Time Frequency Analysis - Winter 2012

Exercise Set 6

6.1. **Prove the following estimate:**

$$I := \sum_{k=0}^{\infty} \int_{I_T^c} \left(1 + 2^k \frac{\operatorname{dist}(x, I_T)}{|I_T|}\right)^{-9} dx \lesssim |I_T|,$$

which was needed to complete the proof on the lectures: One way to do this is:

$$\begin{aligned} I &= \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \int_{\ell|I_T| \le \operatorname{dist}(x,I_T) < (\ell+1)|I_T|} \left(1 + 2^k \frac{\operatorname{dist}(x, I_T)}{|I_T|}\right)^{-9} dx \\ &\lesssim \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \int_{\ell|I_T| \le \operatorname{dist}(x,I_T) < (\ell+1)|I_T|} (1 + 2^k \ell)^{-9} dx \\ &= |I_T| \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} (1 + 2^k \ell)^{-9} \lesssim |I_T|. \end{aligned}$$

6.2. **Prove the other estimate needed in the lectures:**

$$|\langle \phi_P, \phi_{P'} \rangle| \lesssim \left(\frac{|I_P|}{|I_{P'}|}\right)^{1/2} \|v_{I_P} \mathbf{1}_{I_{P'}}\|_1,$$

where $v_I(x) = \frac{1}{|I|}\big(1 + \frac{|x - c(I)|}{|I|}\big)^{-10}$ **and** $|I_{P'}| \le |I_P|$**.** Assume for example that $c(I_{P'}) \le c(I_P)$ and denote by c the center of the segment $(c(I_{P'}), c(I_P))$. We estimate

$$\begin{aligned} \langle \phi_P, \phi_{P'} \rangle &\le |I_{P'}|^{-1/2} |I_P|^{-1/2} \int_{-\infty}^{c} \left(1 + \frac{|x - c(I_P)|}{|I_P|}\right)^{-11} dx \\ &\quad + |I_{P'}|^{-1/2} |I_P|^{-1/2} \int_{c}^{+\infty} \left(1 + \frac{|x - c(I_{P'})|}{|I_{P'}|}\right)^{-11} dx \\ &\simeq |I_{P'}|^{1/2} |I_P|^{-1/2} \left(1 + \frac{|c - c(I_P)|}{|I_P|}\right)^{-10} + |I_P|^{1/2} |I_{P'}|^{-1/2} \left(1 + \frac{|c - c(I_{P'})|}{|I_{P'}|}\right)^{-10} \end{aligned}$$

Now observe that $|c - c(I_P)| = |c - c(I_{P'})| = |c(I_P) - c(I_{P'})|/2$. Thus the estimate above is of the form

$$a^{\frac{1}{2}} b^{-\frac{1}{2}} (1 + \delta/b)^{-10} + b^{\frac{1}{2}} a^{-\frac{1}{2}} (1 + \delta/a)^{-10} \le 2a^{\frac{1}{2}} b^{-\frac{1}{2}} (1 + \delta/b)^{-10}$$

whenever $a \le b$. We get

$$\langle \phi_P, \phi_{P'} \rangle \lesssim |I_{P'}|^{1/2} |I_P|^{-1/2} \left(1 + \frac{|c(I_{P'}) - c(I_P)|}{2|I_P|}\right)^{-10}.$$

Also observe that for $x \in I_{P'}$ we have

$$
\begin{aligned}
1+\frac{|c(I_{P'})-c(I_P)|}{2|I_P|} &\geq 1+\frac{\big||x-c(I_P)|-|x-c(I_{P'})|\big|}{2|I_P|} \geq 1+\frac{\big||x-c(I_P)|-|I_{P'}|\big|}{2|I_P|} \\
&\geq 1+\frac{|x-c(I_P)|}{2|I_P|}-\frac{|I_{P'}|}{2|I_P|} \geq 1+\frac{|x-c(I_P)|}{2|I_P|}-\frac{1}{2} \\
&\gtrsim 1+\frac{|x-c(I_P)|}{|I_P|}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\langle \phi_P, \phi_{P'}\rangle| &\lesssim |I_{P'}|^{1/2}|I_P|^{-1/2}|I_{P'}|^{-1}\int_{I_{P'}}\left(1+\frac{|c(I_{P'})-c(I_P)|}{2|I_P|}\right)^{-10}dx \\
&\lesssim |I_{P'}|^{-1/2}|I_P|^{-1/2}\int_{I_{P'}}\left(1+\frac{|x-c(I_P)|}{|I_P|}\right)^{-10}dx \\
&= |I_{P'}|^{-1/2}|I_P|^{-1/2}|I_P|\|\mathbf{1}_{I_{P'}}v_{I_P}\|_{L^1} = |I_{P'}|^{-1/2}|I_P|^{1/2}\|\mathbf{1}_{I_{P'}}v_{I_P}\|_{L^1}.
\end{aligned}
$$

6.3. **In the lectures we considered a collection $\mathcal{T}$ of trees with the following property:**

$(*)$ **If** $P \in \mathbb{T} \in \mathcal{T}$ **and** $P' \in \mathbb{T}' \in \mathcal{T}$ **satisfy** $\omega_P \subseteq \omega_{P'_d}$, **then** $I_{P'} \cap I_T = \emptyset$.

Prove that under this assumption, every tree $\mathbb{T} \in \mathcal{T}$ can be divided into up-trees $\mathbb{T}_j$, whose top time-intervals I_{T_j} are pairwise disjoint. Let $\mathbb{T} \in \mathcal{T}$ and $\mathbf{T}_j$ be the maximal tiles in $\mathbb{T}$. We define the subtrees $\mathbb{T}_j$ of of $\mathbb{T}$ as

$$
\mathbb{T}_j \stackrel{\text{def}}{=} \{P \in \mathbb{T} : P \leq \mathbf{T}_j\}.
$$

It is immediate that the $\mathbb{T}_j$'s are trees with top $\mathbf{T}_j$. Also we have that $\cup_j \mathbb{T}_j = \mathbb{T}$. Indeed the inclusion $\cup_j \mathbb{T}_j \subseteq \mathbb{T}$ is obvious and on the other hand any tile $P \in \mathbb{T}$ must be $\leq$ than some maximal tile $\mathbf{T}_j$, otherwise it is itself maximal and thus included in $\mathbb{T}_j$. Now for any tile $P \in \mathbb{T}_j$ with $P \neq \mathbf{T}_j$ we have that $\omega_{\mathbf{T}_j} \subsetneq \omega_P$. We conclude that either $\omega_{\mathbf{T}_j} \subsetneq \omega_{P_d}$ or $\omega_{\mathbf{T}_j} \subsetneq \omega_{P_u}$. If the first alternative is true then $(*)$ implies that $I_{\mathbf{T}_j} \cap I_P = \emptyset$ which is clearly impossible since $I_P \subseteq I_{\mathbf{T}_j}$. Thus $\omega_{\mathbf{T}_{j,u}} \subseteq \omega_{\mathbf{T}_j} \subseteq \omega_{P_u}$ for all $P \in \mathbb{T}_j$ with $P \neq \mathbf{T}_j$ and clearly the same is true for $P = \mathbf{T}_j$ so every $\mathbb{T}_j$ is an up-tree. Finally, let $j \neq k$ and consider the tops $I_{\mathbf{T}_j}$ and $I_{\mathbf{T}_k}$. If $I_{\mathbf{T}_j} \cap I_{\mathbf{T}_k} \neq \emptyset$, then since both $\mathbf{T}_j, \mathbf{T}_k \in \mathbb{T}$ we would conclude that the frequency intervals also intersect and thus the tiles $\mathbf{T}_j, \mathbf{T}_k \in \mathbb{T}$ intersect which means they are comparable. However, this contradicts the fact that they were maximal with respect to '$\leq$'.

6.4. **Let**

$$
S := \Big(\sum_{P \in \mathbb{P}} |\langle f, \phi_P\rangle|^2\Big)^{\frac{1}{2}}, \quad S_2 := \sum_{P, P' \in \mathbb{P}, \omega_P \subseteq \omega_{P'_d}} \langle f, \phi_P\rangle\langle \phi_P, \phi_{P'}\rangle\langle \phi_{P'}, f\rangle,
$$

where $\mathbb{P} = \cup_j \mathbb{T}_j$ is a union of trees. Prove that $S_2 \lesssim A^2$ where

$$A := \sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \big(\sum_j |I_{T_j}| \big)^{\frac{1}{2}}.$$

Using this bound and the bound $S^2 \lesssim \sqrt{S^2 + S_2}\|f\|_2$ from the notes, derive a bound for S. Combine these estimates to give an alternative proof of the energy estimate $\sum_j |I_{T_j}| \lesssim \mathcal{E}^{-2}\|f\|_2^2$. We have

$$\begin{aligned} S_2 &\leq \left(\sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \right)^2 \sum_{\substack{P, P' \in \mathbb{P} \\ \omega_P \subseteq \omega_{P'_d}}} |I_P|^{\frac{1}{2}} |I_{P'}|^{\frac{1}{2}} |\langle \phi_P, \phi_{P'} \rangle| \\ &\lesssim \left(\sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \right)^2 \sum_{P \in \mathbb{P}} |I_P| \sum_{\substack{P' \in \mathbb{P} \\ \omega_P \subseteq \omega_{P'_d}}} \|v_{I_P} \mathbf{1}_{I_{P'}}\|_1 \\ &\lesssim \left(\sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \right)^2 \sum_{P \in \mathbb{P}} |I_P| \|v_{I_P} \mathbf{1}_{\mathbb{T}(P)^c}\|_1, \end{aligned}$$

as in the proof of Proposition 7.3 in the notes, where $\mathbb{T}(P)$ denotes the unique tree in $\mathbb{P}$ that contains P. Observe that we have used that all the intervals $I_{P'}$ appearing in the inner sum are disjoint. Now we argue as in the proof of Proposition 7.3 to get

$$S_2 \leq \left(\sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \right)^2 \sum_j \sum_{P \in \mathbb{T}_j} |I_P| \|v_{I_P} \mathbf{1}_{\mathbb{T}(P)^c}\|_1 \leq \left(\sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{\frac{1}{2}}} \right)^2 \sum_j |I_{T_j}| = A^2.$$

We now combine this bound with the estimate $S^2 \lesssim \sqrt{S^2 + S_2}\|f\|_2$ to get

$$S^2 \lesssim \sqrt{S^2 + A^2}\|f\|_2 \lesssim S\|f\|_2 + A\|f\|_2.$$

Now from the proof of the Energy Lemma we have

$$\sum_j |I_{T_j}| \lesssim \frac{1}{\mathcal{E}^2} S^2$$

Now if $S \leq A$ this means that $S^2 \lesssim A\|f\|_2$ so

$$\sum_j |I_{T_j}| \lesssim \mathcal{E}^{-2} A \|f\|_2 \leq \mathcal{E}^{-1} \big(\sum_j |I_{T_j}| \big)^{\frac{1}{2}} \|f\|_2,$$

and rearranging the terms we get

$$\sum_j |I_{T_j}| \lesssim \mathcal{E}^{-2}\|f\|_2^2,$$

as we wanted. If $S > A$ we get $S^2 \lesssim S\|f\|_2$ which implies $S \lesssim \|f\|_2$ and we are done by the proof of the energy lemma.

6.5. **Let $\mathbb{P}$ be a finite collection of tiles. Let $\mathcal{J}$ be the collection of all maximal dyadic intervals J with the property that $3J$ (the interval with the same center and triple the length of J) does not contain any I_P with $P \in \mathbb{P}$. Prove that $\mathcal{J}$ is a partition (a pairwise disjoint cover) of $\mathbb{R}$.** First of all observe that the $J \in \mathcal{J}$ are pairwise disjoint. Indeed, if two of them intersect then one must contain the other since they are dyadic which contradicts the maximality property. Now let $x \in \mathbb{R}$. We need to prove that $x \in J$ for some $J \in \mathcal{J}$. Let I_o be the smallest time interval appearing in $\mathbb{P}$. Consider the dyadic intervals of length $|I_o|/2^{10}$. Then these partition $\mathbb{R}$ so one of them, let us call it I, contains x. The interval $3I$ of course still contains x and we have $|3I| \leq 3 \cdot 2^{-10}|I_o| < |I_P|$ for all $P \in \mathbb{P}$. Thus $3I$ cannot contain any I_P, $P \in \mathbb{P}$, so x is contained in some dyadic interval I such that $3I$ does not contain any $I_P, P \in \mathbb{P}$. It remains to prove that any dyadic interval J such that $3J$ does not contain any $_P$ for any $P \in \mathbb{P}$ is contained in some maximal dyadic interval of the same type. Indeed, suppose that the collection $\mathbb{P}$ is non-empty, otherwise there is nothing to prove and fix such a J. Let $J_0 := J$ and define inductively $J_{k+1} := J_k^{(1)}$ where $I^{(1)}$ denotes the parent of I. It is obvious that $3J_o \subset 3J_1 \subset 3J_2 \subset \cdots$ and that $3J_k \uparrow \mathbb{R}$ as $k \to \infty$. Thus there will be a k such that $3J_k$ does not contain any I_P while $3J_{k+1}$ does, and there is at least one I_P. Then J_k is maximal with the desired property.