Time-frequency analysis — Winter 2012

5. Exercise set

5.1. Investigate the commutation relations between S_ξ and T_y, M_η, D_λ^2: Find a ξ' (possibly different in the different identities below) so that

$$S_\xi T_y = T_y S_{\xi'}, \qquad S_\xi M_\eta = M_\eta S_{\xi'}, \qquad S_\xi D_\lambda^2 = D_\lambda^2 S_{\xi'}.$$

Finally, find a value of $\eta = \eta(\xi, \lambda)$ so that $S_\xi D_\lambda^2 M_\eta = D_\lambda^2 M_\eta S_\xi$.

5.2. We have used several times the identity $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, where $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$ is the L^2 inner product. Prove this identity in the following two ways: (a) Write the identity $\|h\|_{L^2} = \|\hat{h}\|_{L^2}$ for $h = f + ug$, where $u \in \{1, -1, i, -i\}$. (b) In the identity $\int_{\mathbb{R}} f(x)\hat{h}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(x)h(x)\,\mathrm{d}x$, substitute $h(x) = \overline{\hat{f}(x)}$, and manipulate the left side.

5.3. Let $\mathbb{T}$ be an up-tree of tiles. Show that it can be divided into 20 subcollections $\mathbb{T}_i$ so that if $P, P' \in \mathbb{T}_i$ for the same i, then ϕ_P and $\phi_{P'}$ are orthogonal to each other. [Hint: Similar to the orthogonality consideration related to the operator A_ξ.]

5.4. Prove the following fact which was implicitly used in transforming the probabilistic expectation E (integration with respect to the probability measure P) into the Lebesgue integral over $[0, 1]$: For numbers $0 \le a \le b \le 1$, we have

$$\mathsf{P}\Big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in [a, b)\Big) = b - a,$$

where β_j are independent random variables $\mathsf{P}(\beta_j = 0) = \mathsf{P}(\beta_j = 1) = \frac{1}{2}$. [Hint: Investigate first the case that $[a, b) = [0, \frac{1}{2})$, and observe that whether the value of the series belongs to this intervals depends only on the value of β_1. Generalize this idea to any dyadic interval $2^{-k}[j, j+1)$, and finally express an arbitrary interval as a disjoint union of dyadic intervals. Observe that the end-points give no trouble: the probability that the series gets any given single value is zero (why?).]

5.5. Given a function ψ and an interval J, denote $\psi_J := T_{c(J)} D^2_{|J|}\psi$. Let I be a fixed standard dyadic interval. Prove that

$$\mathsf{E}\psi_{I \dot{+} \beta}(x) = (\psi * 1_{[0,1)})_I(x),$$

where E is the expectation over the random choice of the shift parameter $\beta \in \{0, 1\}^{\mathbb{Z}}$. [Hint: transform E into a Lebesgue integral, as in the lectures.]