Time-frequency analysis — Winter 2012

4. Exercise set

4.1. Prove the following relations for the modulation, translation and dilation operators $M_y f(x) := e^{i2\pi xy} f(x)$, $T_y f(x) = f(x-y)$, $D^p_\lambda f(x) = \lambda^{-1/p} f(x/\lambda)$, where $y \in \mathbb{R}$ and $\lambda > 0$:

$$\widehat{M_y f} = T_y \hat{f}, \qquad \widehat{T_y f} = M_{-y} \hat{f}, \qquad \widehat{D^p_\lambda f} = D^{p'}_{1/\lambda} \hat{f}.$$

(p' is the dual exponent, $1/p + 1/p' = 1$.)

4.2. Find the Fourier transforms of $x^\alpha f(x)$ and $\partial^\beta_x f(x)$ in terms of $\hat{f}$, and show that the Fourier transform maps the Schwartz space

$$\mathscr{S}(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}, \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta_x f(x)| < \infty \}$$

into itself (i.e., if $f \in \mathscr{S}(\mathbb{R})$, then also $\hat{f} \in \mathscr{S}(\mathbb{R})$).

4.3. Let ϕ be a "nice" function. Prove that for all $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$|D^1_\varepsilon \phi * f(x)| \leq C_\phi M f(x)$$

where C_ϕ is a constant only depending on ϕ, and

$$M f(x) := \sup_I 1_I(x) \frac{1}{|I|} \int_I |f(y)| \, \mathrm{d}y$$

is the Hardy–Littlewood maximal function (supremum over all intervals $I = [a, b]$, not just dyadic). Formulate more precisely the assumption that ϕ be "nice", so that this estimate works.

4.4. Write down the proof of Heisenberg's uncertainty principle for general x_0 and ξ_0. Then investigate which functions satisfy this inequality with an equality. [Hint: Follow the proof carefully, observe which inequalities were used, and recall the conditions for equality in these estimates. You should arrive at a simple differential equation for f.]

4.5. Prove the existence of a symmetric ($\phi_0(-x) = \phi_0(x)$) nonnegative $\phi_0 \in C^\infty$, which is strictly positive on $(-1, 1)$, zero outside, and satisfies

$$(*) \qquad \sum_{k \in \mathbb{Z}} \phi_0(x + k) \equiv 1.$$

For such a ϕ_0, check that $\phi := \sqrt{\phi_0}$ is also C^∞, and therefore satisfies the properties required for the basic wave packet.

[Hints: Here are possible steps to build ϕ_0:

- Let $\phi_1(x) := 1_{(0,\infty)}(x) e^{-1/x}$ and $\phi_2(x) := \phi_1(x)\phi_1(\frac{1}{3} - x)$.
- Let $\phi_3(x) := \int_{-\infty}^x \phi_2(y) \, \mathrm{d}y$ and $\phi_4(x) := c - \phi_3(1 - x)$.
- For a suitable c, the function $\phi_5(x) := \phi_3(x)$ if $x < \frac{2}{3}$ and $\phi_5(x) := \phi_4(x)$ if $x > \frac{1}{3}$ is well-defined.

Check that all these functions are C^∞, investigate the properties of ϕ_5, and use it to complete the construction of ϕ_0.]