Time Frequency Analysis - Winter 2012

Exercise Set 1

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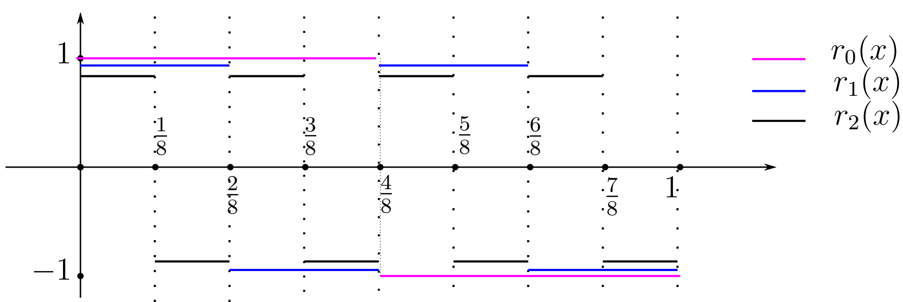
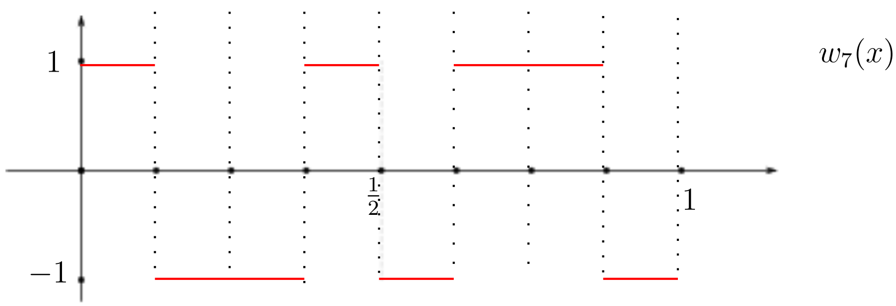
1.1. **Sketch the graph of the function w_7 (the seventh Walsh function).** All the Walsh functions are periodic with period (at most) 1 so it is enough to study what's happening in the unit interval $[0,1)$. The binary expansion of the number 7 is

$$7 = 2^0 + 2^1 + 2^2 =: \sum_{i=0}^{2} 2^i n_i.$$

Thus by the definition of the Walsh functions

$$w_7(x) = \prod_{i=0}^{2} (r_i(x))^{n_i} = r_0(x) \cdot r_1(x) \cdot r_2(x).$$





All the horizontal segments are supposed to be either at 1 or at −1 but they're moved a bit for the sake of clarity.

1.2. **Prove that:**

$$\int_0^1 w_n w_m = \delta_{nm} := \begin{cases} 1, & \text{if} \quad n = m, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

First of all observe that $w_n^2 = \left(\prod_{i=0}^{\infty}(r_i(x))^{n_i}\right)^2 = \prod_{i=0}^{\infty}(r_i(x))^{2n_i} = 1$ where remember that the previous products are actually finite for any non-negative integer n and $r_i^2 \equiv 1$ for any i.

Let us now move to the interesting case $n \neq m$. Observe that for $n = 1$ and $m > 0$ the claim is that every Walsh function w_m with $m > 0$ has zero-mean. Let us first prove this fact. The observation here is that every Rademacher function r_i, $i = 0, 1, 2, \ldots$, is constant on dyadic intervals of length 2^{-i-1} and has zero-mean on every dyadic interval of length 2^{-i}. Now suppose that $m = \sum_{i=0}^{M} 2^i m_i$ with $m_M = 1$ so that

$$w_m(x) = \prod_{i=0}^{M-1} (r_i(x))^{m_i} \cdot r_M(x).$$

The product $\prod_{i=0}^{M-1} (r_i(x))^{m_i}$ is a function which is constant on every dyadic interval of length 2^{-M}. On the other hand r_M has mean-zero on every dyadic interval of length 2^{-M} which implies that the product has mean-zero on $[0, 1)$ (actually on any dyadic interval of length at least 2^{-M}). Here we have used the fact that two dyadic intervals of the same length are either disjoint or they coincide.

The case of a product of two different Walsh functions w_m, w_n with $n \neq m$ is not so different. Indeed suppose that $m = \sum_{i=0}^{M} 2^i m_i$ with $m_M = 1$ and $n = \sum_{j=1}^{N} 2^j n_j$ with $n_N = 1$.

We can assume without loss of generality that $M > N$. Indeed if $M = N$ then denote by N_1 the largest positive integer such that $n_{N_1} \neq m_{N_1}$. The fact that such an integer exists is immediate from the fact that $m \neq n$. We have

$$\begin{aligned} w_n \cdot w_m &= \prod_{i=1}^{N_1} (r_i)^{n_i} \prod_{j=1}^{N_1} (r_j)^{m_j} \prod_{i=N_1+1}^{N_1} (r_i)^{n_i} \prod_{j=N_1+1}^{N_1} (r_j)^{m_j} \\ &= \prod_{i=1}^{N_1} (r_i)^{n_i} \prod_{j=1}^{N_1} (r_j)^{m_j}. \end{aligned}$$

Now one of the two products has at most $N_1 - 1$ factors since $n_{N_1} \neq_{N_1}$. Thus it suffices to show that the function

$$\prod_{i=1}^{N} (r_i)^{n_i} \prod_{j=1}^{M} (r_j)^{m_j}, \quad M > N$$

has mean-zero. We have

$$w_n \cdot w_m = \prod_{i=1}^{N} (r_i)^{n_i} \prod_{j=1}^{M} (r_j)^{m_j} = \prod_{i=1}^{N} (r_i)^{n_i} \prod_{j=1}^{N} (r_j)^{m_j} \prod_{j=N+1}^{M} (r_j)^{m_j}$$

The function $\prod_{i=1}^{N} (r_i)^{n_i} \prod_{j=1}^{N} (r_j)^{m_j}$ is constant on dyadic intervals of length $\leq 2^{-N-1}$. On the other hand the product $\prod_{j=N+1}^{M}$ has mean zero on every dyadic interval of length at least $2^{-M} \leq 2^{-N-1}$. The product thus has mean-zero.

1.3. **Prove that two tiles P, P' are comparable if and only if $P \cap P' \neq \emptyset$.** Suppose that P, P' are comparable, say $P \leq P'$, which means by definition that $I_P \subset I_{P'}$ and $\omega_{P'} \subset \omega_P$. Thus the rectangle $\emptyset \neq I_P \cap I_{P'} \times \omega_P \cap \omega_{P'} \subset P \cap P'$. To prove the other direction, assume that $P \cap P' \neq \emptyset$ so $I_P \cap I_{P'} \neq \emptyset$ and $\omega_P \cap \omega_{P'} \neq \emptyset$. We conclude that we must have $I_P \subset I_{P'}$ or $I_{P'} \subset I_P$ and $\omega_P \subset \omega_{P'}$ or $\omega_{P'} \subset \omega_P$ since all intervals are dyadic and they intersect. Let us fix the case $I_P \subset I_{P'}$ (the other case can be treated in a similar way). Since P, P' are tiles and $|I_P| \leq |I_{P'}|$ the only possibility is that we have $|\omega_{P'}| \leq |\omega_P|$ thus $\omega_{P'} \subset \omega_P$. This means that $P \leq P'$.

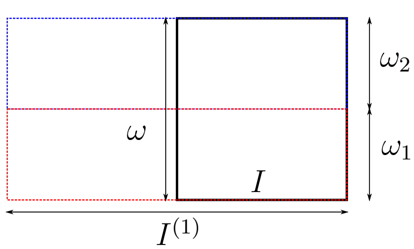
1.4. **Let $\mathbb{T}$ be an up-tree of bitiles. If $P, P' \in \mathbb{T}$ are two different bitiles, show that $P_d \cap P'_d = \emptyset$.** Since $P, P' \in \mathbb{T}$ there exists a bitile T (the top of the tree $\mathbb{T}$) such that $I_P, I_{P'} \subset I_T$ and $\omega_T \subset \omega_{P_u}, \omega_{P'_u}$. Thus $\omega_{P_u} \cap \omega_{P'_u} \supset \omega_T \neq \emptyset$. Since ω_{P_u} and $\omega_{P'_u}$ are dyadic and they intersect, one must contain the other. Suppose that $\omega_{P_u} \subset \omega_{P'_u}$. Then there are two cases:

In the first, $|\omega_{P_u}| = |\omega_{P'_u}|$ and since $\omega_{P_u} \subset \omega_{P'_u}$ the two intervals coincide. This means however that ω_P and $\omega_{P'}$ also coincide and since P, P' are bitiles we get that $|I_P| = |I_{P'}|$. If the intervals $I_P, I_{P'}$ intersect then they must coincide since they have the same length. But then the tiles P, P' would also coincide which is not the case. Thus $I_P \cap I_{P'} = \emptyset$ which means that $P \cap P' = \emptyset$ and a fortiori $P_d \cap P'_d = \emptyset$.

In the second case we have $|\omega_{P_u}| < |\omega_{P'_u}|$ which implies that $|\omega_{P_u}| \leq \frac{1}{2}|\omega_{P'_u}|$ since the intervals are dyadic. This shows that the whole interval $\omega_P \subset \omega_{P'_u}$ so that $P \cap P'_d = \emptyset$. We thus have $P_d \cap P'_d = \emptyset$.

1.5. **Let $P = I \times \omega$ and $P_i = I^{(1)} \times \omega_i$ for $i = 0, 1$, where $I^{(1)}$ is the parent of I and ω_0, ω_1 are the children of ω. Prove that $\omega_P \in \text{span}\{\omega_{P_0}, \omega_{P_1}\}$.** Let $\omega = |I|^{-1}[n, n+1)$ so that the children have the form:

$$\omega_0 = |2I|^{-1}[2n, 2n+1) \quad \text{and} \quad \omega_1 = |2I|^{-1}[2n+1, 2n+2).$$



Thus

$$w_P = w_{I_{\mathbb{P}} \times \omega} = |I|^{-\frac{1}{2}} 1_I w_n(\frac{\cdot}{|I|}),$$

$$w_{I^{(1)} \times \omega_i} = |I^{(1)}|^{-\frac{1}{2}} 1_{I^{(1)}} w_{2n+i}(\frac{\cdot}{|I^{(1)}|}) = \frac{1}{\sqrt{2}} \frac{1}{|I|^{\frac{1}{2}}} 1_{I^{(1)}} w_{2n+i}(\frac{\cdot}{2|I|})$$

Now writing $n = \sum_{j=0}^{\infty} 2^j n_j$ we have $2n = \sum_{j=1}^{\infty} 2^j n_{j-1}$ while of course $i = i2^0$ which means that n and i have 'disjoint' binary expansions. By Lemma 1.1 of the notes we have that $w_{2n+i} = w_{2n} w_i$ and by Lemma 1.3 of the notes $w_{2n} = w_n(2\cdot)$ so that

$$w_{2n+i}(\frac{\cdot}{2|I|}) = w_n(\frac{\cdot}{|I|}) w_i(\frac{\cdot}{2|I|}).$$

Write $I^{(1)} = I \cup I'$ where I' is the dyadic sibling of I. We have that $w_i(x) = (-1)^{i(s+1)}$ on I for some $s \in \{0,1\}$ (in the picture we have that $w_i(x) = (-1)^i$ but observe that I could be the left child of $I^{(1)}$ in which case the signs would be reversed) and also $w_i(x) = (-1)^{i(s+1)+1}$ on I'. Thus

$$w_{P_0} = \frac{1}{\sqrt{2}}\frac{1}{|I|^{\frac{1}{2}}}1_{I'}w_n(\frac{\cdot}{|I|}) + \frac{1}{\sqrt{2}}\frac{1}{|I|^{\frac{1}{2}}}1_I w_n(\frac{\cdot}{|I|})$$
$$w_{P_1} = \frac{1}{\sqrt{2}}\frac{1}{|I|^{\frac{1}{2}}}1_{I'}w_n(\frac{\cdot}{|I|})(-1)^s + \frac{1}{\sqrt{2}}\frac{1}{|I|^{\frac{1}{2}}}1_I w_n(\frac{\cdot}{|I|})(-1)^{s+1}$$

We can now write w_P as a linear combination of w_{P_0}, w_{P_1} as follows

$$w_P = \frac{1}{\sqrt{2}}(w_{P_o} - (-1)^s w_{P_1}).$$

1.6. **For two tiles P, P', prove that $\int_{\mathbb{R}_+} w_P w_{P'} = 0$ if and only if $P \cap P' = \emptyset$.** Let $P = I_P \times \omega_P$ and $P' = I_{P'} \times \omega_{P'}$. Assume first that $P \cap P' = \emptyset$ and we want to show that $\int w_P w_{P'} = 0$. If $I_P \cap I_{P'} = \emptyset$ then the conclusion is obvious so suppose that $I_P \cap I_{P'} \neq \emptyset$ and necessarily $\omega_P \cap \omega_{P'} = \emptyset$.

We need the following notation. For any tile $P = I_P \times \omega_P$ let $I_P^{(1)}$ be the unique parent of I_P and ω_0, ω_1 be the children of ω_P like in Exercise 1.5. We define the collection $\mathcal{P}(P) = \{I_P^{(1)} \times \omega_1, I_P^{(1)} \times \omega_2\}$. Now fix an initial tile P_0 and set $\mathbb{P}_0 := \{P_0\}$ and for any integer $k \geq 1$ let $\mathbb{P}_{k+1}(P_0) := \{\mathcal{P}(Q) : Q \in \mathbb{P}_k(P_0)\}$. Observe that the first collection $\mathbb{P}_1$ contains exactly the tiles constructed in Exercise 1.5 and at each step we repeat the construction for every tile in the previous collection. Exercise 1.5 now implies that for any $k \geq 1$ we have

$$w_P \in \text{span}\{w_Q : Q \in \mathbb{P}_k(P_0)\}.$$

Observe that for any $Q \in \mathbb{P}_k(P_0)$ and any $P \in \mathbb{P}_{k-1}(P_0)$ we have that $|\omega_Q| = \frac{1}{2}|\omega_P|$.

Going back to the exercise, suppose that $|\omega_P| \geq |\omega_{P'}|$ (otherwise rename P and P'). Consider the collection $\mathbb{P}_k(P)$ with k large enough so that $|\omega_Q| = |\omega_{P'}|$ for any $Q \in \mathbb{P}_k(P)$. We then have $|I_Q| = |I_{P'}|$ for all $Q \in \mathbb{P}_k(P)$ and since $\emptyset \neq I_P \cap I_{P'} \subset I_Q \cap I_{P'}$ we must have $I_Q \equiv I_{P'}$. Let $Q = I_Q \times \omega_Q$ be any tile in $\mathbb{P}_k(P)$ and $\omega_Q = |I_Q|^{-1}[s, s+1) = |I_{P'}|^{-1}[s, s+1)$ and also write $\omega_{P'} = |I_{P'}|^{-1}[m, m+1)$. There are two possibilities. First the tile Q is 'below' P' so that

$$|I_{P'}|^{-1}(s+1) \leq |I_{P'}|^{-1}m \Rightarrow s+1 \leq m \Rightarrow s \leq m-1,$$

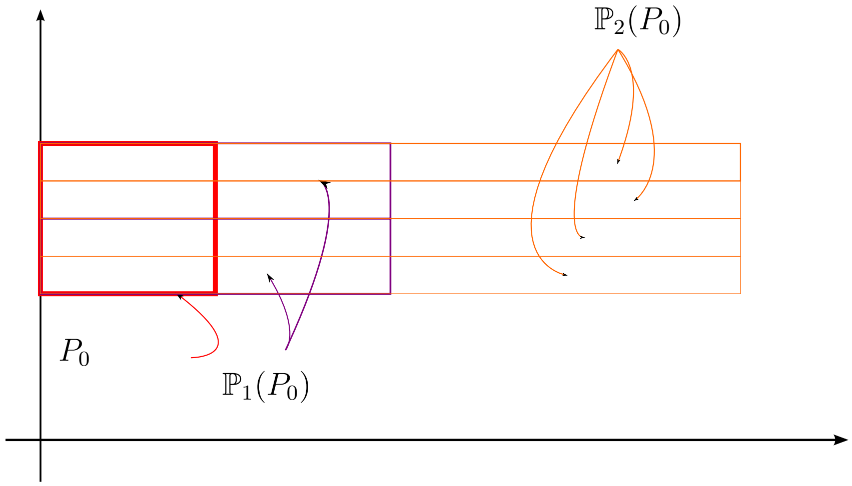
or the tile Q is 'above' P':

$$|I_{P'}|^{-1}(m+1) \leq |I_{P'}|^{-1}s \Rightarrow m+1 \leq s \Rightarrow m \leq s-1.$$

In both cases we must have $m \neq s$ so that

$$\int_{\mathbb{R}_+} w_Q w_{P'} = \frac{1}{|I_Q|}\int_{I_Q} w_s(\frac{x}{|I_Q|})w_m(\frac{x}{|I_Q|})dx = \int_0^1 w_s(x)w_m(x)dx = 0$$

according to Exercise 1.2. Since all the tiles in $\mathbb{P}_k(P)$ are orthogonal to $w_{P'}$ and $w_P \in \text{span}\{w_Q : Q \in \mathbb{P}_k(P)\}$ we conclude that $\int w_P w_{P'} = 0$ as we wanted to show.



For the opposite direction assume that $P \cap P' \neq \emptyset$ thus $\omega_P \cap \omega_{P'} \neq \emptyset$. Without loss of generality we may assume that $\omega_{P'} \subset \omega_P$. Like before, consider the collection $\mathbb{P}_k(P)$ for k large enough so that $|\omega_Q| = |\omega_{P'}|$ for $Q \in \mathbb{P}_k(P)$. Since $\omega_{P'} \subset \omega_P$ we necessarily have that P' is one of the tiles in $\mathbb{P}_k$. It is now not hard to check that $\int w_P w_Q \neq 0$ for every $Q \in \mathbb{P}_k$ and every positive integer $k \geq 1$. Indeed for $k = 1$ this is contained in Exercise 1.5 (you can use the other direction of the current exercise, already proved) while for general k one can show this by a simple inductive argument for example.