Exercises III

1. Using the binomial model for X, $\text{Bin}(N, p)$, assume that we know p and observe X (and always $X \geq 1$), but not what the sample size N was, and that we don't know any upper limit for N. Because we assume $X \geq 1$, also $N \geq 1$. The possible values after observing X are then $N \in \{X, X+1, X+2, \ldots\}$. To describe uncertainty prior to data, choose the improper prior $\pi(N) \propto 1/N$. It is improper, because $\sum_{n=1}^{\infty} 1/n = \infty$. Solve the posterior distribution $\pi(N \mid X, p)$. (Hint: look for Negative Binomial distribution when computing the normalizing constant).

[Note: there are different parameterizations of Negative Binomial distribution in different books, leading to seemingly different expressions].

2. Let $\pi(X \mid N, p) = \text{Bin}(N, p)$ and $\pi(p) = \text{U}(0, 1)$. Show that the prior predictive distribution of X is discrete uniform $\pi(X) = 1/(N+1)$, $\forall X \in \{0, 1, 2, \ldots, N\}$.

3. Compute the variance of beta-binomial distribution, using the law of total variance.

$$V(X) = E(V(X \mid r, N)) + V(E(X \mid r, N))$$

4. Show that if we define $\psi = \log(p/(1-p))$ for $p \in (0, 1)$, and uniform improper prior $\pi(\psi) \propto 1$, then this leads to Haldane's improper prior $\text{Beta}(0, 0)$ for p. (Hint: use transform of variables rule). Let X be binomially distributed as $\text{Bin}(N, p)$. Why the posterior distribution of p is not proper density if $X = 0$ or $X = N$?

5. Give an example of a $\pi(p) = \text{Beta}(\alpha, \beta)$ prior distribution and data X from $\text{Bin}(N, p)$ where the posterior variance is larger than prior variance.