise 6.2 (ii). $\square$ Claim 6.4.1.

But then from Claim 6.4.1 and Definition 6.1 (i) (c) it follows that for all $e \in G - cl(\{c\})$, $e = c^x$ for some $x \in G$. Now the same holds for $b = dZ$ and so we can find $e, x, y \in G$ such that $e = c^x$ and $e = d^y$. But then $d = c^{y^{-1}x}$. In particular, a and b are conjugates in G^*.

Now it is easy to see that G is unstable. By Fact 6.3, there must exist $a = cZ \in G^*$ of order > 2. So $a \neq a^{-1}$ and they are conjugates by some $b = dZ$. Then b does not commute with a but b^2 does (since conjugation is an automorphism of G^*, $ba^{-1}b^{-1} = (bab^{-1})^{-1}$ and so $a^{(b^2)} = (a^b)^b = (a^{-1})^b = (a^b)^{-1} = (a^{-1})^{-1} = a$). Also every element that commutes with b commutes also with b^2 and thus C^*_d is properly contained in $C^*_{d'}$ for any $d' \in G$ such that $b^2 = d'Z$ (notice that $C^*_{d'}$ depends on d' only upto $d'Z$). Now b and b^2 are conjugates by some eZ. For all $i \in \mathbb{N}$, we define $d_i \in G$ as follows: $d_0 = d$ and $d_{i+1} = ed_ie^{-1}$. Again since conjugation is an automorphism, for all $i \in \mathbb{N}$, $C^*_{d_i}$ is properly contained in $C^*_{d_{i+1}}$ and thus for $i < j$, $C^*_{d_i} \subsetneq C^*_{d_j}$. Clearly $C^*_x \subsetneq C^*_y$ is expressible in the first-order logic. $\square$

6.5 Open question. *Are there non-commutative groups that carry an ω-homogeneous pregeometry?*

6.6 Fact. *Commutative groups are stable.*

Now we turn to fields carrying an ω-homogeneous pregeometry. For these we need to recall some basic properties of fields and one not so basic fact.

Let F be a subfield of K i.e. K is a field extension of F. Then K can be seen as a vector space over F as follows: The addition in K is the addition of the field and for $a \in F$ and $x \in K$, $f_a(x) = ax$ (the multiplication of the field). We say that K is a finite dimensional extension of F if the dimension of this field is finite. Now suppose that K is a finite dimensional extension of F and let $\{\xi_i |\ i < n\}$ be a basis for K such that $\xi_0 = 1$.

6.7 Exercise.

(i) Show that $H : K \to F^n$, $H(\Sigma_{i<n} a_i \xi_i) = (a_0, ..., a_{n-1})$ *witnesses that* K *is interpretable in* F.

(ii) Show that every $a \in K$ *is a root of some polynomial* $P \in F[X]$. *In particular, if* F *is algebraically closed then* $F = K$.

(iii) Show that there is a finite $A \subseteq F$ *such that for all* $f \in Aut(F/A)$, *if* $\overline{f} : K \to K$ *is such that* $\overline{f}(\Sigma_{i<n} a_i \xi_i) = \Sigma_{i<n} f(a_i)\xi_i$, *then* $\overline{f} \in Aut(K)$.

(iv) Show that for all $n > 0$, $\{a^n |\ a \in K^\times\}$ *is a subgroup of* $K^\times$ *and if the characteristic of* K *is* $p > 0$, *then* $\{a^p - a |\ a \in K^+\}$ *is a subgroup of* K^+.

6.8 Fact. F *is algebraically closed if for all finite dimensional extension* K *of* F *the following holds: For all* $n > 0$, $\{a^n |\ a \in K^\times\} = K^\times$ *and if the characteristic of* F *is* $p > 0$, *then also* $\{a^p - a |\ a \in K^+\} = K^+$.

6.9 Theorem. *If* (F, cl) *is a field carrying an* ω*-homogeneous pregeometry, then it is algebraically closed.*

Proof. We use Fact 6.8. For this let K be a finite dimensional extension of F and $H : K \to F^n$ as in Exercise 6.7 (i) and $A \subseteq F$ as in Exercise 6.7 (iii). For $A \subseteq X \subseteq F$, we say that $a \in K$ is generic over X if $H(a)$ is generic (in F^n) over X in the pregeometry (F, cl).

6.9.1 Exercise. *Suppose* X *is finite and* $A \subseteq X \subseteq F$.

(i) Show that for all $n > 1$, *if* $a \in K$ *is generic over* X *then so are* a^n *and* $a^n - a$.

(ii) Show that if $a \in K$ *is generic over* X *and* $H(b) \in cl(X)^n$, *then* $a + b$ *and* ab *are generic over* X.

(iii) Show that if $a, b \in K$ *are generic over* X, *then there is an automorphism* $f \in Aut(F/X)$ *such that* $\overline{f}(a) = b$ *(and* $\overline{f} \in Aut(K/H^{-1}(X^n))$*).*

(iv) Show that if G *is an* $H^{-1}(X^n)$*-invariant subgroup of* K^+ *(respectively, of* $K^\times$*) and it contains an element generic over* X, *then* $G = K^+$ *(respectively,* $G = K^\times$*).*

Now let $a \in K$ be generic over A. Then a^n is generic over A by Exercise 6.9.1 (i) and thus the $\emptyset$-invariant subgroup $\{a^n |\ a \in K^\times\}$ of $K^\times$ contains an element generic over A and thus $\{a^n |\ a \in K^\times\} = K^\times$ by Exercise 6.9.1 (iv). If the characteristic of F is $p > 0$, then similarly we can see that $\{a^p - a |\ a \in K^+\} = K^+$.
□

7. M^{eq} and connected components

In order to simplify our arguments, from now on in these notes, we assume that L is countable.

7.1 Definition. *Let $\mathcal{E}$ be the set of all equivalence relations on M^n, $n \in \mathbb{N}$, which are definable without parameters. For all $E \in \mathcal{E}$, by a/E we denote the equivalence class of $a \in M^n$ and (by misusing the notation a bit) by M/E we denote the set of all equivalence classes of E. (Strictly speaking, we choose the objects a/E so that for $E \neq E'$, $a/E \neq a/E'$ even if as sets they are the same.) We let $L^{eq} = L \cup \{F_E, P_E | \ E \in \mathcal{E}\}$, where P_E is a new unary predicate symbol and F_E is a new function symbol of arity n, where n is such that E is an equivalence relation on M^n. Then M^{eq} is defined as follows:*

(i) The universe of M^{eq} is $A^* \cup \bigcup_{E \in \mathcal{E}} M/E$, where A^* is some set of the same cardinality as M of elements outside $\bigcup_{E \in \mathcal{E}} M/E$ (these elements are needed to guarantee that M^{eq} is saturated).

(ii) To simplify notation we identify each $a \in M$ with $a/=$ and then we let the interpretation of every relation and constant symbol from L be the same as that in M.

(iii) For all n-ary function symbols f from L, the interpretation of f on M^n in M^{eq} is the same as the interpretation in M.

(iv) The interpretation of P_E is M/E.

(v) If F_E is n-ary and $a \in M^n$, then the interpretation of F_E maps a to a/E.

(vi) Now we have defined M^{eq} except that the interpretations of function symbols are only partial and we would like to leave it this way. However, we have defined structures so that the interpretations of function symbols must be total. So we make the interpretations total so that no structure is added in the process (i.e. no new definable sets): if f is an n-ary function symbol and $a = (a_1, ..., a_n) \in (M^{eq})^n$ is such that we have not yet defined what $f(a)$ is, then we let $f(a) = a_1$. This does not add structure by Exercise 1.13. (Alternatively we could define M^{eq} as a many sorted structure and then this problem does not appear and the set A is not needed.)

7.2 Exercise.

(i) Show that for every $f \in Aut(M)$ there is $\overline{f} \in Aut(M^{eq})$ such that $\overline{f} \restriction M = f$ (and it is unique up to what it does on A^*). Show also that if $f \in Aut(M^{eq})$, then $f \restriction M \in Aut(M)$.

(ii) Show that M^{eq} is saturated. Hint: Show first that for proving that $\mathcal{A}$ is saturated, it suffices to show that for all $A \subseteq \mathcal{A}$ of cardinality strictly less than the cardinality of $\mathcal{A}$ and for all complete 1-types p over A, if p is finitely realized in $\mathcal{A}$, then it is realized in $\mathcal{A}$.

(iii) Show that if M is ω-stable, then also M^{eq} is ω-stable.

(iv) Show that M^{eq} is stable. Hint: Start by showing that with Ramsay's theorem and (i), one can find an infinite order indiscernible sequence in M^{eq} such that the order is given by some formula, see the lecture notes on model theory.

(v) Show that if $X \subseteq M^n$ is definable in M^{eq}, then it is definable already in M and that if it is definable over A in M, then it is definable over A also in M^{eq}. Conclude that if $P \subseteq M$ is strongly minimal over $A \subseteq M$ in M, then it is strongly minimal over A in M^{eq}. In particular, if F is an uncountable algebraically closed field, then it is strongly minimal over $\emptyset$ in F^{eq}.

(vi) Suppose $X \subseteq M^n$ is definable over $A \subseteq M$ and that $E \subseteq X^2$ is an equivalence relation also definable over A. Then there is $F : X \to M^{eq}$ definable over A such that for all $x, y \in X$, xEy holds iff $F(x) = F(y)$.

By Exercise 7.2 (vi), even if an equivalence relation is not on M^n and/or definable over $\emptyset$, as long as it is definable, we can use M^{eq} as if the equivalence relation is definable without parameters.

Before studying interpretable groups from the point of view of model theory, let us look at some examples.

7.3 Examples. *Let F be an uncountable algebraically closed field of characteristic 0 (e.g. $\mathbb{C}$) and let S be a finite set of polynomials from $F[X_0, ..., X_n]$. Now if $V = V_S \subseteq F^{n+1}$ is the vanishing set of these polynomials, it is called an affine variety. If A contains the coefficients of the polynomials from S, then we also say that V is an affine variety over A. A topology on F^{n+1} in which affine varieties are the closed sets is called Zariski topology.*

7.3.1 Fact. *The family of affine varieties $\subseteq F^{n+1}$ is closed under intersections (since there are no infinite properly decreasing sequences of affine varieties). The family is also closed under finite unions (exercise).*

A subset of a topological space is called irreducible if it is not a union of two closed proper subsets (in the literature, sometimes irreducibility is included in the definition of a variety).

7.3.2 Exercise.

(i) Show that if V is an irreducible affine variety over A, then for all finite $A \subseteq B \subseteq F$, if $a, b \in V$ are generic over B (i.e. a and b are generic over A, $a \downarrow_A B$ and $b \downarrow_A B$), then $t(a/B) = t(b/B)$.

(ii) Show that if V is an irreducible affine variety over A and

$$rk(V/A) = max\{dim^F_{acl^F_A}(a) |\ a \in V\} = 1,$$

then V is strongly minimal over A.

Hint for (i): Suppose not. Then there are generic $a^i = (a^i_0, ..., a^i_n) \in V$, $i < m$ and $m \geq 2$, over B such that for all $b \in V$ generic over B, $t(b/B) = t(a^i/B)$ for some $i < m$ and for $i < j < m$, $t(a^i/B) \neq t(a^j/B)$. But then (by elimination of quantifiers) we can find polynomials $f^p_j \in F[X_0, ..., X_n]$, $j < m$ and $p < m-1$, such that the coefficients are from $acl(B)$ and $f^p_j(a^i) = 0$ for all $p < m-1$ iff $i = j$. Let $k = rk(V)$. To simplify this hint we assume that for all $i < m$, $dim((a^i_0, ..., a^i_{k-1})/B) = k$. Then by saturation we can find non-zero polynomials $g_l \in F[X_0, ..., X_{k-1}]$, $l < m'$, such that all the coefficients are from $acl(B)$ and for all $a = (a_0, .., a_n) \in V$, either there is $l < m'$ such that $g_l(a_0, ..., a_{k-1}) = 0$

or there is $j < m$ such that $f_j^p(a) = 0$ for all $p < m-1$ (or both). Now it is easy to find proper subvarieties V_i, $i < m$, such that $V = \cup_{i<m} V_i$ and deduce a contradiction.

Let us now look at projective spaces.

7.3.3 Exercise. *Show that $P_n(F)$ is interpretable in F^{eq}. Furthermore, show that the one-to-one function from Definition 1.14 can be chosen so that on the set of points it is identity.*

From now on in this example, $P_n(F)$ denotes the set of point of $P_n(F)$ (cf. the discussion after Definition 4.4). If all the polynomials in S are homogeneous ($f \in P[X_0, ..., X_n]$ is homogeneous if all monomials in f have the same degree), then $U = U_S = \{[x_0, ..., x_n] | \ (x_0, ..., x_n) \in V_S\} \subseteq P_n(F)$ is called a projective variety. Notice that the homogeneity of polynomials in S quarantee that U_S is well-defined. Irreducibility of such varieties is defined exactly as it was defined for affine varieties. If by regular functions (see the literature, regular functions are definable) one can define a group law on an irreducible projective variety U, then U is called an abelian variety. A good reason to call these varieties abelian is the fact that the group law is necessarily commutative (although this is probably not the reason why the word abelian was originally chosen). In fact we have already seen this in one special case:

If $S = \{X_0^3 - X_1^2 X_2 + a X_0 X_2^2 + b X_2^3\}$ and $4a^3 + 27b^2 \neq 0$, then $U = U_S \subseteq P_2[F]$ is called an elliptic curve and these are abelian varieties.

7.3.4 Exercise. *Show that the elliptic curve U is strongly minimal over $\{a, b\}$ in F^{eq}. Hint: Show first that it suffices to show that the affine variety $V = \{(x, y) \in F^2 | \ y^2 = x^3 + ax + b\}$ is strongly minimal over $\{a, b\}$. Then either one can use algebra and show that it is irreducible and apply Exercise 7.3.2 (ii) or one can argue as follows: Suppose that V is not strongly minimal over $\{a, b\}$ and show that there are a generic $(c, d) \in V$ over some $\{a, b\} \subseteq B \subseteq F$ and a polynomial $f \in F[X_0, X_1]$ with coefficients in B such that $f(c, d) = 0$ but $f(c, -d) \neq 0$. Now consider what powers of X_1 appear in monomials in f and show that $d = g(c)/h(c)$ where g and h are polynomials with coefficients from the field generated by B. Finish by showing that this is not possible (look at the degrees of the three polynomials).*

Thus by Exercise 7.3.4 and Theorem 6.4 the group law is commutative.

Now we are ready to introduce the setup that, excluding the last section, will be studied for the rest of these notes.

7.4 Assumptions. *We assume that there are $A, Q \subseteq M$ and $P \subseteq M^m$ such that*

(i) A is finite,
(ii) P and Q are definable over A,
(iii) P is strongly minimal over A,
(iv) (P, acl_A^P) is a geometry,
(v) there is $n \in \mathbb{N} - \{0\}$ such that

(a) *for all (finite)* $B \subseteq Q$ *and* $a = (a_1, ..., a_n) \in P^n$ *generic over* A,

$$dim(a/AB)(= dim^P_{acl^P_A}(a/B)) = n,$$

(b) *for some (finite)* $B \subseteq Q$ *and* $a = (a_1, ..., a_{n+1}) \in P^{n+1}$ *generic over* A,

$$dim(a/AB)(= dim^P_{acl^P_A}(a/B)) = n.$$

We have made the assumption (iv) i.e. that (P, acl^P_A) is a geometry to guarantee that certain groups are interpretable. An easy modification shows that we can loosen this to: for all $x \in P$, $acl^P_A(\{x\})$ is finite. This is because then the equivalence relation $x \sim y$ if $acl^P_A(\{x\}) = acl^P_A(\{y\})$ on $P^* = P - acl^P_A(\emptyset)$ is definable and so $(P^*/\sim, (acl^P_A)^*)$, see Exercise 5.3 (iv), can be found from M^{eq} and we can use this instead of P. This weaker assumptions holds e.g. if $Th(M)$ is ω-categorical, see the lecture notes on model theory or Section 12.

7.5 Exercise. *Let* F *be an uncountable algebraically closed field and* k *and* l_∞ *be distinct lines in* $P_2(F)$. *Show that* $M = P_2(F)$, $P = k - l_\infty$, $Q = l_\infty$ *and* $A = \{k, l_\infty\}$ *satisfy all the assumptions we have made on* M, P, Q *and* A. *Hint: Exercise 1.18.*

Now we can define the group G that will be our main object of study in these notes. Let $G' = Aut(M/AQ)$. Then G' acts the natural way on P and let K be the kernel of this action. Then we let $G = G'/K$. Probably the first question on G that comes to ones mind is: Does it contain more than one element. It does, but for this one fact from stability theory is needed (a proof can be found from [HLS]):

7.6 Fact. *Suppose* $X \subseteq M$ *is definable over some finite* $B \subseteq M$ *and* $a, b \in M^m$ *are such that* $t(a/BX) = t(b/BX)$. *Then there is* $f \in Aut(M/BX)$ *such that* $f(a) = b$.

Now from Fact 7.6 it follows:

7.7 Exercise. *Show that the following claims hold without assuming Assumptions 7.4 (iv)*

(i) *For all generic* $a, b \in P^n$ *there is* $f \in G$ *such that* $fa = b$ *(i.e. the action of* G *on* P *has rank* $(\geq)$ n*).*

(ii) *If* $n \geq 2$, *then* (P, acl^P_A) *is a geometry iff the action of* G *on* P *is* 2*-transitive.*

The reason why abelian varieties behave nicely (as a group) is that they have 'unique generics' i.e. the claim in Exercise 7.3.2 (i) holds for them and we would like the same to be true for G. However, this is not true, from [HK] a counter example can be found. What we do is that we will find a normal subgroup G° of G such that G/G° is small and G° have unique generics. G° will be called the connected component of G.

The key fact behind finding G° is the following theorem from stability theory which is also one of the reasons why we (partially) work in M^{eq} (when we want to press the fact that we take algebraic closure in M^{eq} we write acl^{eq} instead of just acl):

7.8 Fact. *Suppose $A \subseteq B \subseteq C \subseteq M^{eq}$ and $a, b \in P^m$. If $B = acl^{eq}(B)$, $t(a/B) = t(b/B)$, $a \downarrow_B C$ and $b \downarrow_B C$, then $t(a/C) = t(b/C)$.*

But some work is needed before we can apply this fact.

7.9 Lemma. *The action of G on P is $n+1$-determined i.e. if $a = (a_0, ..., a_n) \in P^{n+1}$ is generic and $f, g \in G$ are such that for all $i \leq n$, $fa_i = ga_i$, then $f = g$.*

Proof. Clearly it is enough to show that if for all $i \leq n$, $fa_i = a_i$, then $fc = c$ for all $c \in P$. In fact, it is enough to show that if $c \in P$ is generic over Aa, then $fc = c$ (exercise). For $i \leq n$, let $a^i = (a_0, ..., a_{i-1}, a_{i+1}, ..., a_n)$. We start by showing that

(*) fc is not generic over $Aa^i c$.

Clearly it is enough to show this for $i = 0$. For a contradiction, suppose it is. Let finite $C \subseteq Q$ be such that $a^0 c$ is not generic over AC. Let $d \in P$ be generic over $ACa^0 c$ and choose $F \in Aut(M/Aa^0 c)$ such that $F(d) = fc$. Then $a^0 c$ is not generic over $AF(C)$ but by the choice of d, $(fc)a^0 = (fc, fa_1, ..., fa_n)$ is generic over $AF(C)$. This contradicts the choice of G'.

Suppose then that $fc \neq c$. Since P is a geometry, $fc \notin acl^P_A(\{c\})$. Thus by (*) there is $0 < i \leq n$ such that $a_i \in acl^P_A(\{c, fc, a_1, ..., a_{i-1}, a_{i+1}, ..., a_n\})$. Applying (*) again, $fc \in acl^P_A(\{c, a_0, ..., a_{i-1}, a_{i+1}, ..., a_n\})$. Thus the dimension of $c(fc)a$ is $n+1$, but this contradicts the choice of c. □

7.10 Exercise.

(i) Show that for all $f \in G$ and generic $a = (a_0, ..., a_n) \in P^{n+1}$

$$dim((a_0, ..., a_n, fa_0, ..., fa_n)) =\leq 2n+1.$$

Hint: Use Assumption 7.4 (v) (b).

(ii) Show that for all $f \in G$ and generic $a = (a_0, ..., a_n) \in P^{n+1}$, $fa_n \in acl^P_A(\{a_0, ..., a_n, fa_0, ..., fa_{n-1}\})$. Hint: Suppose not and show first that there are a generic $(a_0, ..., a_{n+1}) \in P^{n+2}$ and $g \in G$ such that $ga_i = a_i$ for all $i < n$ and $ga_n = a_{n+1}$.

We write $A^* = acl^{eq}(A)$ and notice that for all $X \subseteq P$, $acl^{eq}(XA^*)^m \cap P = acl^P_A(X)$. Let $\Sigma = Aut(M^{eq}/A^*)$. Now Σ acts on $M^{eq} \cup G$ as follows: For $\delta \in \Sigma$ and $x \in M^{eq}$, $\delta x = \delta(x)$ and for $g = h/K \in G$, $\delta g = h^\delta/K$. We say that $f, g \in G$ have the same type if there is $\delta \in \Sigma$ such that $\delta f = g$. This is an equivalence relation and the equivalence class of $f \in G$ is called the type of f.

7.11 Exercise. *Show that for all $\delta \in \Sigma$, $f, g \in G$ and $x \in P$, $(\delta g)(\delta x) = \delta(gx)$ and $\delta(fg) = (\delta f)(\delta g)$ (i.e. the action of each $\delta \in \Sigma$ is an automorphism of the action of G on P).*

We say that $a = (a_0, ..., a_{2n+1}) \in P^{2n+2}$ is a quasicode of $f \in G$ if $(a_0, ..., a_n)$ is generic and for all $i \leq n$, $a_{n+i+1} = fa_i$.

7.12 Exercise. *Show that if $\delta \in \Sigma$, $f, g \in G$, a is a quasicode of f, b is a quasicode of g and $\delta(a) = b$, then $\delta f = g$.*

We say that a quasicode a is generic over $B \subseteq M^{eq}$ if $dim(a/AB) = 2n+1$ We say that $f \in G$ is generic over $B \subseteq M^{eq} \cup G$ if there is $B \cap M^{eq} \subseteq C \subseteq M^{eq}$ such that every $g \in B \cap G$ has a quasicode in C^{2n+2} and f has a quasicode that is generic over C. We say that $f \in G$ is generic if it is generic over $\emptyset$. Similarly $a \in P^m$ is generic over $A \subseteq B \subseteq M^{eq} \cup G$ if there is $B \cap M^{eq} \subseteq C \subseteq M^{eq}$ such that every $g \in B \cap G$ has a quasicode in C^{2n+2} and a is generic over C. We say that a quasicode $a = (a_0, ..., a_{2n+1}) \in P^{2n+2}$ of $f \in G$ is a semicode of f if $(a_0, ..., a_n)$ is generic over $\{f\}$.

7.13 Exercise.

(i) Show that the number of types of generic $f \in G$ is at most maximum of ω and the cardinality of L.

(ii) Show that if $a = (a_0, ..., a_{2n+1}) \in P^{2n+2}$ and $b = (b_0, ..., b_{2n+1}) \in P^{2n+2}$ are semicodes of f, then there is $\delta \in \Sigma$ such that for all $i \leq 2n+1$, $\delta a_i = b_i$.

(iii) Show that if a is a semicode of a generic f, then $dim(a) = 2n+1$.

(iv) Show that if $g \in G$ and $f \in G$ is generic over $g \in B \subseteq M^{eq} \cup G$, then f^{-1}, fg and gf are generic over B. Hint: E.g. for fg generic over B, let $a = (a_0, ..., a_n) \in P^{n+1}$ be generic over $B \cup \{f\}$ and study $dim^P_{acl^P_A}(fa_0...fa_n/Ba)$ and $dim^P_{acl^P_A}((fg)a_0...(fg)a_n/Ba)$.

(v) Show that if $a \in P$ is generic over $\{f, g\} \subseteq G$ and $fa = ga$, then $f = g$.

Let S be the set of all types of generic elements $f \in G$. Then G acts on S as follows: For $p \in S$ and $g \in G$, gp is the type of gf, where f is any generic element of G over $\{g\}$ such that its type is p.

7.14 Exercise. *Show that the definition above gives a well-defined action of G on S.*

We let G° be the kernel of the action of G on S and call it the connected component of G.

7.15 Lemma. *G° is a normal subgroup of G, for all $\delta \in \Sigma$, $\delta G^\circ = G^\circ$, G° contains a generic element of G and the cardinality of G/G° is at most (the maximum of) ω (and the cardinality of L). In particular the cardinality of G/G° is strictly less than the cardinality of M.*

Proof. The first claim follows from Exercise 3.2 (i). The second claim follows from the observation that if f is generic over $\{g\}$ then δf is generic over δg and $\delta(gf) = (\delta g)(\delta f)$ and thus gf and $(\delta g)(\delta f)$ have the same type i.e. if the type of f is $p \in S$, then $(\delta g)p = gp$.

For the third claim, let $g \in G$ be generic. From what we showed above, it follows that if $f \in G$ is such that its type is the same as that of g, $f^{-1}g \in G^\circ$ (f and g act on S the same way). So by choosing f so that it is in addition generic over $\{g\}$, by Exercise 7.13 (iv), $f^{-1}g$ is a generic element of G°.

For the last claim, we notice that by what we have seen above, if $f, g \in G$ are generic and have the same type, then $f/G^\circ = g/G^\circ$. Also since G° contains a generic element, by Exercise 7.13 (iv), for all $g \in G$ there is generic $f \in G$ such that, $f/G^\circ = g/G^\circ$. Thus the claim follows from Exercise 7.13 (i). □

7.16 Lemma. *If $f,g \in G^\circ$ are generic, then they have the same type.*

Proof. Let $f,g \in G^\circ$ be generic and we need to show that they have the same type. W.o.l.g. we may assume that f is generic over $\{g,g^{-1}\}$. Then by Exercise 7.13 (iv), fg^{-1} is generic over $\{g,g^{-1}\}$. But then g is generic over $\{fg^{-1}\}$ (check) and since $fg^{-1} \in G^\circ$, the type of $(fg^{-1})g$ is the same as the type of g. Since $(fg^{-1})g = f$, the claim follows. □

7.17 Exercise.

(i) Show that G° has unique generics i.e. if $g,f \in G^\circ$ are generic over B, $A \subseteq B \subseteq M^{eq}$ and the cardinality of B is less than the cardinality of M, then there is $\delta \in \Sigma_B$ such that $\delta f = g$.

(ii) Show that if $a = (a_0,...,a_{n-1})$ and $b = (b_0,...,b_{n-1})$ are elements of P^n and generic over A, then there is $f \in G^\circ$ such that for all $i < n$, $fa_i = b_i$.

8 M^{eq} and interpretability of groups

From now on in these notes we will assume that M is ω-stable. We will use this assumption only to be able to use Fact 8.2 below. A proof for Fact 8.2 can be found from [Po] (Corollary 5.19).

8.1 Definition.

(i) We say that $X \subseteq M^{eq}$ is ∞-definable if there are a countable $B \subseteq M^{eq}$ and a type q over B such that X is the set of all realizations of q.

(ii) Suppose $X \subseteq M^{eq}$ is ∞-definable and $R \subseteq X^k$. We say that R is definable inside X if there is a definable R' such that $R = R' \cap X^k$. For functions $f : X^k \to X$, definability inside X is defined similarly.

(iii) We say that an L^*-structure $\mathcal{A}$ is ∞-interpretable in M^{eq} if there is a one-to-one function $F : \mathcal{A} \to M^{eq}$ such that $F(\mathcal{A})$ is ∞-definable, for all $R \in L^*$, $F(R^{\mathcal{A}})$ is definable inside $F(\mathcal{A})$ and for all $f \in L^*$, $F \circ f^{\mathcal{A}} \circ F^{-1}$ is definable inside $F(\mathcal{A})$. As before, we say that a structure is ∞-definable in $\mathcal{A}$ if id witnesses that it is interpretable in $\mathcal{A}$.

8.2 Fact. *Every group ∞-definable in M^{eq} is definable in M^{eq}. In particular, every group ∞-interpretable in M^{eq} is interpretable in M^{eq}.*

Now with the help of Fact 8.2, we will show that G° is interpretable in M^{eq}.

8.3 Exercise. *Show that there is a formula $\phi(x,y,z,d)$, d is a finite sequence of elements of A^*, such that for all semicodes a of generic elements $f \in G^\circ$ and $b \in P$ generic over Aa the following holds: for all $c \in M^{eq}$, $M^{eq} \models \phi(c,b,a)$ iff $fb = c$.*

We let P^* be the set of all semicodes of generic elements of G°. We let E be a binary relation on M^{4n+4} such that for all $(a,b),(a',b') \in (M^{4n+4})^2$ the following holds: $(a,b)E(a',b')$ holds if $a = a'$ and $b = b'$ or for all $c \in P$ generic over $Aa a' b b'$, we have: there are some $d,d',e,e' \in P$ such that

$$M^{eq} \models \phi(d,c,a) \wedge \phi(d',c,a') \wedge \phi(e,d,b) \wedge \phi(e',d',b'),$$

and for any such elements $d, d', e, e' \in P$, $e = e'$. Notice that E is an equivalence relation on M^{4n+4}. By Lemma 5.17 E is definable over A^* in M^{eq}. By Exercise 7.2 (v), E is definable already in M.

Keeping in mind Exercise 7.2 (vi), we let $P^\circ = (P^*)^2/E \subseteq M^{eq}$.

8.4 Exercise.

(i) Show that there is a type over A^* such that P° is the set of all realizations of p. Hint: Suppose $q(x,y)$ is a type. If for all all finite $q' = \{\phi_i(x,y) | \, i < m\} \subseteq q$, a realizes $\exists x \wedge_{i<m} \phi_i(x,y)$, then there is b such that (b,a) realizes q.

(ii) Show that for all $f \in G^\circ$ there are generic $g, h \in G^\circ$ such that $f = gh$.

(iii) Show that if $(a_i, b_i) \in (P^*)^2$, $i < 2$, $(a_0, b_0)E(a_1, b_1)$ and a_i is a semicode of f_i and b_i is a semicode of g_i, then $f_0 g_0 = f_1 g_1$. Hint: Exercise 7.13 (v).

(iv) Show that there is $R \subseteq (M^{eq})^3$ definable over A^* such that for all $c_i = (a_i, b_i)/E \in P^\circ$, $i < 3$, if a_i is a semicode of f_i and b_i is a semicode of g_i, then $(c_0, c_1, c_2) \in R$ iff $f_0 g_0 f_1 g_1 = f_2 g_2$. Hint: Using the idea from (i), show first that some such R is ∞-definable. Then notice that for all c_0 and c_1 from P° there is exactly one $c_2 \in P^\circ$ such that $(c_0, c_1, c_2) \in R$. Now use the saturation of M^{eq} to find the formula that defines (good enough) R (i.e. not necessarily exactly the same R).

8.5 Theorem. *G° is interpretable in M^{eq}.*

Proof. From Exercise 8.4 it follows immediately that G° is ∞-interpretable (let F from Definition 8.1 (iii) be: $F(f)$ is $(a,b)/E$ for any semicodes a of $g \in G^\circ$ and b of $h \in G^\circ$ such that g and h are generic and $f = gh$). Thus by Fact 8.2, G° is interpretable im M^{eq}. □

8.6 Exercise. *Suppose H' is a subgroup of H and that both are definable in M^{eq}. Show that if the cardinality of H/H' is strictly less than the cardinality of M, then H/H' is finite. Hint: See the hint for Exercise 7.2 (iv).*

9 The case n=1 and corollaries

If $n = 1$, we define a closure operation cl on G° as follows: For $f \in G^\circ$ and $X \subseteq G^\circ$, $f \in cl(X)$ if there are some $g_1, ..., g_k \in X$, $k \in \mathbb{N}$, and $a \in P$ generic over $\{f, g_1, ..., g_k\}$ such that $fa \in acl^P_A(\{a, g_1 a, ..., g_k a\}$.

9.1 Exercise. *Suppose $n = 1$.*

(i) Show that cl is a pregeometry on G°.

(ii) Show that (G°, cl) is a group carrying an ω-homogeneous pregeometry.

9.2 Theorem. *If $n = 1$, then G° is commutative and the action of G° on P is regular.*

Proof. The claim that G° is commutative follows immediately from Exercise 9.1, Theorem 8.5 and Theorem 6.4. So we prove that the action is regular. Since the action is transitive, it suffices to show that if $f, g \in G^\circ$ and $fx = gx$ for some $x \in P$, then $f = g$. Notice that the proof for this does not use the assumption that (P, acl^P_A) is a geometry (in fact it does not use acl^P_A at all). This is important because later we often need a similar fact in the case when the pregeometry is not a geometry and there we want to say just that the proof goes as the one in here, see also Remark 9.3 below.

Clearly it suffices to show that if $f \in G^\circ$ is such that $fx = x$ for some $x \in P$, then $f = id$. So for a contradiction, suppose that there are $f \in G^\circ$ and some $x, y \in P$ such that $x \neq y$, $fx = x$ but $fy \neq y$. Since G° acts transitively on P, there is $g \in G^\circ$ such that $gx = y$. Then obviously $(fgf^{-1})x = fy \neq y$. But by commutativity, $fgf^{-1} = g$, a contradiction. □

9.3 Exercise. *Assume $n = 1$. Show that the action of G° on P is isomorphic with the first of the two possible actions of G° on G° from Example 3.3.*

As corollaries of Theorem 9.2, we will show that the action of G° on P is n-determined and that $n \leq 3$.

Suppose $n > 1$. Let $x = (x_1, ..., x_{n-1}) \in P^{n-1}$ be generic. Let $H = (G^\circ)_x$ and $P' = P - acl_A^P(x)$. Then H acts on P' by the action induces from the action of G° on P (i.e. if π is the action of G° on P, then $f \mapsto \pi(f) \restriction P'$ gives the action of H on P'). Clearly, since the action of G° on P has rank n, see Exercise 7.17 (ii), the action of H is transitive. Also $P' = (P', acl_{Ax}^{P'})$ is a pregeometry, see Exercise 5.3 (iii), and so since the action of G is $n+1$-determined, the action of H on P' is 2-determined. Thus exactly as in the case of G, we can find H°, the connected component of H, and show that it is interpretable in M^{eq} and that H/H° is countable (in fact, from the proof of Corollary 9.5, it follows that $H^\circ = H$). And so, still as in the case of G, H° is commutative and the action of it is regular.

9.4 Remark. *Since there is no guarantee that P' is a geometry but we had assumed that P is a geometry, one may wonder, if it really is so that for H everything goes as for G? It does: So far we have used the assumption that P is a geometry only once, when we showed that the action of G is $n+1$-determined and this does imply that the action of H is 2-determined. In the next section we will analyze the case $n = 2$ and there the assumption is used again. And when we analyze the case $n = 3$, we will use results from the case $n = 2$, again by way of a localization, and there we need to show that the localization is indeed a geometry. Another difference is that P is definable but P' need not be. But this was not used in the analysis of G° (check), e.g. in the proof of ∞-interpretability of G°, everything was based on Lemma 5.17 (and uniqueness of generics) and P' inherits the property from P.*

9.5 Corollary. *The action of G° on P is n-determined, i.e. if $y = (y_i)_{i<n} \in P^n$ is generic and $f, g \in G^\circ$ are such that $fy_i = gy_i$ for all $i < n$, then $f = g$.*

Proof. In the case $n = 1$, we have already shown this so we assume that $n > 1$. Choose $h \in H^\circ$ so that for some $(a_0, b_0) \in P$ generic over x (i.e. a generic pair of elements of P') $ha_0 = b_0$ (h exists since the action of H° is transitive). Notice that since the action of H° on P' is regular, for all $y \notin acl_A^P(x)$, $hy \neq y$ (i.e. the dimension of the set $\{y \in P |\ hy = y\}$ is $n - 1$).

Now choose $a_i \in P$, $0 < i < n$, so that letting $b_i = ha_i$, for all $0 < i < n$,

$$a_i \notin acl^P_{Ax}(\{a_j, b_j |\ j < i\}).$$

9.5.1 Claim. *h is a generic element of G°.*

Proof. It suffices to show that $dim^P_{acl^P_A}(\{a_i, b_i |\ i < n\}) = 2n$. If not, then since

$$dim^P_{acl^P_A}(\{x_1, ..., x_{n-1}\} \cup \{a_i, b_i |\ i < n\}) \geq$$

$$dim^P_{acl^P_A}(\{x_1, ..., x_{n-1}, a_0, b_0\}) + n - 1 = n + 1 + n - 1 = 2n,$$

there is $0 < j < n$ such that $x_j \notin acl^P_A(\{a_i, b_i |\ i < n\})$. Choose $c \in P - acl^P_A(\{x_j\} \cup \{a_i, b_i |\ i < n\})$. By Exercise 7.10 (ii), letting $d = hc$, $x_j \notin acl^P_A(\{c, d\} \cup \{a_i, b_i |\ i < n\})$. But then $hy = y$ for all $y \in P$ generic over $\{h\}$ and thus $h = id$, a contradiction. □ Claim 9.5.1.

Now for a contradiction suppose that G° is not n-determined i.e. that there are $x_0 \in P$ and $g \in G^\circ$ such that $(x_i)_{i<n} \in P^n$ is generic (x_i, $0 < i < n$ are as above), $gx_i = x_i$ for all $i < n$ and $g \neq id$. Again, choose $a_i \in P$, $i < n$, so that for all $i < n$, letting $b_j = ga_j$,

$$a_i \notin acl^P_A(\{x_j |\ j < n\} \cup \{a_j, b_j |\ j < i\}).$$

Then

$$dim^P_{acl^P_A}(\{x_0, ..., x_{n-1}\} \cup \{a_i, b_i |\ i < n\}) \geq 2n$$

and so as in the proof of Claim 9.5.1, since $g \neq id$, g must be a generic element of G°. Since G° has unique generics, there is $\delta \in \Sigma$ such that $\delta g = h$. But by the choice of Σ, this is not possible since the sets $\{y \in P |\ gy = y\}$ and $\{y \in P |\ hy = y\}$ have different dimensions. □

9.6 Corollary. $n \leq 3$.

Proof. For a contradiction, assume that $n > 3$. Let $(x_i)_{i<n} \in P^n$ be generic and let $f \in G^\circ$ be such that $fx_0 = x_1$, $fx_1 = x_0$ and $fx_i = x_i$ for $1 < i < n$. Notice that by Corollary 9.5, f is an involution and the set $X = \{x \in P | fx = x\}$ has dimension $\leq n - 1$. Also if $g \in G^\circ$ commutes with f, then for all $1 < i < n$, $g(x_i) \in X$ (otherwise $gx_i \neq (f \circ g \circ f^{-1})x_i$ although $g = f \circ g \circ f^{-1}$).

Now choose $(a_0, a_1) \in P^2$ generic over $X \cup \{x_0, x_1\}$. Let $b_i = fa_i$ for $i < 2$. Then the dimension of $\{x_i |\ i < n\} \cup \{a_i, b_i |\ i < 2\}$ at least $n + 2$ and so the dimension of $\{x_i |\ i < n - 2\} \cup \{a_i, b_i |\ i < 2\}$ must be $n + 2$ since otherwise, as in the proof of Corollary 9.5, either x_{n-2} or x_{n-1} is generic over $\{f\}$ and since $fx_{n-2} = x_{n-2}$ and $fx_{n-1} = x_{n-1}$, $f = id$ (which it is not). But then by Exercise 7.17 (ii), there is $g \in G^\circ$ such that $gx_i = x_i$, for all $i < n - 2$, $ga_0 = a_1$ and $gb_0 = b_1$.

Since f is an involution it is easy to see that letting $h = gfg^{-1}f^{-1}$, $hx_i = x_i$ for all $i < n - 2$, $hb_1 = b_1$ and $ha_1 = a_1$. By Corollary 9.5, $h = id$ i.e. g and f commute. Thus $gx_{n-2} \in X$ and $gx_{n-1} \in X$. So by Exercise 7.10 (ii), the dimension of (a_0, a_1) over $X \cup \{x_0, x_1\}$ is at most 1. But this contradicts the choice of the pair (a_0, a_1). □

10 The case n=2

Through out this section we assume that $n = 2$. Since P was assumed to be a geometry, this means that the action of G° on P is 2-regular (by Corollary 9.5).

The following lemma was proved in Claim 9.5.1 (only for elements of $H^\circ \subseteq G^\circ$ but this was not used in the proof of the claim).

10.1 Lemma. *Suppose $(a, b, c) \in P^3$ is generic and $g \in G^\circ$ is such that $ga = a$ and $gb = c$. Then g is generic.* □

So since G° has unique generics, for every generic g there is $a \in P$ such that $ga = a$. Now fix $0 \in P$ and let $G^\circ_0 = (G^\circ)_0$. Then G°_0 acts on $P - \{0\}$ regularly and thus $G^\circ_0 = (G^\circ_0)^\circ$ i.e. $= H^\circ$, see Section 9, and thus by Section 9, G°_0 is commutative and interpretable in M^{eq}.

10.2 Exercise. *Let $F : G^\circ \to M^{eq}$ be as in the proof of Theorem 8.5. Show that $F \restriction G^\circ_0$ witnesses that G°_0 is interpretable in M^{eq}.*

Now we let I be the set of all involutions of G°. Notice that

(*) for all distinct $b, c \in P$, there is exactly one $f \in I$ such that $fb = c$.

10.3 Lemma. *If $f, g \in I$, then f and fg are not generic elements of G°.*

Proof. For a contradiction, suppose f is generic. Then we can find $b, c \in P$ such that $dim^P_{acl^P_A}(\{b, c, fb, fc\}) = 4$. Choose $\delta \in \Sigma$ so that $\delta \restriction \{b, c, fb\} = id$ and $\delta(fc) \neq fc$. So $\delta f \neq f$ and δf is an involution but $(\delta f)b = fb \neq b$, a contradiction with (*) above.

Again for a contradiction, suppose fg is generic and by Lemma 10.1 find $b \in P$ such that $(fg)b = b$. If $gb \neq b$, then $f = g^{-1}$ and thus $fg = id$, a contradiction. So $gb = b$ and thus $fb = b$. Choose any distinct $c, d \in P - \{b\}$. Since g is not generic, $dim^P_{acl^P_A}(\{b, c, gc\}) = 2$ and similarly for d and thus

$$acl^P_A(\{b, c, d, gc, gd\}) \subseteq acl^P_A(\{b, c, d\}).$$

By repeating the argument for the non-generic element $f \in I$ and distinct $gc, gd \in P - \{b\}$, we get

$$acl^P_A(\{b, gc, gd, (fg)c, (fg)d\}) \subseteq acl^P_A(\{b, gc, gd\}).$$

But then $dim^P_{acl^P_A}(\{b, c, d, (fg)c, (fg)d\}) \leq 3$ and since this holds for all distinct $c, d \in P - \{b\}$ (including (c, d) generic over fg) fg is not generic, a contradiction. □

We let N be the set of all $g \in G^\circ$ such that for all but countably many $f \in I$, $gf \in I$. We say that $f \in I$ is generic over $B \subseteq M^{eq} \cup G$ in I if there is $B \cap M^{eq} \subseteq C \subseteq M^{eq}$ such that every $g \in B \cap G$ has a quasicode in C^6 and f has a quasicode $c \in P^6$ such that $dim^P_{acl^P_A}(c/C) = 4$ (i.e. the largest possible).

10.4 Exercise.

(i) Show that $f \in I$ *is generic over* $B \subseteq M^{eq} \cup G$ *in* I *iff there are* $B \cap M^{eq} \subseteq C \subseteq M^{eq}$ *and* $b \in P$ *such that* C^6 *contains a quasicode of every* $g \in B \cap G$ $(b, fb) \in P^2$ *is generic over* C.

(ii) Show that if $f, g \in I$ *are generic over countable* $B \subseteq M^{eq} \cup G$ *in* I*, then there is* $\delta \in \Sigma$ *such that* $\delta \restriction B = id$ *and* $\delta f = g$*. Hint: Use (*) above.*

(iii) Show that for all $g \in G^\circ$ *the following are equivalent:*

(a) $g \in N$,

(b) for all $f \in I$ *generic over* $\{g\}$ *in* I, $gf \in I$,

(c) for some $f \in I$ *generic over* $\{g\}$ *in* I, $gf \in I$.

10.5 Lemma. *If* $f, g \in N$ *and* $fb = gb$ *for some* $b \in P$*, then* $f = g$.

Proof. Choose $h \in I$ such that both fh and gh belong to I and $c = hb \neq fb$. Then $(fh)c = fb = gb = (gh)c \neq c$ and since $fh, gh \in I$, $fh = gh$ and so $f = g$. □

10.6 Lemma. N *is a normal subgroup of* G°.

Proof. Clearly N is fixed as a set under all automorphisms of G° and thus under all inner automorphism and so if it is a subgroup, it is also normal. So it is enough to show that for all $f, g \in N$, $fg, f^{-1} \in N$.

For $fg \in N$, choose any finite $B \subseteq P$ so that B^6 contains quasicodes of f, g and fg and then choose $c \in P$ so that $c, gc \notin acl_A^P(B)$ (e.g. by the pigeon hole principle). Finally choose $d \in P$ generic over $B \cup \{c, gc\}$. Let $h \in I$ be such that $hd = c$. Then By Exercise 10.4, $gh \in I$ and it is generic over $\{f\}$ in I. Thus $f(gh) \in I$. But $f(gh) = (fg)h$ and h is generic over $\{fg\}$ and so $fg \in N$.

For $f^{-1} \in N$, choose $h \in I$ generic over $\{f, f^{-1}\}$ in I. Then $fh \in I$ and thus $(fh)^2 = id$ i.e. $f^{-1} = hfh$. Since $h = h^{-1}$, $hf^{-1}h = f$ But then $(f^{-1}h)^2 = f^{-1}f = id$ i.e. $f^{-1}h \in I$. Since h is generic over f^{-1} in I, $f^{-1} \in N$ (see Exercise 10.4). □

10.7 Exercise. *For all* $\delta \in \Sigma$, $\delta N = \{\delta f | \ f \in N\} = N$.

10.8 Lemma. *The action of* N *on* P *is regular.*

Proof. By Lemma 10.5, it suffices to show that for any $(a, b) \in P^2$ there is $f \in N$ such that $fa = b$. Since $id \in N$, it suffices to show this in the case $a \neq b$ i.e. (a, b) is generic. For this choose $c \in P - acl_A^P(\{a, b\})$ and let $g, h \in I$ be such that $ha = c$ and $gc = b$. We claim that $f = gh$ is as wanted. Clearly $fa = b$ and since $fh = g \in I$, it suffices to show that h is generic over $\{f\}$ in I.

Choose $(x_1, x_2, y) \in P^3$ generic over $\{a, b, c, h\}$. Since by Lemma 10.3, f is not generic, for $i \in \{1, 2\}$, $fx_i \in acl_A^P(\{a, b, x_1, x_2\})$. So it suffices to show that (y, hy) is generic over $\{a, b, x_1, x_2\}$. If not, then $hy \in acl_A^P(\{a, b, x_1, x_2, y\})$ and since h is not generic by Lemma 10.3, $hy \neq y$ by Lemma 10.1. Then since $\delta h = h$ for all $\delta \in \Sigma$ such that $\delta y = y$ and $\delta(hy) = hy$, $c \in acl_A^P(\{a, b, x_1, x_2, y\})$. But this contradicts the choice of the elements a, b, c, x_1, x_2 and y. □

10.9 Exercise. *N is interpretable in M^{eq}. Hint: Let $H : G^\circ \to M^{eq}$ witness that G° is interpretable in M^{eq}. Show that $H(N)$ in ∞-definable by using Exercise 10.4 (iii). (Alternatively one can use Lemma 10.8 and repeat the argument from Section 9.)*

Now we define a closure operation on N as in Section 9: For $f \in N$ and (finite) $X \subseteq N$, $f \in cl_N(X)$ if there are a finite $Y \subseteq X$ and $a \in P$ generic over $Y \cup \{f\}$ such that $fa \in acl(\{a\} \cup \{ga|\ g \in Y\})$.

10.10 Exercise. *Show that cl_N is well-defined and that (N, cl_N) is a group carrying an ω-homogeneous pregeometry.*

10.11 Corollary. *N is commutative.*

Proof. Immediate by Exercise 10.9 and 10.10 and Theorem 6.4. □

For every $x \in P$, let $f_x \in N$ be such that $f_x 0 = x$. By Lemma 10.8, $x \mapsto f_x$ is a one-to-one function from P onto N. Now pick $1 \in P - \{0\}$. For all $x \in P - \{0\}$, let $g_x \in G_0^\circ$ be such that $g_x 1 = x$. Since G_0° acts regularly on $P - \{0\}$, $x \mapsto g_x$ is a one-to-one function from $P - \{0\}$ onto G_0°.

10.12 Lemma. *For all $x \in P - \{0\}$ and $g \in G_0^\circ$, $g = g_x$ iff $f_1^g (= g f_1 g^{-1}) = f_x$.*

Proof. By Lemma 10.6, $f_1^g \in N$. Thus by Lemma 10.8, it suffices to show that $g1 = x$ iff $(f_1^g)0 = x$ (i.e. $f_1^g = f_x$). But $(gf_1g^{-1})0 = (gf_1)0 = g1$ and thus the claim follows. □

Now we can define an addition $+$ and multiplication $\times$ on N as follows: The addition is the group operation of N. Also for all $f \in N$, $f_0 \times f = f \times f_0 = f_0$. And finally, for $x, y \in P - \{0\}$, $f_x \times f_y = f_y^{g_x}$ $(= f_1^{g_x g_y})$.

10.13 Exercise.

(i) Show that $F_N = (N, +, \times, f_0, f_1)$ is a field.

(ii) Show that $(F_N, (cl_N)_{f_1})$ is a field carrying an ω-homogeneous pregeometry. Conclude that F_N is algebraically closed.

(iii) Show that $g \mapsto f_1^g$ is an isomorphism from G_0° onto $F_N^\times$.

By Lemma 10.6, G_0° acts on N by conjugation and so $G_0^\circ \rtimes N$ can be formed.

10.14 Exercise. *Show that $(f, g) \mapsto (x \mapsto g(fx))$ is an action of $G_0^\circ \rtimes N$ on P (where $f \mapsto (x \to fx)$ is the action of G_0° on P induced from the action of G° on P and similarly for N).*

Let $H : G_0^\circ \rtimes N \to G^\circ$ be such that $H((f, g)) = gf$, $H^* : G_0^\circ \rtimes N \to (F_N)^\times \rtimes (F_N)^+$ be such that $H^*(f, g) = (h, g)$, where $h \in N$ is the unique element such that $h0 = f1$, and $F : P \to N$ be such that $F(x) = f_x$.

10.15 Exercise.

(i) Show that (H, id) witnesses that the actions of $G_0^\circ \rtimes N$ and G° on P are isomorphic.

(ii) Show that (H^*, F) witnesses that the action of $G_0^\circ \rtimes N$ on P and the action of $(F_N)^\times \rtimes (F_N)^+$ on F_N are isomorphic. Conclude that the action of G°

on P and the action of $(F_N)^\times \rtimes (F_N)^+$ on F_N are isomorphic. Hint: $(gf)x = (gff_x)0 = (gff_xf^{-1})0 = (gf_x^f)0$ and if $h \in N$ is such that $h0 = f1$, then $H^*(fg)F(x) = g + hf_x = g + f_x^f = gf_x^f$.

(iii) Show that F_N is definable in M^{eq}.

11 The case n=3 and the conclusion

Excluding the conclusion below, throughout this section we assume that $n = 3$. Fix some $\infty \in P$. As already mentioned in Remark 9.4, the main point here is to show that (misusing the notation a bit) $P_\infty = P - \{\infty\}$ with $acl^{P_\infty}_{A\cup\{\infty\}}$ is a geometry. Because if this is the case, by using the case $n = 2$ and Exercise 4.14, it is easy to work out what is going on. In fact we do this first and then later prove that P_∞ indeed is a geometry.

11.1 Theorem. *Suppose that P_∞ is a geometry. Then there an an algebraically closed field F interpretable in M^{eq} such that the action of G° on P is isomorphic with the action of $PGL_2(F)$ on $P_1(F)$.*

Proof. We let Σ_∞ be $Aut(M^{eq}/acl^{eq}(A \cup \{\infty\}))$. Denote $G^\circ_\infty = (G^\circ)_\infty$. Now we can define $(G^\circ_\infty)^\circ$ using Σ_∞ as G° was defined in Section 7. Since P_∞ is a geometry, G°_∞ acts 2-regularly on P_∞ and so $(G^\circ_\infty)^\circ = G^\circ_\infty$. Now, again since P_∞ is a geometry, we are exactly in the same situation as in the case $n = 2$, when P is replaced by P_∞, G° is replaced by G°_∞ and Σ is replaced by Σ_∞. Thus we can find $0, 1 \in P_\infty$, a normal subgroup N_∞ of G°_∞ and the subgroup $G^\circ_{\infty 0} = (G^\circ_\infty)_0 = (G^\circ)_{\{\infty,0\}}$ of G°_∞ and go on defining the addition and multiplication on N_∞ to get an algebraically closed field F_{N_∞} such that it is interpretable in M^{eq} and the action of G°_∞ on P_∞ is isomorphic with the action of $(F_{N_\infty})^\times \rtimes (F_{N_\infty})^+$ on F_{N_∞}. Let (H, H') be the isomorphism (keep in mind that $H'(x)$ is the unique $f_x \in N_\infty$ such that $f_x 0 = x$ and for $g \in G^\circ_{\infty 0}$ and $h \in N_\infty$, $H(hg) = (f, h)$ where $f \in N_\infty$ is the unique element such that $f0 = g1$). So by Exercise 4.14 (v), if we identify $x \in P_\infty$ with $H'(x)$, it is enough to find $\alpha \in G^\circ$ such that $\alpha 0 = \infty$, $\alpha\infty = 0$, $\alpha 1 = 1$ and

(*) for all $x \in P_\infty - \{0\}$, $\alpha x = x^{-1}$.

Since the action of G° on P is 3-regular, there is unique $\alpha \in G^\circ$ such that $\alpha 0 = \infty$, $\alpha\infty = 0$ and $\alpha 1 = 1$. We show that this α satisfies (*). Notice that α is an involution.

To prove (*), it suffices to show that for all $g \in G^\circ_{\infty 0}$, $g^\alpha = g^{-1}$: Clearly, for all $g \in G^\circ_{\infty 0}$, $g^\alpha \in G^\circ_{\infty 0}$ and since $g^\alpha_x 1 = (\alpha g_x)1 = \alpha x$, $g^\alpha_x = g_{\alpha x}$ and so if $g_{\alpha x} = g_x^{-1}$, $f_{\alpha x} = f_x^{-1}$ (see Exercise 10.12 (iii)).

Since every element of $G^\circ_{\infty 0}$ is a product of two elements of $G^\circ_{\infty 0}$ both generic over $\{\infty, 0, 1\}$ and $G^\circ_{\infty 0}$ is commutative, it is enough to show that for every $g \in G^\circ_{\infty 0}$ generic over $\{\infty, 0, 1\}$, $g^\alpha = g^{-1}$. But for every $\delta \in \Sigma_\infty$, if $\delta 0 = 0$ and $\delta 1 = 1$, then $\delta\alpha = \alpha$, and thus by the uniqueness of generics of $G^\circ_{\infty 0}$ (i.e. since the action of G°_∞ on P_∞ is 2-regular), it is enough to find one $g \in G^\circ_{\infty 0}$ generic over $\{\infty, 0, 1\}$ such that $g^\alpha = g^{-1}$. An easy calculation shows that $((g^\alpha)g^{-1})^\alpha = ((g^\alpha)g^{-1})^{-1}$ (exercise, keep in mind that $G^\circ_{\infty 0}$ is commutative and that α is an involution). And so if there is $g \in G^{\circ}_{\infty 0}$ such that $g^{\alpha}g^{-1}$ is generic over $\{\infty, 0, 1\}$, we are done.

So for a contradiction, assume that for all $g \in G^{\circ}_{\infty 0}$, $g^{\alpha}g^{-1}$ is not generic over $\{\infty, 0, 1\}$. As in the previous sections, one can see that $f \in G^{\circ}_{\infty 0}$ is generic over $\{\infty, 0, 1\}$ iff $f1$ is generic over $\{\infty, 0, 1\}$. Since $f1$ determines f, for all $g, h \in G^{\circ}_{\infty 0}$ generic over $\{\infty, 0, 1\}$,

(**) $g^{\alpha}g^{-1} = h^{\alpha}h^{-1}$

(if $g \in G^{\circ}_{\infty 0}$ is generic over the set $\{\infty, 0, 1\}$, then it is generic also over the set $\{\infty, 0, 1, (g^{\alpha}g^{-1})1\}$). Choose these so that in addition $g^{-1}h$ is generic over $\{\infty, 0, 1\}$. An easy calculation shows that (**) implies that $(g^{-1}h)^{\alpha} = g^{-1}h$. Since $g^{-1}h$ is generic, $(g^{-1}h)1$ is generic over $\{0, 1, \infty\}$ and thus because $\alpha \neq id$, $(g^{-1}h)^{\alpha}1 = (\alpha g^{1}h)1 = \alpha((g^{-1}h)1) \neq (g^{-1}h)1$, a contradiction. □

To prove the following theorem we need to go through much of the theory from Section 10 without the assumption that the pregeometry is a geometry.

11.2 Theorem. *P_{∞} is a geometry.*

Proof. By Exercise 5.3, P_{∞} is a pregeometry and since P is a geometry, $acl^{P_{\infty}}_{A\infty}(\emptyset) = \emptyset$. Thus it suffices to show that for all $a \in P_{\infty}$, $acl^{P_{\infty}}_{A\infty}(\{a\}) = \{a\}$. Clearly it is enough to show this for some $a \in P_{\infty}$.

We denote by cl the closure operation $acl^{P_{\infty}}_{A\infty}$ and we write $dim(x_1, ..., x_n)$ for $dim^{P_{\infty}}_{cl}(\{x_1, ..., x_n\})$. We denote $H = G^{\circ}_{\infty}$ and notice the following:

11.2.1 Exercise.

(i) If the pairs $(a, b), (c, d) \in (P_{\infty})^2$ have dimension 2, then there is $g \in H$ such that $ga = c$ and $gb = d$.

(ii) Suppose $(a, b) \in (P_{\infty})^2$ has dimension 2 and $fa = ga$ and $fb = gb$ for $f, g \in H$, then $f = g$.

Fix $0 \in P_{\infty}$. In the proof of Corollary 9.5 we showed the following:

11.2.2 Claim. *Suppose $b \notin cl(\{0\})$ and $g \in H$ is such that $g0 = 0$ and $gb \notin cl(\{b, 0\})$. Then g is a generic element of H i.e. there are $x_1, x_2 \in P_{\infty}$ such that $dim(x_1, x_2, gx_1, gx_2) = 4$.* □

11.2.3 Claim. *If $g \in H$ is generic and $g0 = 0$, then $ga = a$ for all $a \in cl(\{0\})$.*

Proof. By Exercise 11.2.1 and Claim 11.2.2, it is enough to find $b, c \in P_{\infty}$ and $h \in H$ such that $dim(0, b, c) = 3$, $h0 = 0$, $hb = c$ and $ha = a$ for all $a \in cl(\{0\})$. Fix $e \notin cl(\{0\})$ and for all $d \notin cl(\{0, d\})$ let $g_d \in H$ be such that $g_d 0 = 0$ and $g_d e = d$. By the pigeon hole principle and ω-stability, there are $d, d' \in P_{\infty}$ such that $dim(d, d', e, 0) = 4$ and the actions of g_d and $g_{d'}$ on $cl(\{0\})$ are the same (exercise, hint: if x and y are semicodes of g and f and $t(x/cl(\{0\})) = t(y/cl(\{0\}))$, there is $\delta \in \Sigma$ such that $\delta(x) = \delta(y)$ and $\delta \restriction cl(\{0\}) = id$ and then for $a \in cl(\{0\})$, $fa = \delta g \delta^{-1} a = ga$). Then $g = g_d g^{-1}_{d'}$, $b = d'$ and $c = d$ are as wanted. □ Claim 11.2.3.

11.2.4 Claim. *If $g \in H$, $a \in P_{\infty} - cl(\{0\})$, $g0 = 0$ and $ga \in cl(\{a\}) - \{a\}$, then g is generic.*

Proof. Again choose $x_1, x_2 \in P_\infty$ so that $dim(0, a, x_1, x_2) = 4$ and for a contradiction suppose $dim(x_1, x_2, gx_1, gx_2) \leq 3$. If $0 \notin cl(\{x_1, x_2, gx_1, gx_2\})$, then $g = id$, a contradiction. So $a \notin cl(\{x_1, x_2, gx_1, gx_2\})$. Thus if $y_1, y_2 \in P_\infty$ are such that $dim(y_1, y_2, x_1, x_2, gx_1, gx_2) = dim(x_1, x_2, gx_1, gx_2) + 2$, then $dim(y_1, y_2, gy_1, gy_2) = 2$. But then $0 \notin cl(\{y_1, y_2, gy_1, gy_2\})$ and so $g = id$, a contradiction. □ Claim 11.2.4.

As before, we let I_∞ be the set of all involutions of H. Exactly as in Section 10, one can prove the following claim.

11.2.5 Claim. *If $dim(a, b) = 2$, then there is a unique $g \in I_\infty$ such that $ga = b$. Furthermore, if $f, g \in I_\infty$, then f and fg are not generic elements of H.* □ *Claim 11.2.5.*

Again we let N_∞ to be the set of all $g \in H$ such that for all but countably many $h \in I$, gh is an involution. Still essentially as in Section 10, one can show the following claim.

11.2.6 Claim.
(i) For all $g, h \in N_\infty$ and $a \in P_\infty$, if $ga = ha$, then $g = h$.
(ii) N_∞ is a normal subgroup of H.
(iii) For all $a, b \in P_\infty$ there is $g \in N_\infty$ such that $ga = b$ (and so the action of N_∞ on P_∞ is regular). □ *Claim 11.2.6.*

11.2.7 Claim. *Suppose $a \in cl(\{0\}) - \{0\}$ and $b \notin cl(\{0\})$. Let $f \in N_\infty$ be such that $f0 = a$ and $h \in I_\infty$ be such that $ha = b$. Then $fb \in cl(\{b\}) - \{b\}$, $fh = hf$ and $f \in I_\infty$.*

Proof. It is easy to see using the definition of N_∞ and Claim 11.2.6, that $fh \in I_\infty$. Thus $f = hf^{-1}h$ (since h and fh are involutions, $fh = (fh)^{-1} = h^{-1}f^{-1} = hf^{-1}$) and so $fb = (hf^{-1})a$. Since $a \in cl(\{0\})$, $f^{-1}a \in cl(\{a\})$ and so $fb = hf^{-1}a \in cl(\{b\})$. By Claim 11.2.6 again, $fb \neq b$ (since otherwise $f = id$) and we have prove the first of the three claims.

11.2.7.1 Exercise. *Show that for $fh = hf$, it is enough to show that $fg = gf$ for any $g \in H$ such that $dim(a, b, ga, gb) = 4$. Conclude that it is enough to show this for some such g. Hint: $(gh)f = f(gh) = (gf)h$.*

Let g, g' be as g in Exercise 11.2.7.1. As above, $f(ga) \in cl(\{ga\})$ and thus $(g^{-1}fg)a \in cl(\{a\})$ and similarly for g'. So by the pigeon hole principle, we can find these so that in addition $dim(a, b, (g'g^{-1})a, (g'g^{-1})b) = 4$ and $(g^{-1}fg)a = (g'^{-1}fg')a$. Then by Claim 11.2.6, $g^{-1}fg = g'^{-1}fg'$ and so $g'g^{-1}$ commutes with f and we have proved the second claim.

Finally, $id = (fh)(fh) = f^2h^2 = f^2$ and so $f \in I_\infty$. □ Claim 11.2.7.

Now we are ready to show that P_∞ is a geometry. As pointed out in the beginning of this proof, it is enough to show that $cl(\{0\}) = \{0\}$. For a contradiction, suppose that this is not the case. Then there is $a \in cl(\{0\}) - \{0\}$. By Claim 11.2.6, there is $f \in N_\infty$ such that $f0 = a$. Pick $b \notin cl(\{0\})$ (and choose $h \in I_\infty$ such that $ha = b$). Then by Claim 11.2.7, $f \in I_\infty$ and $fb \in cl(\{b\}) - \{b\}$. Since $fb \notin cl(\{0\})$, there is $g \in H$ such that $gb = fb$ and $g0 = 0$. By Claim 11.2.4,

g is a generic element of H and thus it fixes $cl(\{0\})$ pointwise by Claim 11.2.3. Thus $(gfg^{-1})0 = f0$. Since N_∞ is normal, $gfg^{-1} \in N_\infty$. So by Claim 11.2.6, $gfg^{-1} = f$ i.e. f and g commute. So $g^2b = g(fb) = f(gb) = f^2b = b$. Thus $g^2 = id$ (since also $g^2 0 = 0$). So $g \in I_\infty$, which contradicts Claim 11.2.5. ▫

So we have proved:

11.3 Conclusion. *Suppose M is an uncountable saturated ω-stable structure (in a countable vocabulary) and $P \subseteq M^m$, $m \in \mathbb{N}$, and $Q \subseteq M$ are infinite sets definable over finite $A \subseteq M$ such that*

(i) P is strongly minimal over A and (P, acl_A^P) is a geometry,

(ii) there is $n \in \mathbb{N} - \{0\}$ such that

(a) for all generic $a \in P^n$ and finite $B \subseteq Q$, $dim^P_{acl_A^P}(a/B) = n$,

(b) for some generic $a \in P^{n+1}$ and finite $B \subseteq Q$, $dim^P_{acl_A^P}(a/B) \le n$.

Let $G = Aut(M/QA)/K$, where K is the kernel of the natural action of the group $Aut(M/QA)$ on P and let G° be the connected component of G. Then $n \le 3$ and G° is interpretable in M^{eq}. In addition,

- *if $n = 1$, then G° is commutative and the action of G° on P is regular,*
- *if $n = 2$, then there is an algebraically closed field F interpretable in M^{eq} such that the action of G° on P is isomorphic to the action of $F^\times \rtimes F^+$ on F,*
- *if $n = 3$, then there is an algebraically closed field F interpretable in M^{eq} such that the action of G° on P is isomorphic to the action of $PGL_2(F)$ on $P_1(F)$.*

As pointed out in Section 1, the assumption that M is uncountable and saturated is to a large extension without loss of generality. We demonstrate this by giving one easy consequence of Conclusion 11.3 to the case when the model is not assumed to be saturated nor uncountable.

Suppose $A \subseteq \mathcal{A}$ is finite and $P \subseteq \mathcal{A}^m$ is definable over A. For an elementary extension $\mathcal{B}$ of $\mathcal{A}$, we write $P^{\mathcal{B}}$ for the set $\phi(\mathcal{B}, c)$, where $\phi(x, c)$ is any formula that defines P in $\mathcal{A}$ ($P^{\mathcal{B}}$ does not depend on the choice of $\phi(x, c)$, exercise). For the following fact, see e.g. the lecture notes on model theory.

11.4 Fact. *If $\mathcal{A}$ is an infinite ω-stable structure, then it has a saturated uncountable elementary extension (which by the definition is ω-stable).*

11.5 Exercise. *Suppose $\mathcal{A} \preceq \mathcal{B}$, $P \subseteq \mathcal{A}^m$ is strongly minimal over finite $A \subseteq \mathcal{A}$, $B \subseteq P$ is finite and $a \in P$. Then the following hold:*

(i) $a \in acl_A^P(B)$ iff $a \in acl_A^{P^{\mathcal{B}}}(B)$.

(ii) (P, acl_A^P) is a pregeometry.

(iii) If $Q \subseteq \mathcal{A}$ is definable over A and there is finite $C \subseteq Q^{\mathcal{B}}$ such that $a \in acl_{AC}^{P^{\mathcal{B}}}(B)$, then there is finite $D \subseteq Q$ such that $a \in acl_{AD}^P(B)$.

11.6 Exercise. *Suppose $\mathcal{A}$ is an infinite ω-stable structure (in a countable vocabulary) and $P \subseteq \mathcal{A}^m$, $m \in \mathbb{N}$, and $Q \subseteq \mathcal{A}$ are infinite sets definable over finite $A \subseteq \mathcal{A}$ such that*

(i) P is strongly minimal over A and (P, acl_A^P) is a geometry,

(ii) there is $n \in \mathbb{N} - \{0\}$ such that

(a) for all generic $a \in P^n$, $dim^P_{acl^P_A}(a/Q) = n$,

(b) for some generic $a \in P^{n+1}$, $dim^P_{acl^P_A}(a/Q) \leq n$.

Then $n \leq 3$ and

- *if $n = 1$, then a commutative group is interpretable in $\mathcal{A}^{eq}$,*
- *if $n \geq 2$, then an algebraically closed field interpretable in $\mathcal{A}^{eq}$.*

Hint: First show by using Fact 11.4 and Exercise 11.5, that one can find an elementary extension M of $\mathcal{A}$ such that the the assumptions of Conclusion 11.3 hold for P^M and Q^M. The by going through the proofs, one can see that all the interpretable objects are interpretable over $A \cup \{\infty, 0, 1\}$ and that one can choose ∞, 0 and 1 so that they belong to $\mathcal{A}$. Then just apply the fact that $\mathcal{A}$ is an elementary submodel of M.

12 On local modularity

In this section we sketch a proof of a special case of a result by B. Zilber from [Zi]. Our proof uses ideas also from [Hr2] and [Hy2]. The result is only a special case because again we want to avoid the use of stability theory.

12.1 Assumptions. *We assume that the vocabulary L is countable and that M satisfies the following:*

(i) M is uncountable, strongly minimal and (M, acl) is a geometry.

(ii) M is ω-categorical i.e. all countable elementary submodels of M are isomorphic (and thus by Ryll-Nardzewski, see the lecture notes on model theory, $acl(A)$ is finite for all finite $A \subseteq M$, exercise).

(iii) M has the elimination of imaginaries i.e. for all $a \in M^n$, $n \in \mathbb{N}$, and (finite) B there is $e = e^a_B \in acl(B)^m$, $m \in \mathbb{N}$, called a canonical basis of $p = t(a/acl(B))$ (denoted also $cb(p)$), such that it depends only on $t(a/acl(B))$ and

(a) $a \downarrow_e B$ (i.e. $dim^M_{acl}(a/B) = dim^M_{acl}(a/e)$),

(b) $e = acl(e)$ (for convenience),

(c) $t(a/e)$ is stationary i.e. for all (finite) $B \subseteq C \subseteq M$ and b, if $t(b/e) = t(a/e)$, $a \downarrow_e C$ and $b \downarrow_e C$, then $t(b/C) = t(a/C)$,

(d) if $B' \subseteq M$ is such that, $a \downarrow_B B'$ and $a \downarrow_{B'} B$, then $cb(t(a/acl(B'))) = cb(t(a/acl(B)))$.

12.2 Fact. *If L is countable, then Assumptions 12.1 (i) and (ii) imply that Assumptions 1.16 are satisfied.*

Assumption 12.1 (iii) is made just to avoid the use of stability theory, suitable sequences e can always be found from M^{eq}.

12.3 Exercise. *Let V be an uncountable vector space over a finite field F (i.e. for some uncountable I, the universe of V consists of all $f : I \to F$ such that $\{i \in I |\ f(i) \neq 0\}$ is finite, $(f+g)(i) = f(i)+g(i)$ and $(\lambda f)(i) = \lambda f(i)$ for all $\lambda \in F$). Show that excluding the requirement that (M, acl) is a geometry (but see Example 12.5), V satisfies Assumptions 12.1. Hint: Let $cb(t(a/acl(A)))$ be $span(a) \cap span(A)$.*

12.4 Fact. *Algebraically closed fields satisfy the assumption (iii) above if in (iii), $e \in acl(B)^m$ is replaced by $e = acl(e')$ for some $e' \in acl(B)^m$ (in algebraically closed fields $acl(e)$ is not finite even if e is, of course).*

Also Assumption 12.1 (i) is 'w.o.l.g.':

12.5 Exercise. *Suppose M satisfies Assumptions 1.16, is ω-categorical and $P \subseteq M^n$ is strongly minimal over finite $A \subseteq M$. Let $P^*/\sim \subseteq M^{eq}$ and $(acl_A^P)^*$ be as in Exercise 5.3 (iv). Show that there is M' that satisfies Assumptions 12.1 (i) and (ii), the universe of M' is $P^*/\sim$ and $acl_\emptyset^{M'} = (acl_A^P)^*$. Hint: Code the types over A that are realized in $P^*/\sim$ as predicates and keep in mind Ryll-Nardzewski.*

12.6 Definition.

(i) We say that a pregeometry (S, cl) is modular if $dim(AB) = dim(A) + dim(B) - dim(cl(A) \cap cl(B))$ for all finite $A, B \subseteq S$.

(ii) We say that a pregeometry (S, cl) is locally modular if for some $a \in S$, $(S, cl_{\{a\}})$ is modular.

12.7 Exercise.

(i) Let F be a field and $V = V_n(F)$, $n > 2$, an n-dimensional vector space over F. We say that $A \subseteq V$ is an affine subspace if there are a subspace $S \subseteq V$ and $a \in V$ such that $A = \{x + a |\ x \in S\}$. We define a closure operation cl on V as follows: $cl(\emptyset) = \emptyset$ and for non-empty $X \subseteq V$, we let $cl(X)$ be the least affine subspace that contains X. Show that cl is well-defined (i.e. the least affine subspace exists) and that (V, cl) is a geometry that is locally modular but not modular.

(ii) Let (V, cl) be as in (i) above. Describe $(V/\sim, (cl_{\{0\}})^*)$ (see Exercise 5.3 (iv)).

(iii) Show that a pregeometry (S, cl) is locally modular iff the geometry $(S^*/\sim, cl^*)$ is locally modular.

12.8 Fact. *If F is an algebraically closed field, then (F, acl) is not locally modular.*

The result of which proof we will sketch is the following:

12.9 Theorem. *Under Assumptions 12.1, (M, acl) is locally modular.*

Proof. (Sketch) For a contradiction, we suppose that $(P, acl_\emptyset^P)$ is not locally modular.

12.9.1 Definition. *Suppose $Q \subseteq M^n$, $n \in \mathbb{N}$, is strongly minimal over finite $A \subseteq M$. We say that Q is k-pseudolinear if for all $B \subseteq Q$ and $a, b \in Q - acl_A^Q(B)$ such that $dim_{acl_A^Q}^Q(ab/B) = 1$,*

$$dim_{acl}^M(cb(t(ab/acl(AB)))/A) \leq k.$$

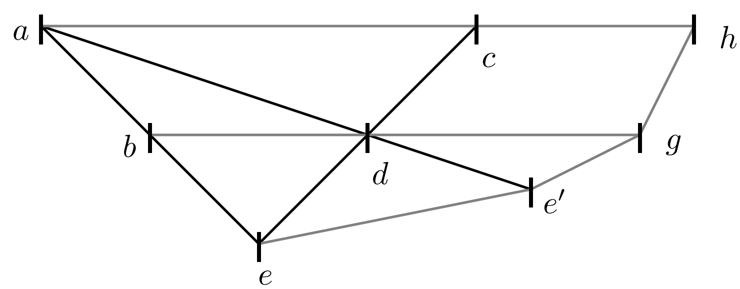
The proofs of the following facts can be found from [Pi].

12.9.2 Facts.

(i) If $Q \subseteq M^n$, $n \in \mathbb{N}$, *is strongly minimal over finite* $A \subseteq M$, *then* Q *is* 2*-pseudolinear.*

(ii) (M, acl) *is not* 1*-pseudolinear (*1*-pseudolinearity implies local modularity).*

So we can find $e \in M^m$, $m \in \mathbb{N}$, and $a, b \in M - e$ such that $dim^M_{acl}(ab/e) = 1$, $e = cb(t(ab/e))$ and $dim^M_{acl}(e) = 2$. Now choose e' so that $t(e'/\emptyset) = t(e/\emptyset)$ and $e' \downarrow_{\emptyset} eab$. Choose $d \in M$ so that $t(ade'/\emptyset) = t(abe/\emptyset)$ and finally choose $c \in M$ so that $t(cd/e) = t(ab/e)$.



Let $e^* = cb(t(abcd/acl(ee')))$ and notice that for any $x \in \{a, b, c, d\}$,

$$dim^M_{acl}(xee') = dim^M_{acl}(abcdee') = 5,$$

$dim^M_{acl}(ee') = 4$ and $dim^M_{acl}(e^*/e) = dim^M_{acl}(e^*/e') = 2$ in fact $e^* = acl(ee')$ (exercise).

12.9.3 Claim. *cd is definable from* abe^*.

Proof. Let $c'd' \in M$ be such that $t(c'd'/abe^*) = t(cd/abe^*)$. By saturation of M, it suffices to show that $d' = d$ and $c' = c$. We prove first that $d' = d$. Since (M, acl) is a geometry, it suffices to show that $d' \in acl(d)$.

Let $g = cb(t(bd/e^*)) \subseteq e^* = acl(ee')$. We start by some remarks on g: First, $a \in acl(ge'b)$ and so $e \subseteq acl(ge')$. Thus $dim^M_{acl}(g/e') = 2$. By 2-pseudolinearity, $dim^M_{acl}(g) \le 2$ and so $dim^M_{acl}(g) = 2$ and $g \downarrow_{\emptyset} e'$.

Now for a contradiction, suppose $dim^M_{acl}(d'd) = 2$. Since $d, d' \in acl(e'a)$, $dim^M_{acl}(e'/dd') = 1$. Similarly, by the remarks above, $dim^M_{acl}(g/dd') = 1$. Thus $dim^M_{acl}(ge'dd') \le 4$. But $acl(ge'dd')$ contains e and thus $dim^M_{acl}(ge'dd') = 5$, a contradiction.

Now to see that $c' = c$, let $h = cb(t(ab/e^*))$. Exactly as with g and e', we can see that $dim^M_{acl}(h) = 2$ and $h \downarrow_{\emptyset} e$. Thus exactly as above, if $dim(cc') = 2$, $dim(h/cc') = 1$. Since $d' = d$, $c' \in acl(ed)$ and so as above, also $dim(e/cc') = 1$. Thus $dim(hecc') \le 4$ which is impossible since $acl(he)$ contains e'. □ Claim 12.9.3

Similarly:

12.9.4 Claim. *ab is definable from cde^*.* □

We let D be the set of realizations of $t(ab/e)$, where a, b and e are as above. Since $acl(A)$ is finite for all finite $A \subseteq M$, D is definable over e. By Assumption 12.1 (iii) (c), D is strongly minimal.

We let G' be the set of all triples (a, b, e^*) such that $a, b \in D$, b is definable from ae^* and a is definable from be^*, $e^* = cb(t(ab/e^*))$, $e \subseteq e^*$ and $dim^M_{acl}(ab/e^*) = 1$. Notice that for all $(a, b, e^*) \in G'$, $dim(e^*/e) \leq 2$ by Fact 12.9.2 (i). Let $\sim$ be the following equivalence relation on G': as sets $e^*_0 = e^*_1$ and $t(a_0b_0/e^*_0) = t(a_1b_1/e^*_0)$. And let $G = G'/\sim$. For simplicity we write (a, b, e^*) also for $(a, b, e^*)/\sim$.

We say that $c \in D^n$ is generic over AB, where $A = \{(a_i, b_i, e^*_i) | \ i \in I\} \subseteq G'$ and $B \subseteq M^{eq}$, if $dim^D_{acl^D_e}(c/B \cup \bigcup_{i \in I} e^*_i) = n$. Now G 'acts generically' on D as follows: If $g = (a, b, e^*) \in G$ and $c \in D$ is generic over $\{g\}$, then gc is the unique $d \in D$ such that $t(cd/e^*) = t(ab/e^*)$.

12.9.5 Exercise. *Show that if $c \in D$ is generic over $\{g, g'\} \subseteq G$ and $gc = g'c$, then $g = g'$.*

Now we define a group structure on G according to this 'action' i.e. $gh = f$ if for some (any) $c \in D$ generic over $\{g, h, f\}$, $g(hc) = fc$.

12.9.6 Exercise. *Show that G is a group.*

We say that $g = (a, b, e^*) \in G$ is generic over $A \subseteq M^{eq} \cup G$ if $dim^M_{acl}(e^*/A') = 2$ where $A' = (A \cap M^{eq})e \cup \bigcup_{(a',b',e') \in G \cap A} e'$ (keep in mind that $dim^M_{acl}(e^*/e) \leq 2$ always). Now as in Section 7 we can find a connected component G° of G so that if both $g = (a_0, b_0, e^*_0)$ and $h = (a_1, b_1, e^*_1) \in G^\circ$ are generic over finite $A \subseteq M^{eq}$, then there is $F \in Aut(M^{eq}/Aacl^{eq}(e))$ such that $F(e^*_0) = e^*_1$ and $t(F(a_0)F(b_0)/e^*_1) = t(a_1, b_1/e^*_1)$ i.e. '$F(g) = h$'.

12.9.7 Exercise. *G° is interpretable in M^{eq}. Hint: One can use method from Section 8 but there is also an easy way (keep in mind that $acl(A)$ is finite for all finite $A \subseteq M$).*

12.9.8 Claim.
(i) There are $c, d \in D$ and $g \in G^\circ$ such that $dim^D_{acl^D_e}(c, d, gc, gd) = 4$. Thus if $(c_0, c_1), (d_0, d_1) \in D^2$ are generic over $\emptyset$, then there is $g \in G^\circ$ such that $gc_i = d_i$ for $i < 2$.

(ii) There are no $g = (a, b, e^*) \in G^\circ$ and $(c_i)_{i<3} \in D^3$ such that for all $i < 3$, c_i is generic over g and $dim^D_{acl^D_e}(c_0, c_1, c_2, gc_0, gc_1, gc_2) = 6$.

Proof. (i) Let $g = (a, b, e^*) \in G$ be such that $dim^M_{acl_e}(e^*) = 2$ and $c, d \in D$ such that $dim^D_{acl^D_e}((c, d)/e^*) = 2$. For a contradiction, suppose

$$gd \in acl^D_e(\{c, d, gc\}.$$

Then $(d, gd) \downarrow_{\{e,c,gc\}} e^*$ and thus since $e^* = cb(t((d, gd)/e^*))$, $e^* \in acl(\{e, c, gc\})$. Since $c \downarrow_e e^*$, $dim(e^*/e) \leq 1$, a contradiction.

(ii) For a contradiction, suppose there are such elements. Since $dim^M_{acl_e}(e^*) \leq 2$ and $dim^M_{acl_e}(c_0, c_1, e^*) = dim^M_{acl_e}(c_0, c_1, gc_0, gc_1, e^*)$, $e^* \in acl_e(\{c_0, c_1, gc_0, gc_1\})$. Thus $dim^M_{acl_e}((c_2, gc_2)/e^*) \geq dim^M_{acl_e}((c_2, gc_2)/\{c_0, c_1, gc_0, gc_1\}) = 2$, a contradiction. □ Claim 12.9.8

Now we would like to analyze G° as a group was analyzed in Section 10. But for this we need a real action, not just 'action' i.e. we need to make the generic action total. For this we need to change D and G° a bit.

Let C' be the set of pairs (g, a) such that $g \in G^\circ$ and $a \in D$. Let $\approx$ be an equivalence on C' such that $(g, a) \approx (f, b)$ if for some (any) $h \in G^\circ$ generic over $\{g, a, f, b\}$, $(hg)a \in acl^D_e(\{(hf)b\})$. Let C be the set of all $\approx$-equivalence classes and as before we write (g, a) also for $(g, a)/\approx$. Notice that C is a definable set in M^{eq} (and even definable over e). On C we define a closure operation cl as follows: $(g_n, a_n) \in cl(\{(g_1, a_i) |\ i < n\})$ if for some (any) $h \in G^\circ$ generic over $\{g_i, a_i |\ i < n\}$, $(hg_n)a_n \in acl^D_e(\{(hg_i)a_i |\ i < n\})$. We let G° act on C the obvious way i.e. $f(g, a) = (fg, a)$.

12.9.9 Exercise.

(i) Show that (C, cl) is a well-defined geometry and that for all $a_i \in D$, $i \leq n$, $(id, a_0) \in cl(\{(id, a_i) |\ 1 \leq i \leq n\})$ iff $a_0 \in acl^D_e(\{a_i |\ 1 \leq i \leq n\})$.

(ii) Show that if $a \in D$ is generic over $h \in G^\circ$ and $g \in G^\circ$, then $(g, a) \approx (gh^{-1}, ha)$ (and so if a is generic over g, $(g, a) \approx (id, ga)$).

(iii) Show that the action is well-defined and, indeed, an action.

(iv) Show that for all $(g_i, a_i) \in C$, $i \leq n$, and $h \in G^\circ$,

$$(g_n, a_n) \in cl(\{(g_1, a_i) |\ i < n\})$$

iff

$$((hg_n), a_n) \in cl(\{((hg_1), a_i) |\ i < n\}).$$

(v) Show that the action of G° on C is 2-transitive.

We let K be the set of all $g \in G^\circ$ such that for some (any) $a \in D$ generic over g, $ga \in acl^D_e(\{a\})$. We let $H = G^\circ/K$.

12.9.10 Exercise.

(i) Show that H is a finite normal subgroup of G°, it is interpretable in M^{eq} and if $(f, b) = (h, c)$ and $g/K = g'/K$, then $(gf, b) = (g'h, c)$. Hint for the last claim: Using Exercise 12.9.9 (ii), show that we may assume that $f = h$, see also the proof of Claim 12.9.11 below.

(ii) Show that for $f, g \in G^\circ$, and $a \in D$ generic over $\{f, g\}$, $f/K = g/K$ iff $f(id, a) = g(id, a)$.

12.9.11 Claim. *If $x_i = (g_i, a_i) \in C$, $i < 3$, are such that $dim^C_{cl}(\{x_i |\ i < 3\}) = 3$ and $f, h \in G^\circ$ are such that for all $i < 3$, $fx_i = hx_i$, then $f/K = g/K$.*

Proof. Using Exercise 12.9.9 (ii), it is easy to see that then w.o.l.g we may assume that for all $i < j < 3$, $g_i = g_j = g$. Also since it is enough to show that $fg/K = hg/K$, we may assume that $g = id$. By Exercise 12.9.9 (i),

$dim_e^D(\{a_0, a_1, a_2\}) = 3$. Now choose $h' \in G^\circ$ such that it is generic over the set that contains all the elements mentioned in the assumptions. Then for all $i < 3$, $(h'f)a_i \in acl_e^D((h'h)a_i)$. Denote $b_i = (h'f)a_i$. Since a_i is generic over $h'h$, $(h^{-1}h'^{-1})b_i \in acl_e^D(a_i)$. Denote $c_i = (h^{-1}h'^{-1})b_i$. Then $h^{-1}f(id, a_i) = h^{-1}h'^{-1}h'f(id, a_i) = h^{-1}h'^{-1}(id, b_i) = (id, c_i)$. Since $c_i \in acl_e^D(a_i)$ and a_i is generic over $h^{-1}f$ for some $i < 3$, $h^{-1}f/K = id/K$ by Exercise 12.9.10 (ii). Thus $f/K = h/K$. □ Claim 12.9.11

12.9.12 Claim. *There are only finitely many $x \in C$ such that for all $a \in D$, $x \neq (id, a)$.*

Proof. Suppose not. Let (g_i, a_i), $i < \omega$, witness this. By Exercise 12.9.9 (ii), we may assume that there is a such that for all $i < \omega$, $a_i = a$ (exercise). Let $f = (c, d, e^*)$ be generic over $\{g_i | \ i < \omega\} \cup \{a\}$. For a contradiction, it suffices to show that $(fg_i)a \in acl_e^D(e^*)$ for all $i < \omega$. But this is clear, since otherwise, f^{-1} is generic over $(fg_i)a$ and so by Exercise 12.9.9 (ii), $(g_i, a) \approx (f^{-1}, (fg_i)a) \approx (id, f^{-1}((fg)a))$. □ Claim 12.9.12

12.9.13 Exercise.

(i) Show that there is finite $A^c \subseteq C$ such that for all finite $B \subseteq C - A^c$ $cl(AB) = Acl(B)$ and for all $a, b \in C - cl(AB)$, there is $F \in Aut(M^{eq}/ABe)$ such that $F(a) = b$. Hint: Let A^c be set set of $(g, x) \in C$ such that for all $y \in D$, $(g, x) \not\approx (id, y)$. Then to find F, choose $h \in G^\circ$ generic enough and find F so that $F(h) = h$, $F(ha) = hb$ and for all $c \in AB$, $F(hc) = hc$.

(ii) Show that there are no $a, b, c \in C$ and $g \in G^\circ$ such that

$$dim_{cl}^C(\{a, b, c, ga, gb, gc\}) = 6.$$

Hint: For any $f \in G^\circ$, $dim_{cl}^C(\{fa, fb, fc, (fg)a, (fg)b, (fg)c\}) = 6$ if and only if $dim_{cl}^C(\{a, b, c, ga, gb, gc\}) = 6$.

12.9.14 Claim. *M^{eq} interprets an algebraically closed field.*

Proof. Now we have everything we needed in Sections 9 and 10, except that the property in Exercise 12.9.13 (i) is weaker than what we had earlier. However, it is good enough to make the proofs go through (to show that if $fx = gx$, $fy = gy$ and $x \neq y$, then $f = g$, notice first that it is enough to prove this under the additional assumption that $x, y \in C - A^c$, see Exercise 12.9.13, and then, working in $C - A^c$, apply the proofs from Section 9.) □ Claim 12.9.14

But now the following exercise gives the needed contradiction:

12.9.15 Exercise. *M^{eq} does not interpret an algebraically closed field. Hint: Use the fact that algebraically closed fields are not ω-categorical and Ryll-Nardzewski (and keep in mind that M^{eq} is not ω-categorical - but it is very close of being ω-categorical).*

This finishes the proof of Theorem 12.9. □

12.10 Remark. *One reason why B. Zilber studied these questions was the following. He looked at totally categorical theories T (see the literature). After knowing that the pregeometries of strongly minimal sets in models of such theories are locally modular, he was able to coordinatize the structure with elements of a strongly minimal set (not completely unlike in Fact 4.13) and show that every first-order sentence true in a model of T is true already in some finite substructure. He concluded that the theory is not finitely axiomatizable i.e. there is no finite theory T' with exactly the same models as T.*

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