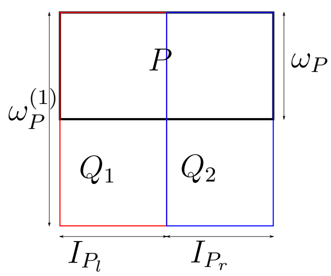
Time Frequency Analysis - Winter 2012

Exercise Set 3

3.1. **Let $\mathbb{P}_1, \mathbb{P}_1$ be collections of tiles. Recall that $\cup_{P\in\mathbb{P}_1} P = \cup_{P\in\mathbb{P}_2} P$, then also** $\text{span}\{w_P : P \in \mathbb{P}_1\} = \text{span}\{w_P : P \in \mathbb{P}_2\}$**. Give an example to show that it is possible to have the second "$=$" even if the first "$=$" j is replaced by "$\subseteq$".** Let P be any bitile and write $P = I_P \times \omega_P$, $I_P = I_{P_l} \cup I_{P_r}$ where I_{P_l} and I_{P_r} are the dyadic children of I and let $\omega_P^{(1)}$ be the dyadic parent of ω_P. Set $Q_1 = I_{P_l} \times \omega_P^{(1)}$ and $Q_2 = I_{P_r} \times \omega_P^{(1)}$. Now define (fo example) the collections $\mathbb{P}_1 \stackrel{\text{def}}{=} \{P, Q_2\}$ and $\mathbb{P}_2 \stackrel{\text{def}}{=} \{Q_1, Q_2\}$. Here is a picture:



Observe that $P \cup Q_2 \subsetneq Q_1 \cup Q_2$. Now Exercise 1.5 implies that there exists non-zero real numbers c_1, c_2 such that $w_P = c_1 w_{Q_1} + c_2 w_{Q_2}$. Thus on the one hand we have $\text{span}\{w_P : P \in \mathbb{P}_1\} \subseteq \text{span}\{w_P : P \in \mathbb{P}_2\}$. On the other hand since $c_1 \neq 0$ we can write $w_{Q_1} = c_1' w_P + c_2' w_{Q_2}$ for some non-zero constants c_1', c_2'. So we have that $w_{Q_1} \in \text{span}\{w_P : P \in \mathbb{P}_2\}$ and obviously $w_{Q_2} \in \text{span}\{w_P : P \in \mathbb{P}_2\}$ so we also get that $\text{span}\{w_P : P \in \mathbb{P}_2\} \subseteq \text{span}\{w_P : P \in \mathbb{P}_1\}$ se the two subspaces are the same.

3.2. **Let $\mathbb{P}$ be a finite collection of bitiles with** $\text{density}(\mathbb{P}) \cdot \text{energy}(\mathbb{P}) = 0$**. Check that then $\sum_{P\in\mathbb{P}} \langle f, w_{P_d}\rangle \langle w_{P_d}, \mathbf{1}_{E_{P_u}}\rangle = 0$.** First recall the definitions

$$\text{density}(\mathbb{P}) \stackrel{\text{def}}{=} \sup_{P\in\mathbb{P}} \sup_{P'\geq P} \frac{|I_{P'} \cap E_{P'}|}{|I_{P'}|}, \quad E_{P'} \stackrel{\text{def}}{=} \{x \in E : N(x) \in \omega_{P'}\},$$

$$\text{energy}(\mathbb{P}) \stackrel{\text{def}}{=} \sup_{\mathbb{T}\subseteq\mathbb{P},\text{up-tree}} \left(\frac{1}{|I_T|} \sum_{P\in\mathbb{T}} |\langle f, w_{P_d}\rangle|^2 \right)^{\frac{1}{2}}.$$

If $\text{density}(\mathbb{P}) = 0$ then taking $P' = P$ in the definition we have that $I_P \cap E_P = \emptyset$ for every $P \in \mathbb{P}$. Since w_{P_d} is supported on I_P we get that $\langle w_{P_d}, \mathbf{1}_{E_P}\rangle = 0$ for every $P \in \mathbb{P}$ and thus the sum is zero. If $\text{energy}(\mathbb{P}) = 0$ then taking the tree consisting of the single tile P in the definition we conclude that $\langle f, w_{P_d}\rangle = 0$ for all $P \in \mathbb{P}$ thus the sum is again zero. Note that the tree lemma also immediately implies that the sum is zero whenever $\text{energy}(\mathbb{T}) \cdot \text{density}(\mathbb{T}) = 0$.

3.3. **Prove the following corollary of the Density lemma and the Energy lemma: If $\mathbb{P}_n$ is a finite collection of bitiles with**

$$\text{density}(\mathbb{P}_n) \leq 4^n |E|, \quad \text{energy}(\mathbb{P}_n) \leq 2^n \|f\|_{L^2},$$

then we can decompose it as $\mathbb{P}_n = \mathbb{P}_{n-1} \cup \bigcup_j \mathbb{T}_{n,j}$, where $\mathbb{P}_{n-1}$ satisfies the same bounds as $\mathbb{P}_n$ with $n-1$ in place of n, each $\mathbb{T}_{n,j}$ is a tree with top $\mathbb{T}_{n,j}$, and the top time intervals satisfy $\sum_j |I_{T_{n,j}}| \leq C4^{-n}$. If density$(\mathbb{P}_n) \leq 4^{n-1}|E|$ and energy$(\mathbb{P}_n) \leq 2^{n-1}\|f\|_{L^2}$ then we do nothing and just set $\mathbb{P}_{n-1} \stackrel{\text{def}}{=} \mathbb{P}_n$. There are no trees in this case but we can always consider the empty set as a tree.

Initialize $\mathbb{Q} := \mathbb{P}_n, \mathcal{T}_n := \{\emptyset\}$.
IF

$$\text{energy}(\mathbb{Q}) \geq 2^{n-1}\|f\|_{L^2}$$

THEN
apply the energy lemma to $\mathbb{Q}$ to write

$$\mathbb{Q} = \mathbb{Q}_o \cup \bigcup_j \mathbb{T}^*_{n,j},$$

where

$$\text{energy}(\mathbb{Q}_o) \leq \frac{1}{2}\,\text{energy}(\mathbb{Q}) \leq 2^{n-1}\|f\|_{L^2},$$

and

$$\sum_j |I_{T^*_{n,j}}| \leq \text{energy}(\mathbb{Q})^{-2}\|f\|^2_{L^2} \leq 4 \cdot 4^{-n}.$$

SET $\mathbb{Q} := \mathbb{Q}_o$ and $\mathcal{T}_n = \{\mathbb{T}^*_{n,j}\}_j$ and observe that the density did not increase:

$$\text{density}(\mathbb{Q}) \leq 4^n|E|.$$

ELSE
IF

$$\text{energy}(\mathbb{Q}) \leq 2^{n-1}\|f\|_{L^2}$$

THEN do nothing.
ENDIF

IF

$$\text{density}(\mathbb{Q}) \geq 4^{n-1}|E|$$

THEN
apply the density lemma to $\mathbb{Q}$ to write

$$\mathbb{Q} = \mathbb{Q}_1 \cup \bigcup_j \tilde{\mathbb{T}}_{n,j},$$

where

$$\text{density}(\mathbb{Q}_1) \leq \frac{1}{4}\,\text{energy}(\mathbb{Q}) \leq 4^{n-1}|E|,$$

and

$$\sum_j |I_{\tilde{T}_{n,j}}| \leq \text{density}(\mathbb{Q})^{-1}|E| \leq 4 \cdot 4^{-n}.$$

SET $\mathbb{Q} := \mathbb{Q}_1$ and $\mathcal{T}_n := \mathcal{T}_n \cup \{\tilde{\mathbb{T}}_{n,j}\}_j$ and observe that the energy did not increase:

$$\text{energy}(\mathbb{Q}) \leq 2^{n-1}\|f\|_{L^2}.$$

ELSE
IF

$$\text{density}(\mathbb{Q}) \leq 4^{n-1}|E|$$

THEN do nothing.
ENDIF
SET $\mathbb{P}_{n-1} := \mathbb{Q}$ and observe

$$\text{density}(\mathbb{P}_{n-1}) \leq 4^{n-1}|E|, \quad \text{energy}(\mathbb{P}_{n-1}) \leq 2^{n-1}\|f\|_{L^2},$$

and

$$\sum_{\mathbb{T}\in\mathcal{T}_n} |I_T| = \sum_j |I_{T^*_{n,j}}| + \sum_j |I_{\tilde{T}_{n,j}}| \leq 4\cdot 4^{-n} + \cdot 4 \cdot 4^{-n} \leq 8\cdot 4^{-n}.$$

3.4. **Let $\mathbb{P}$ be a finite collection of tiles with the property that $\sum_{P\in\mathbb{P}} \mathbf{1}_P \leq n$. Show that there exists a decomposition $\mathbb{P} = \bigcup_{j=1}^n \mathbb{P}_j$, where each $\mathbb{P}_j$ is a pairwise disjoint collection of tiles. Give an example to show that the claim is not true if the tiles are replaced by arbitrary sets.** Consider all the tiles that have an n-intersection. Among the choose the longest ones, remove them from $\mathbb{P}$ and put them in $\mathbb{P}_1$. Repeat the process until there are no more n-intersections. Set $\mathbb{P}' \stackrel{\text{def}}{=} \mathbb{P}\setminus\mathbb{P}_1$. By construction we have that $\sum_{P\in\mathbb{P}'} \mathbf{1}_P \leq n-1$. Now we claim that all the tiles in $\mathbb{P}_1$ are disjoint. Assume on the contrary that $P, P' \in \mathbb{P}_1$ and $P \cap P' \neq \emptyset$. We must have $|I_P| \neq |I_{P'}|$ otherwise we would have $P = P'$ since they are both dyadic and they intersect. We can thus assume that $I_{P'} \subsetneq I_P$ and $\omega_P \subsetneq \omega'_P$. Since $|I_P| > |I_{P'}|$ the tile P was chosen first. On the other hand, when the tile P' was chosen it had an n-intersection with some tiles Q_j with $|I_{Q_j}| < |I_{P'}|$. Thus for all the Q_j we had $I_{Q_j} \subset I_{P'} \subset I_P$. On the other hand we have $|\omega_{Q_j}| > |\omega_{P'}|$ so that all the Q_j's satisfied $\omega_P \subset \omega_{P'} \subset \omega_{Q_j}$. We conclude that the Q'_j, P' and P had an $(n+1)$-intersection which is a contradiction. Continuing inductively we define the remaining collections $\mathbb{P}_2, \ldots, \mathbb{P}_n$, all of the consisting of pairwise disjoint tiles.

3.5. **Show that we have $\|Mf\|_{L^2} \leq 2\|f\|_{L^2}$ for the dyadic maximal function.** Recall the definition

$$Mf(x) \stackrel{\text{def}}{=} \sup_{J\in\mathcal{D}} \frac{\mathbf{1}_J(x)}{|J|}\int_J |f(y)|dy.$$

Since M only takes into account the modulus of f it obviously suffices to consider non-negative functions f. Assume also that f is a simple function supported on a finitely many dyadic intervals. Then in the definition of the maximal function there are only finitely many dyadic intervals involved and the supremum is actually a maximum.

Thus for any $x \in \mathbb{R}$ we can define $J(x)$ to be the maximal dyadic interval $J(x) \ni x$ such that

$$\frac{1}{|J(x)|}\int_{J(x)} f(y)dy = Mf(x).$$

The claim is that it is equivalent to consider the linear operator

$$\tilde{M}f(x) = \sum_{J\in\mathcal{D}} \frac{\mathbf{1}_{E(J)}(x)}{|J|}\int_J f(y)dy,$$

for some arbitrary selection of sets $E(J) \subset J$ which are pairwise disjoint, that is $E(J)\cap E(J') = \emptyset$ if $J \neq J'$. Indeed for any $x \in \mathbb{R}$ there is a unique $E(J') \ni x$ since the $E(J)'s$ are disjoint so that that

$$\sum_{J\in\mathcal{D}} \frac{\mathbf{1}_{E(J)}(x)}{|J|}\int_J f(y)dy = \frac{\mathbf{1}_{E(J')}}{|J'|}\int_{J'} f(y)dy \leq Mf(x).$$

On the other hand, set $E(J) = \{x \in \mathbb{R} : J = J(x)\} \subset J$. It is not hard to see that $E(J) \cap E(K) = \emptyset$ if $J \neq K$. Indeed, this is clear if $J \cap K = \emptyset$. If $J \subset K$ say and $E(J) \cap E(K) \ni y$ then we have that $J(y) = K = J$ which is impossible since $J(y)$ was maximal with that property. Also the collection $\{E(J) : J \in \mathcal{D}\}$ covers $\mathbb{R}$ since for every $x \in \mathbb{R}$ we have that $x \in E(J(x))$. Thus for every $x \in \mathbb{R}$ there is a unique J such that $x \in E(J)$ and

$$\tilde{M}f(x) \geq \frac{\mathbf{1}_{E(J)}(x)}{|J|}\int_J f(y)dy \geq Mf(x).$$

Let us find a formula for the adjoint M^*. We have

$$\begin{aligned}\int Mfg &= \sum_{J\in\mathcal{D}}\int \frac{\mathbf{1}_{E(J)}(x)}{|J|}\int_J f(y)dy\; g(x)dx \\ &= \sum_{J\in\mathcal{D}}\int\int \frac{\mathbf{1}_{E(J)(x)}\mathbf{1}_J(y)}{|J|} f(y)g(x)dxdy \\ &= \int f(y)\Big(\sum_{J\in\mathcal{D}} \frac{\mathbf{1}_J(x)}{|J|}\int_{E(J)} g(x)dx\Big)dy,\end{aligned}$$

so that

$$\tilde{M}^*g(x) = \sum_{J\in\mathcal{D}} \frac{\mathbf{1}_J(x)}{|J|}\int_{E(J)} g(y)dy.$$

Thus

$$\tilde{M}\tilde{M}^*f(x) = \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} \mathbf{1}_{E(I)}(x)\frac{|I\cap J|}{|I|\cdot|J|}\int_{E(J)} f(y)dy.$$

If $I \cap J = \emptyset$ then the previous sum is zero. If $I \subseteq J$ then $|I \cap J| = |I|$ thus

$$\begin{aligned}\sum_J \sum_{I \subseteq J} \mathbf{1}_{E(I)}(x) \frac{|I \cap J|}{|I| \cdot |J|} \int_{E(J)} f(y)dy &= \sum_I \sum_{I \subseteq J} \frac{\mathbf{1}_{E(I)}(x)}{|J|} \int_J f(y)dy \\ &\leq \sum_J \left(\mathbf{1}_{\cup_{I \subset J} E(I)}(x) \right) \frac{1}{|J|} \int_J f(y)dy \\ &\leq \sum_J \frac{\mathbf{1}_J(x)}{|J|} \int_J f(y)dy = \tilde{M} f(x).\end{aligned}$$

Similarly, if $J \subsetneq I$ then $|I \cap J| = |J|$ and we get

$$\begin{aligned}\sum_I \sum_{J \subsetneq I} \mathbf{1}_{E(I)}(x) \frac{|I \cap J|}{|I| \cdot |J|} \int_{E(J)} f(y)dy &= \sum_I \mathbf{1}_{E(I)}(x) \frac{1}{|I|} \int f(y) \big(\sum_{J \subsetneq I} \mathbf{1}_{E(J)}(y) \big) dy \\ &\leq \sum_I \frac{\mathbf{1}_{E(I)}(x)}{|I|} \int f(y) \mathbf{1}_I(y) dy = \tilde{M}^* f(x).\end{aligned}$$

Summing the estimates gives $\tilde{M}\tilde{M}^* f \leq \tilde{M} f(x) + \tilde{M}^* f(x)$ so for every $f \in L^2$ which is a step function supported on finitely many dyadic intervals we have

$$\|\tilde{M}\tilde{M}^* f\|_2 \leq \|\tilde{M} f\|_2 + \|\tilde{M}^* f\|_2 \leq 2\|\tilde{M}\|_{L^2 \to L^2},$$

where we used the general fact that $\|T^*\| = \|T\|$ whenever $T : H \to H$ is a linear operator and H is a Hilbert space. Using the also general identity $\|TT^*\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2$ we have

$$\|\tilde{M}\|^2_{L^2 \to L^2} = \|\tilde{M}\tilde{M}^*\|_{L^2 \to L^2} \leq 2\|\tilde{M}\|_{L^2 \to L^2},$$

which gives the claim of the exercise after dividing by $\|\tilde{M}\|_{L^2 \to L^2}$.

Remark 1. We have used the following:

Lemma 2. *Let $T : H \to H$ be a linear operator on a Hilbert space H and let $T^* : H \to H$ be its adjoint. Then*

$$\|T\|_{H \to H} = \|T^*\|_{H \to H} = \|TT^*\|^{\frac{1}{2}}_{H \to H}.$$

Proof. The first identity is just duality. For any $y \in H$ and $x^* \in H$ we have

$$|\langle Ty, x^* \rangle| = |\langle y, T^* x^* \rangle| \leq \|y\|_H \|T^*\|_{H \to H} \|x^*\|_H \tag{1}$$

$$|\langle Ty, x^* \rangle| = |\langle y, T^* x^* \rangle| \leq \|y\|_H \|T\|_{H \to H} \|x^*\|_H \tag{2}$$

Taking the supremum in (1) over $x^* \in H$ with $\|x^*\|_H = 1$ we get

$$\|Ty\|_X \leq \|T^*\| \|y\|_H$$

thus $\|T\|_{H \to H} \leq \|T^*\|$. Taking the supremum in (2) over $y \in H$ with $\|y\|_H = 1$ we get

$$\|T^* x^*\|_H \leq \|T\| \|x^*\|_H,$$

that is $\|T^*\|_{H \to H} \leq \|T\|_{H \to H}$.

The two inequalities prove the first identity. For the second observe that

$$\|TT^*\|_{H\to H} \leq \|T\|_{H\to H}\|T^*\|_{H\to H} = \|T^*\|^2_{H\to H}$$

by the first identity we already proved. To show the opposite inequality note that for every $x^* \in H$ we have

$$\begin{aligned}\|T^*x^*\|_H^2 &= \langle T^*x^*, T^*x^*\rangle = \langle x^*, TT^*x^*\rangle \leq \|x^*\|_{X^*}\|TT^*x^*\|_X \\ &\leq \|TT^*\|_{H\to H}\|x^*\|_H^2.\end{aligned}$$

Taking square roots and then supremum for $\|x^*\|_H \leq 1$ proves

$$\|T^*\|_{H\to H} \leq \|TT^*\|^{\frac{1}{2}}_{H\to H}.$$

□