Microlocal Analysis. (Gunther)

('96, '97)

① WF u $\quad u \in \mathcal{D}'(X)$, $(x_0, \xi_0) \notin WFu$

$\iff$ • $\exists\ \varphi \in C_c^\infty(X)$, $\varphi(x_0) \neq 0$

• $\exists\ V$ conic nbhd of ξ_0

s.t. $|\widehat{\varphi u}(\xi)| = O(|\xi|^{-N})$ $\forall N$.

② WF (Composition) $A : C_c^\infty(Y) \to \mathcal{D}'(X)$

$B : C_c^\infty(Z) \to \mathcal{D}'(Y)$

If defined, (see p20)

$WF'(A \circ B) \subseteq WF'A \circ WF'B$

$\cup (0 \times WF_Z' B) \cup (WF_X' A \times 0)$

③ ΨDO: $A \in \Psi_{cl}^0(X)$

$WFu \subseteq WF(Au) \cup \operatorname{Char} A$

$WF\, Au \subseteq WFu$.

④ Prop$^{\underline{n}}$ of Sing I $\quad P$: diff. operator

If $u \in \mathcal{D}'(X)$ & $Pu \in C^\infty(X)$,

$(x_0, \xi_0) \in WFu$

$\partial_\xi p \neq 0$ on null-bich. thru (x_0, ξ_0)

then whole null-bich. is in WFu.

⑤ $WF(I_{a,\varphi}) \subseteq \{(x, \xi) : \xi = d_x \varphi(x, \theta)$

$d_\theta \varphi(x, \theta) = 0 \}$.

⑥ $WF(Au)$ $\quad A : C_c^\infty(Y) \to \mathcal{D}'(X)$ linear, cts.

$WF_x A = WF_y A = \phi$

then $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$

and $WF(Au) \subseteq WF'(A) \circ WFu$

(9/30) MICROLOCAL ANALYSIS

Texts J.J. Duistermaat: F.I.O.'s
A. Grigis & J. Sjöstrand: Microlocal Analysis...

References
1. M. Taylor: ΨD.O.'s
2. F. Treves: Intro. to... Vols I, II.
3. L. Hörmander: F.I.O. I (Acta Math 1971)
4. J. Duistermaat, L. Hörmander: F.I.O. II (")

Consider ($n=3$)

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$$
$$u|_{t=0} = 0$$
$$\frac{\partial u}{\partial t}\Big|_{t=0} = \delta_0$$

has dist^{n}al sol^n $u(t,x) = \frac{1}{4\pi} \frac{t\,\delta(t-|x|)}{|x|} = K(t,\cdot)$

i.e. $u(t,\cdot)(\varphi) = \frac{t}{4\pi}\int_{S^2} \varphi(t\omega)\,d\omega$

$$\operatorname{singsupp} u(t,\cdot) = \{x \in \mathbb{R}^3 : |x| = |t|\}.$$

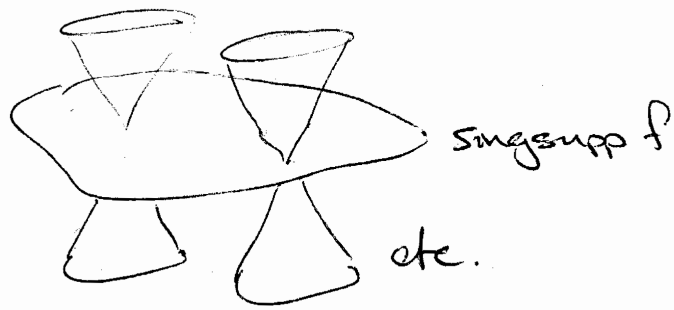
Now, $\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$

$$u|_{t=0} = 0$$
$$\frac{\partial u}{\partial t}\Big|_{t=0} = f \in \mathcal{E}'(\mathbb{R}^3)$$

The sol^{n} is given by

$$u(t,\cdot) = (K(t,\cdot) * f)$$

$$\text{singsupp}\, u(t,\cdot) \subseteq \{\text{singsupp}\, K(t,\cdot)\} + \{\text{singsupp}\, f\}$$



Microlocal analysis "looks" at "directions of sing's", not just "points" where the dist^{n} is singular.
Analysis of singularities of dist^{n}s will be on

$$\mathbb{R}^n_x \times \mathbb{R}^n_\xi = T^*(\mathbb{R}^n).$$

<u>Def^{n}</u> <u>Wave Front Set</u> of a dist^{n}.
Let $X \subseteq \mathbb{R}^n$, open, $u \in \mathcal{E}'(X)$.

<u>Prop^{n}</u>: $u \in C^\infty(X) \Leftrightarrow |\hat{u}(\xi)| = O(|\xi|^{-N})\ \forall N \in \mathbb{N}$

i.e. given $N \in \mathbb{N}$, $\exists\, C_N > 0$ st.

$$|\hat{u}(\xi)| \leq C_N |\xi|^{-N}, \quad |\xi| \geq 1.$$

(Hörmander) Let $u \in \mathcal{D}'(X)$, $(x_0, \xi_0) \in X \times (\mathbb{R}^n - 0)$ $\notin$ (WFu) (wave front set of u) if $\exists$ nbhds $\mathcal{U}$ of x_0, V of ξ_0 so that

$$|\widehat{\varphi u}(t\xi)| = O(t^{-N}) \quad \forall N \in \mathbb{N},$$
$$\forall \xi \in V, \quad \forall \varphi \in C_0^\infty(\mathcal{U}).$$

Later:

$$\begin{array}{ccc} WF\,u & \subseteq & X \times \mathbb{R}^n \setminus 0 \\ \downarrow \pi & & \downarrow \pi \\ \text{singsupp}\, u & \subseteq & X \end{array}$$

Examples

1. $u = \delta_0$, singsupp $\delta_0 = \{0\}$.
For any nbhd $\mathcal{U}$ of 0, let $\varphi \in C_0^\infty(\mathcal{U})$.

$$\widehat{\varphi u}(t\xi) = (\varphi u)(e^{-i\langle \cdot, t\xi\rangle})$$

$$= u(\varphi e^{-i\langle \cdot, t\xi \rangle}) = \varphi(0).$$

i.e. no decay in any dir$^{\underline{n}}$, so
$WF\,u = \{(0,\xi) : \xi \in \mathbb{R}^n - 0\}$.

2. $\delta_{x_1=0}$ in $\mathbb{R}^2$ $\quad \delta_{x_1=0}(\varphi) \triangleq \int \varphi(0,x_2)\,dx_2$.

singsupp $\delta_{x_1=0} = \{x_1 = 0\}$.
Take $x_0 = (0, x_2^0)$, $\mathcal{U}$ any nbhd of x_0, $\varphi \in C_0^\infty(\mathcal{U})$

Let $\xi_0 = (\xi_1^0, \xi_2^0)$.

$$(\varphi\delta_{x_1=0})^\wedge(t\xi_0) = (\varphi\delta_{x_1=0})(e^{-i\langle\cdot, t\xi_0\rangle})$$

$$= \delta_{x_1=0}(\varphi e^{-i\langle\cdot, t\xi_0\rangle})$$

$$= \int \varphi(0,x_2)\, e^{-i(x_2 \cdot t\xi_2^0)}\, dx_2$$

$$= \hat{\varphi}_0(t\xi_2^0) \quad \text{w/ } \varphi_0(x) = \varphi(0,x)$$

which is rapidly decreasing as long as $\xi_2^0 \neq 0$.
Thus $WF(\delta_{x_1=0}) \subseteq \{(x,\xi) : x = (0,x_2),\ \xi = \underset{(\xi_1 \neq 0)}{(\xi_1, 0)}\}$.

Exercise $WF_d(\delta_{x_1=0}) = \{(x,\xi) : x = (0, x_2),\ \xi = (\xi_1, 0)\}$
$=$ conormal bundle of $x_1 = 0$.

Exercise: Let F be a closed-conic set of $X \times \mathbb{R}^n - 0$
$((x,\xi) \in F \Rightarrow (x, \lambda\xi) \in F,\ \lambda > 0)$
Then $\exists\ u \in \mathcal{D}'(X)$ s.t. $WFu = F$.

"Theorem" (Propogation of Singularities).

$$\left(\frac{\partial^2}{\partial t^2} - c^2(x)\Delta\right) u = 0 \qquad c(x) > 0,\ c \in C^\infty(\mathbb{R}^n)$$
(sound speed of medium).

Let $(y_0, \eta_0) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus 0$ $\quad y_0 = (x_0, t_0)$ etc.
$\in WFu$

Then the bicharacteristic curve through (y_0, η_0) is also in WFu

Let $p(t, x, \tau, \xi) = \tau^2 - c^2(x)|\xi|^2$,
the principal symbol of $\partial_t^2 - c^2(x)\Delta$.
Hamiltonian vector field associated to p is

$$H_p = \sum_{j=1}^{n+1} \frac{\partial p}{\partial \eta_j}\frac{\partial}{\partial y_j} - \frac{\partial p}{\partial y_j}\frac{\partial}{\partial \eta_j}$$

Bicharacteristic curves are simply the integral curves of H_p.

Example $c(x) = 1$. The bicharacteristics are straight lines whose proj^{n}s are called characteristics

For general $c(x)$, the proj^n of bicharacteristics onto x-space are the geodesics of the metric

$$g = c^2(x)\,|dx|^2.$$

(10/2)

<u>Objectives</u>

(1) Propogation of singularities:

$Pu = 0$, $(x_0, \xi_0) \in WF\,u \Rightarrow$ Bichar. curve thru' (x_0,ξ_0) is in $WF\,u$

(cond^{n}s on P).

(2) Want to construct, as explicitly as possible, solutions of

$$(\partial_t^2 - c^2(x)\Delta)u = 0$$
$$u|_{t=0} = 0$$
$$\partial_t u|_{t=0} = f, \quad f \in \mathcal{E}'(X).$$

We will find $u = E(f)$ where E is a F.I.O..

Local sol^{n}s (near (x_0, t_0)) were known earlier (Lax '57). Using symplectic geometry one can in fact construct global sol^{n}s. (under minor cond^{n}s on $c(x)$).

<u>Example</u>. $c(x) = 1$.

$$(\partial_t^2 - \Delta)u = 0$$
$$u|_{t=0} = 0$$
$$\partial_t u|_{t=0} = f.$$

$$\hat{u}(t,\xi) = \int e^{-ix\xi} u(t,x)\,dx$$

$$\Rightarrow \partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0, \quad \hat{u}(0,\xi) = 0, \quad \partial_t \hat{u}(0,\xi) = \hat{f}(\xi)$$

$$\Rightarrow \hat{u}(t,\xi) = e^{it|\xi|}A(\xi) + e^{-it|\xi|}B(\xi)$$

$$A(\xi) = -B(\xi)$$

$$2i|\xi|\,A(\xi) = \hat{f}(\xi)$$

i.e.

$$A(\xi) = \frac{\hat{f}(\xi)}{2i|\xi|}.$$

Thus $u(t,x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\,\hat{u}(t,\xi)\,d\xi$

$$= \frac{1}{(2\pi)^n}\left(\int e^{i(x\xi + t|\xi|)}\,\frac{\hat{f}(\xi)}{2i|\xi|}\,d\xi + \int e^{i(x\xi - t|\xi|)}\cdot\frac{-\hat{f}(\xi)}{2i|\xi|}\,d\xi\right)$$

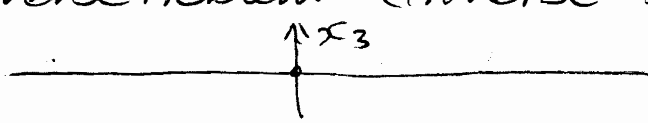
$$= \int e^{i\varphi_+(t,x,\xi)}a_+(t,x,\xi)\hat{f}(\xi)\,d\xi + \int e^{i\varphi_-(t,x,\xi)}a_-(t,x,\xi)\hat{f}(\xi)\,d\xi.$$

$$= E(f).$$

Also compute WFu for $(\partial_t^2 - c^2(x)\Delta)u = 0$.

<u>Applications</u>:

(A) Inverse Problem (Inverse seismic problem).



$x_3 = 0$ (flat surface of earth).

One model is $(\partial_t^2 - c^2(x)\Delta)u = \delta_0 \quad x_3 < 0$

$u = 0$ for $t \ll 0$

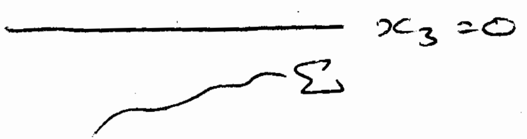
Measure $u|_{x_3=0}$

Problem: Determine $c(x)$ from $u|_{x_3=0}$.

(largely unsolved)

It is unknown if $F : c(x) \mapsto u_c|_{x_3=0}$ is injective.

If $c(x)$ has discontinuities along a surface Σ,



can one recover Σ from $u_c|_{x_3=0}$ & can one determine the jump across Σ?

In fact, one looks at $WF(u_c|_{x_3=0})$

(B) Suppose X compact manifold with Riemannian metric $g(x)$, Δ_g Laplace-Beltrami operator.

Consider $\Delta_g u_i = \lambda_i u_i$ in X

$\{\lambda_i\}$ e-values of Δ_g.

$S(t) = \sum e^{it\sqrt{\lambda_j}} \in \mathcal{D}'(\mathbb{R})$

Poisson summation formula:

$\text{singsupp } S(t) \subseteq \{0\} \cup \{\mathcal{L}\} \cup \{-\mathcal{L}\}$

where $\mathcal{L}$ = periods of closed geodesics.

This comes from...

Poisson summation formula $\quad X = S^1$

$$\sum_{n\in\mathbb{Z}} f(n) = c \cdot \sum_{n\in\mathbb{Z}} \hat{f}(n) \qquad \Delta_g = -\partial_\theta^2$$

e-values of $-\partial_\theta^2$ are $\lambda_j = 2\pi j^2$

Now

$$\sum e^{it\sqrt{\lambda_j}} = \sum \delta(t - 2\pi j)$$

$$\text{singsupp}\left(\sum e^{it\sqrt{\lambda_j}}\right) = \{t = 2\pi j : j \in \mathbb{Z}\}.$$

A byproduct of (A) is a method to invert the generalized Radon transform.

$$H_{\omega,s} = \{x \in \mathbb{R}^n \mid x\cdot\omega = s\} \quad \omega \in S^{n-1}$$

dH = Lebesgue measure on $H_{\omega,s}$.

$$Rf(s,\omega) = \int_{H_{s,\omega}} f(x)\,dH. \qquad \text{Radon Transform.}$$

To invert,

$$(-\Delta)^{\frac{n-1}{2}} R^t R f = c_n f.$$

$$R^t : C^\infty(\mathbb{R}_s \times S^{n-1}_\omega) \to C^\infty(\mathbb{R}^n)$$

$$R^t g(x) = \int_{S^{n-1}} g(x\cdot\omega, \omega)\,d\omega.$$

To generalize, we take families of surfaces more general than planes.
Q: Is this transform still invertable?

(10/4)

<u>Prop$^{\underline{n}}$</u> (Restatement of NFu)
$(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus 0$, $(x_0, \xi_0) \in NF(u)^c \triangleq G_u$
if $\exists$ $\mathcal{U}$ nbhd of x_0
V conic nbhd of ξ_0 in $\mathbb{R}^n \setminus 0$
s.t. given $N \in \mathbb{N}$ $\exists$ $C_{N,\mathcal{U},V}$ s.t.

$$|\widehat{\varphi u}(\xi)| \le C_N (1+|\xi|)^{-N}$$

$\xi \in V$, $\forall \varphi \in C_0^\infty(\mathcal{U})$.

<u>Prop^n 1</u> $(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus 0$, $(x_0, \xi_0) \in G(u)$
if $\exists\ \varphi \in C_0^\infty(X)$, $\varphi(x_0) \neq 0$, $\exists\ V$ conic nbhd of ξ_0 in $\mathbb{R}^n \setminus 0$ s.t.
$$|\widehat{\varphi u}(\xi)| = O(|\xi|^{-N}).$$

<u>Prop^n 2</u> $\pi : X \times \mathbb{R}^n \setminus 0 \longrightarrow X$
$$(x, \xi) \longmapsto x$$
Then $\pi(WF\, u) = \text{singsupp}\, u$.

Proof// (2). $x_0 \notin \text{singsupp}\, u$. Let $\mathcal{U}$ be a nbhd of x_0 s.t. $\mathcal{U} \cap \text{singsupp}\, u = \phi$

$\Rightarrow \varphi u \in C^\infty(X)\ \ \forall\ \varphi \in C_c^\infty(\mathcal{U})$.

$\Rightarrow \widehat{\varphi u}(\xi) = O(|\xi|^{-N})\ \ \forall\ \xi \in \mathbb{R}^n - 0,\ \forall N \in \mathbb{N}$.

$\Rightarrow (x_0, \xi) \in Gu\ \ \forall\ \xi \in \mathbb{R}^n - 0$.

Now let $x_0 \notin \pi(WFu) \Rightarrow (x_0, \xi) \in Gu\ \ \forall\ \xi \in \mathbb{R}^n - 0$
$\exists$ conic nbhd V_ξ of ξ in $\mathbb{R}^n \setminus 0$ with
$$|\widehat{\varphi u}(\eta)| \leq C_{N, V_\xi} (1 + |\eta|)^{-N}, \quad \eta \in V(\xi).$$

Then $\{V_\xi \cap S^{n-1}\}$ covers S^{n-1}, compact so take finite subcover $\{V_{\xi_j} \cap S^{n-1}\}$, $j = 1, \dots, M$
Thus we obtain
$$|\widehat{\varphi u}(\eta)| = O(|\xi|^{-N})\ \ \forall\ \xi \in \mathbb{R}^n \setminus 0$$
$\Rightarrow x_0 \notin \text{singsupp}\ u$. since $\varphi u \in C^\infty(X)$.
(use seminorm estimates) ◻

<u>Proof</u> (of prop^n 1)
Let $\psi \in C_0^\infty(X)$: $\psi(x_0) \neq 0$.
We want to prove $\exists\ V$ a conic nbhd of ξ_0 so that $|\widehat{\psi u}| = O(|\xi|^{-N})\ \ \forall\ \xi \in V$.
True trivally.

OTOH,

We know $\exists$ V... s.t.

$$|\widehat{\psi u}(\xi)| = O(|\xi|^{-N}), \quad \xi \in V$$

$$\psi \in C_0^\infty(X) \quad \psi(x_0) \neq 0$$

We must prove $\exists$ nbhd $\mathcal{U}$ of x_0, $\tilde{V}$ conic of ξ_0 so that $|\widehat{\varphi u}(\xi)| = O(|\xi|^{-N}) \ \forall \ \xi \in \tilde{V}$

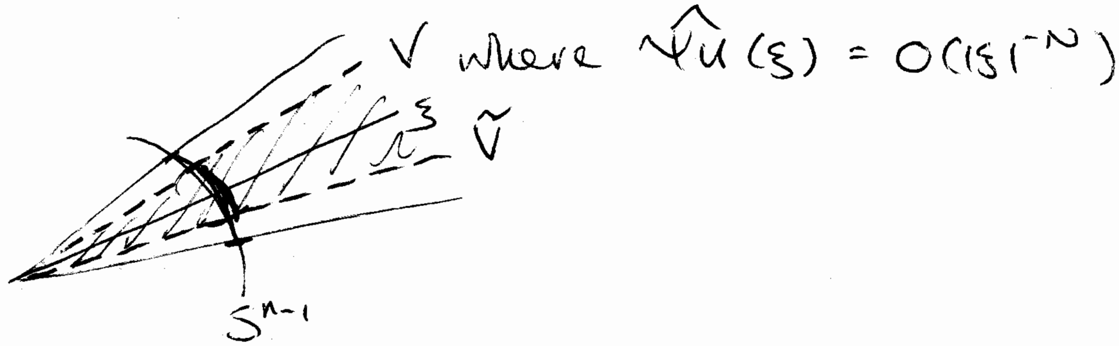
& $\forall \varphi \in C_0^\infty(\mathcal{U})$.

$\psi(x_0) \neq 0$, so let $\mathcal{U}$ be nbhd of x_0 s.t. $\psi \neq 0$ on $\mathcal{U}$.

Let $\varphi \in C_0^\infty(\mathcal{U})$.

Then
$$\varphi u = (\varphi \tfrac{1}{\psi})(\psi u)$$
$$= \alpha(\psi u), \quad \text{where } \alpha = \varphi \tfrac{1}{\psi} \in C_0^\infty(\mathcal{U}).$$

$$\widehat{\varphi u}(\xi) = \hat{\alpha} * \widehat{\psi u}(\xi)$$
$$= \int \hat{\alpha}(\xi - \eta)\, \widehat{\psi u}(\eta)\, d\eta = \int \hat{\alpha}(\eta)\, \widehat{\psi u}(\xi - \eta)\, d\eta$$



Let $\tilde{V}$ be a conic nbhd ξ (in V) s.t. for ε suff. small, $|\eta| \le \varepsilon |\xi|$

$$\Rightarrow \xi - \eta \in V \quad \forall \ \xi \in \tilde{V}$$

Write...

$$\varphi\hat{u}(\xi) = \int_{|\eta|\le\varepsilon|\xi|} \hat{\alpha}(\eta)\widehat{\psi u}(\xi-\eta)\,d\eta + \int_{|\eta|\ge\varepsilon|\xi|} \hat{\alpha}(\eta)\widehat{\psi u}(\xi-\eta)\,d\eta.$$

Now

$$\xi\in\tilde{V} \Rightarrow |\widehat{\psi u}(\xi-\eta)| \le C_N(1+|\xi-\eta|)^{-N}$$
$$(|\eta|\le\varepsilon|\xi|).$$

But
$$\begin{aligned}1+|\xi-\eta| &\ge 1+|\xi|-|\eta| \\ &\ge 1+|\xi|-\varepsilon|\xi| = 1+(1-\varepsilon)|\xi| \\ &\ge C(1+|\xi|) \quad \text{for } |\eta|\le\varepsilon|\xi|.\end{aligned}$$

Thus

$$\left|\int_{|\eta|\le\varepsilon|\xi|} \hat{\alpha}(\eta)\widehat{\psi u}(\xi-\eta)\,d\eta\right| \le C(1+|\xi|)^{-N}\int_{|\eta|\le\varepsilon|\xi|}|\hat{\alpha}(\eta)|\,d\eta$$
$$\le C(1+|\xi|)^{-N}$$

for $\xi\in\tilde{V}$.

Next,

$$(A)\quad \int_{|\eta|\ge\varepsilon|\xi|} |\hat{\alpha}(\eta)||\widehat{\psi u}(\xi-\eta)|\,d\eta \qquad \text{for any } M>0,$$
$$\le C_M\int_{|\eta|\ge\varepsilon|\xi|}(1+|\eta|)^{-M}(1+|\xi-\eta|)^k\,d\eta \quad \text{for some } k>0.$$

$$1+|\xi-\eta| \le 1+|\xi|+|\eta| \le C(1+|\eta|)$$
$$\Rightarrow (1+|\xi-\eta|)^k \le C_k(1+|\eta|)^k.$$

Given $N\in\mathbb{N}$ choose M so that

$$A \le C_N\int_{|\eta|\ge\varepsilon|\xi|}(1+|\eta|)^{-N}(1+|\eta|)^{-2n}\,d\eta$$
$$\le C_N(1+|\xi|)^{-N}\underbrace{\int_{|\eta|\ge\varepsilon|\xi|}(1+|\eta|)^{-2n}\,d\eta}_{\text{so that is is finite}}$$

$\blacksquare$

Corollary Let $f \in C^\infty(X)$, $u \in \mathcal{D}'(X)$.
$WF(fu) \subseteq WFu$.

Prop^n 3 WFu is a closed conic set in $X \times \mathbb{R}^n - 0$
(exercise).

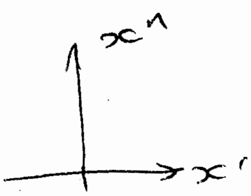
Exercise Let F be a closed conic set in $X \times \mathbb{R}^n \setminus 0$
Find $u \in \mathcal{D}'(X)$ s.t. $WFu = F$.

Operations on Distributions

$u \in \mathcal{D}'(X)$: **Restrict** $u|_\Sigma$, Σ a submanifold.

If $u \in H^s(\mathbb{R}^n)$ & $s > \frac{1}{2}$, then $u|_{x_n=0}$ makes sense and $Ru \in H^{s-1/2}(\mathbb{R}^{n-1})$.
We can also restrict to $x_n = 0$ if

$$u \in C^0(\mathbb{R}_{x_n}; \mathcal{D}'(\mathbb{R}^{n-1}))$$



$$Ru \in \mathcal{D}'(\mathbb{R}^{n-1}).$$

We will show that if $u \in \mathcal{D}'(\mathbb{R}^n)$ and
$WFu \cap \{(x', 0); (0, \xi_n); \xi_n \neq 0\} = \phi$.

Products $u, v \in \mathcal{D}'(\mathbb{R}^n)$, $u \cdot v$ doesn't nec. make sense.

eg. in $\mathbb{R}^2$, if $u = \delta_{x_1=0}$, $v = \delta_{x_2=0}$,
then $uv = \delta_0$

(10/7)

Intuitively, we will show that R is well defined & cts for dist^{n}s u such that

$$WF u \cap \{(x',0),(0,\xi_n) : \xi_n \neq 0\} = \emptyset.$$

<u>Multiplication of Distributions.</u>

eg. $\delta_0 \cdot \delta_0$ doesn't make sense

$$\delta_{x_1=0} \cdot \delta_{x_2=0} = \delta_{(0,0)} \quad \text{in } \mathcal{D}'(\mathbb{R}^2)$$

$$WF(\delta_{x_1=0}) = \{(0,x_2);(\xi_1,0),\ \xi_1 \neq 0\}$$

$$WF(\delta_{x_2=0}) = \{(x_1,0);(0,\xi_2),\ \xi_2 \neq 0\}.$$

Notice the WF sets have no intersection, even though the singsupp's intersect @ 0.

If $u,v \in C^\infty(\mathbb{R}^n)$ define

$$(u \otimes v)(x,y) := u(x)v(y)$$

define

$$\Delta : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

$$\Delta(x) = (x,x).$$

& define $\boxed{uv = \Delta^*(u \otimes v) = (u \otimes v) \circ \Delta}$

$$\mathbb{R}^{n-1} \xrightarrow{\ i\ } \mathbb{R}^n$$

$$x' \xmapsto{\ i\ } (x',0)$$

$u \in C^\infty(\mathbb{R}^n)$, $\quad Ru = i^* u = u \circ i$

More generally,

$X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open.

$\phi : X \to Y$, smooth.

Let $u \in C^\infty(Y)$; $\quad \phi^* u \in C^\infty(X)$

<u>Question</u> Under what cond^{n}s can we extend ϕ^* to dist^{n}s??

We consider the restriction map

$$i : \mathbb{R}^{n-1} \to \mathbb{R}^n$$
$$x' \to (x';0)$$

△ show that $i^* u$ is well defined for $u \in \mathcal{D}'(\mathbb{R}^n)$ s.t. $WFu \cap \underbrace{\{(x',0);(0,\xi_n),\xi_n \neq 0\}}_{\Gamma, \text{ say}} = \Pi = \phi$.

Let $\alpha \in C_0^\infty(\mathbb{R}^n)$, if $u \in C^\infty(\mathbb{R}^n)$,

$$R(\alpha u)(x') = (\alpha u)(x';0)$$
$$= \int e^{ix'\cdot\eta'} \widehat{\alpha u}(\eta)\,d\eta$$

$$(R(\alpha u)(x'))(\varphi) = \iint e^{ix'\eta'} \widehat{\alpha u}(\eta)\,d\eta\, \varphi(x)\,dx'$$
$$\psi \in C_0^\infty(\mathbb{R}^{n-1})$$

In general the integral in η is not cvgt.
We know that if $\alpha(x';0) \neq 0$,
$\widehat{\alpha u}(\eta)$ is rapidly decreasing near $\eta' = 0, \eta_n \neq 0$
(by cond^n. on Π).
Write integral

$$= \iint_V e^{ix'\eta'} \widehat{\alpha u}(\eta)\,d\eta\, \varphi(x)\,dx + \iint_{V^c} e^{ix'\eta'} \widehat{\alpha u}(\eta)\,d\eta\, \varphi(x)\,dx$$

w/ V a conic nbhd of $\eta' = 0$.
this is well defined.
In V^c, the point is that $\eta' \neq 0$; then
since $\Delta_{x'} e^{ix'\eta'} = -|\eta'|^2 e^{ix'\eta'}$
or

$$e^{ix'\eta'} = \frac{(-1)^N \Delta_{x'}^N e^{ix'\eta'}}{|\eta'|^{2N}} \qquad \eta' \neq 0$$

$$\iint_{V^c} e^{ix'\cdot\eta'} \alpha\hat{u}(\eta)\,d\eta\,\varphi(x)\,dx$$
$$= \iint_{V^c} \frac{(-1)^N \Delta_{x'}^N e^{ix'\eta'}}{|\eta'|^{2N}} \alpha\hat{u}(\eta)\,d\eta\,\varphi(x)\,dx$$
$$= \iint_{V^c} e^{ix'\eta'} \frac{\alpha\hat{u}(\eta)}{|\eta'|^{2N}}\,d\eta\,(-1)^N(\Delta_{x'})^N\varphi(x)\,dx.$$

which makes sense for suff. high N & cond^n on Π.

Now let $\varphi : X \to Y$ be smooth, $\varphi \in C^\infty(Y)$,
$\alpha \in C_0^\infty(Y)$.

Define $\varphi^*(\alpha u)(x) = ((\alpha u)\circ\varphi)(x)$.

$$\varphi^*(\alpha u)(\psi) = \int e^{i\langle\varphi(x),\eta\rangle}\alpha\hat{u}(\eta)\,d\eta\,\psi(x)\,dx \qquad \psi \in C_0^\infty(X)$$

__Def^n__ The set of normals of φ

$$N_\varphi = \{(y,\eta) \in Y \times \mathbb{R}^m - 0 ;\ y = \varphi(x) \ \exists\, x \in X,\ (d\varphi)^t|_{\varphi(x)}\,\eta = 0\}$$

(Exercise) If $\varphi = i$ inclusion,
$N_\varphi = \Gamma$ as before

Point is ... for points $(x,\eta) \in X \times \mathbb{R}^m - 0$ so that
$(\varphi(x),\eta) \notin N_\varphi$
we construct an operator L

$$L = \frac{1}{i}\sum_{j=1}^{n} L_j(x,\eta)\frac{\partial}{\partial x_j}$$

w/ L_j smooth in $X \times \mathbb{R}^m \setminus \{0\}$
homogeneous of degree -1 in η for large $|\eta|$.
so that

$$L e^{i\langle \varphi(x), \eta\rangle} = e^{i\langle \varphi(x), \eta\rangle}$$

Then $\varphi^*(\alpha u)(\psi) = \iint e^{i\langle \varphi(x), \eta\rangle} \alpha \hat{u}(\eta)\, d\eta\, \psi(x)\, dx$

$$= \iint_V + \iint_{V^c} (\text{——})$$

w/ V a conic nbhd of N_φ.

(10/9)

Define $N' = \{(x,\eta) \in X \times \mathbb{R}^m \setminus \{0\};\ \sum_{k=1}^{n} \frac{\partial \varphi_k(x)}{\partial x_j} \eta_k = 0$, $j = 1, \dots, n$ (i.e. $(d\varphi_x)^t_\eta = 0$ $\}$.

On N'^c we'll find the L above.

$$\frac{\partial}{\partial x_j}\left(e^{i\langle \varphi(x), \eta\rangle}\right) = i \sum \frac{\partial \varphi_k}{\partial x_j}(x)\, \eta_k$$

Put $\quad L_j(x,\eta) := \dfrac{\sum_{k=1}^{n} \frac{\partial \varphi_k}{\partial x_j} \eta_k}{\sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \varphi_k}{\partial x_j} \eta_k \right)^2}$

$(\eta \neq 0)$.

Notice that L_j is hom. of deg. -1 in η, $|\eta| \geq 1$.

<u>Cutoffs</u>: $\quad X \xrightarrow{\varphi} Y$

Let $y \in Y$, & $u \in C^\infty(Y)$.
Choose an open nbhd V of y so that $\bar{V}$ is compact and define

$$\varphi^*(\alpha u), \quad \alpha \in C_0^\infty(V)$$

so that it extends to $u \in \mathcal{D}'(Y)$,

$WF\, u \subseteq \Gamma$, Γ conic & closed in $Y \times \mathbb{R}^m \setminus \{0\}$.

$$\Gamma \cap N_\varphi = \emptyset$$

Let $\pi : (X \times \mathbb{R}^n - 0) \longrightarrow X$

$\qquad (Y \times \mathbb{R}^m - 0) \longrightarrow Y.$

& $\rho : (X \times \mathbb{R}^n - 0) \longrightarrow \mathbb{R}^n - 0$

$\qquad (Y \times \mathbb{R}^m - 0) \longrightarrow \mathbb{R}^m - 0.$

<u>Claim</u> $\exists$ open nbhd V_1 of y, $\bar{V}_1$ compact so that

$$\rho(\pi^{-1}(V_1) \cap \Gamma) \cap \rho(\pi^{-1}(V_1) \cap N_\varphi) = \emptyset$$

(Hint of proof of claim):

Suppose not. Then $\exists\ \eta \in \mathbb{R}^m - 0$

$$\eta \in \rho(\pi^{-1}(V_1) \cap \Gamma) \cap \rho(\pi^{-1}(V_1) \cap N_\varphi).$$

Then $\exists$ $(y_1, \eta) \in \pi^{-1}(V_1) \cap \Gamma$

$\qquad (y_2, \eta) \in \pi^{-1}(V_1) \cap N_\varphi.$

(depending on the nbhd V_1) for any nbhd V_1 of y.

Since the set is conic,

$$\frac{\eta_{V_1}}{\|\eta_{V_1}\|} \in (\sim) \cap (\sim)$$

By compactness, & choosing a cvgt. subseq., we find $(y, \tilde{\eta}) \in \Gamma \cap N_\varphi$, $\tilde{\eta} \neq 0$ a contradiction.

<u>Not$^{\underline{n}}$</u> $C = \rho(\pi^{-1}(V_1) \cap N_\varphi)$

$\qquad D = \rho(\pi^{-1}(V_1) \cap \Gamma)$

Now we choose $g \in C^\infty(\mathbb{R}^m - 0)$, g hom. of degree 0 for $|\eta| \geq 1$ and $g = 1$ on a conic nbhd E of D, $g = 0$ near $\eta = 0$, & $E \cap C = \emptyset$.

The operator

$\tilde{L} = f_1 g_1 L$, L as before

Choose $f \in C_0^\infty(V_1)$

choose a nbhd V of y, $\bar{V} \subseteq V_1$

& then $f_1 \in C_0^\infty(X)$ so that

$\operatorname{supp} f_1 \subseteq \varphi^{-1}(\operatorname{supp} f)$.

$g_1 \in C^\infty(\mathbb{R}^n - 0)$, g hom. of deg. 0 $|\eta| \geq 1$

$g_1 = 1$ on $\operatorname{supp} g$,

$g_1 = 0$ near $|\eta| = 0$

g_1 supp. on nbhd of $\operatorname{supp} g$,

$\operatorname{supp} g_1 \cap N_\varphi = \phi$.

Finally $\psi \in C_0^\infty(V)$,

$$(\alpha u)^* \psi = \iint e^{i\langle\varphi(x),\eta\rangle} f_1(x) g(\eta)\, \widehat{\alpha u}(\eta)\, d\eta\, \psi(x)\, dx$$

$$+ \iint e^{i\langle\varphi(x),\eta\rangle} f_1(x)(1-g(\eta))\, \widehat{\alpha u}(\eta)\, d\eta\, \psi(x)\, dx$$

What we have now is $f_1(x) g(\eta) \widehat{\alpha u}(\eta)$ is rappidly dec. in a conic nbhd V in $\mathbb{R}^n - 0$.
On $\operatorname{supp} f_1 \times \operatorname{supp} g$,

$$L e^{i\langle\varphi(x),\eta\rangle} = e^{i\langle\varphi(x),\eta\rangle}$$

$$\iint e^{i\langle\varphi(x),\eta\rangle} (1-g)(\eta)\, f_1(x)\, \widehat{\alpha u}(\eta)\, d\eta\, \psi(x)\, dx$$

$$:= \iint e^{i\langle\varphi(x),\eta\rangle} (\widehat{\alpha u})(\eta)(1-g)(\eta)\, ({}^tL)^k (f_1(x)\psi(x))\, dx\, d\eta.$$

What we did was find $f_1 \in C_0^\infty(X)$ & $g(\eta)$,
V a nbhd of y so that
$f_1(x)g(\eta)\widehat{\alpha u}(\eta)$ is rapidly decreasing $\forall\, x$ & η
Also, on supp $f_1(x)(1-g(\eta))$

$$({}^t d\varphi_x(\eta)) \neq 0$$

(10/11) We try again...

$$\phi : X \to Y \qquad u \in C^\infty(Y) \qquad \varphi^* u = u \circ \varphi.$$

Extend to dist^{n}s.

$WFu \subseteq \Gamma$, Γ closed conic set in $Y \times \mathbb{R}^n - 0$

$\Gamma \cap N_\varphi = \emptyset$.

<u>Def^n</u>

$$\mathcal{D}'_\Gamma(Y) = \{u \in \mathcal{D}'(Y) ;\ WFu \subseteq \Gamma\}.$$

Goal: Extend φ^* ctsly to $\mathcal{D}'_\Gamma$.

<u>Prop^n</u> $\varphi^* : C^\infty(Y) \to C^\infty(X)$
extends ctsly to

$$\varphi^* : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'(X).$$

<u>Step 1</u> Let $y \in Y$; find nbhd V_y of y and define (continuously) $\varphi^*(\alpha u)$ $\forall\, \alpha \in C_0^\infty(V_y)$ and $u \in \mathcal{D}'_\Gamma(Y)$

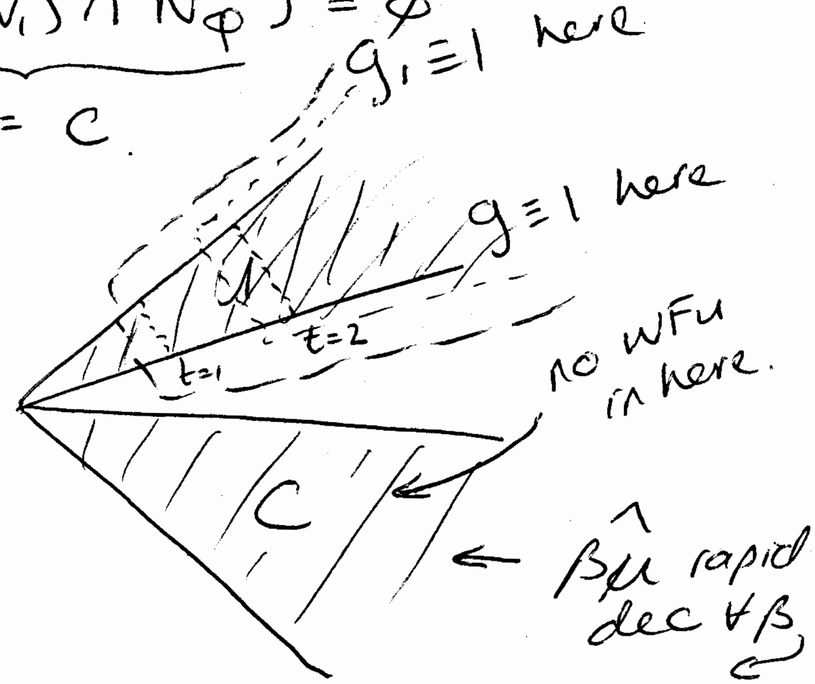
First we take $u \in C^\infty(Y)$.

<u>Claim</u> $\exists\, V_1$ nbhd of y, $\overline{V_1}$ compact so that

$$\rho(\pi^{-1}(V_1) \cap \Gamma) \cap \rho(\pi^{-1}(V_1) \cap N_\varphi) = \emptyset$$

Let U = conic nbhd of $\Gamma_1 := \rho(\pi^{-1}(V_1) \cap \Gamma)$
so that

$$U \cap \underbrace{\rho(\pi^{-1}(V_1) \cap N_\phi)}_{:= C} = \emptyset$$



· Remark: We have: if K compact, $K \subseteq Y$,
$\forall \beta \in C_0^\infty(K)$,

$$\widehat{\beta u}(\eta) = O(|\eta|^{-N}) \quad \forall \eta \in C, \forall N$$

Now define g & g_1:

$g, g_1 \in C^\infty(\mathbb{R}^n - 0)$

$g, g_1 \equiv 0$ on some nbhd of 0 & C.

$g \equiv 1$ on a conic nbhd of $\bigcup_{t \geq 2} U(t)$

$$U(t) := \{\eta \in U : \|\eta\| = t\}.$$

$g_1 \equiv 1$ on conic nbhd of supp g & $\bigcup_{t \geq 1} U(t)$

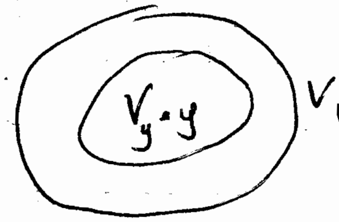
Now cut-offs in X-space:

Let $f_1, f \in C^\infty(X)$ such that: Choose V_y, $\overline{V_y} \subseteq V_1$ s.t. $f \equiv 1$ on $\varphi^{-1}(V_y)$.

$\operatorname{supp} f \subseteq \varphi^{-1}(V_1)$

& $f_1 \equiv 1$ on $\operatorname{supp} f$

$\operatorname{supp} f_1 \subseteq \varphi^{-1}(V_1)$.



——— η ———

$$L = \frac{1}{i} \sum L_j(x,\eta) \frac{\partial}{\partial x_j}$$

well defined on $\{(x,y) \in X \times \mathbb{R}^n - 0$ s.t. ${}^t d\varphi_x \eta \neq 0\}$.

Define: $M = f_1(x)\, g_1(\eta)\, L$

<u>Prop</u>n Let $\alpha \in C_0^\infty(V_y)$, let $u \in C^\infty(Y)$.

Then $(\alpha u \circ \varphi)(\psi) =$

$$= [\varphi^*(\alpha u)](\psi) = \iint e^{i\langle\varphi(x),\eta\rangle} g(\eta)\, \widehat{\alpha u}(\eta) (M^t)^k (f(x)\psi(x))\, dx\, d\eta$$

$$+ \iint e^{i\langle\varphi(x),\eta\rangle} (1-g(\eta))\, \widehat{\alpha u}(\eta)\, f(x)\, \psi(x)\, dx\, d\eta.$$

for k suff. large.

<u>Proof</u>: $\varphi^*(\alpha u)(\psi)$

$$= \iint e^{i\langle\varphi(x),\eta\rangle} \widehat{\alpha u}(\eta)\, \psi(x)\, dx\, d\eta$$

$$= \iint e^{\sim} g(\eta)\, \widehat{\alpha u}(\eta)\, \psi(x)\, dx\, d\eta + \iint e^{\sim} (1-g(\eta))\, \widehat{\alpha u}(\eta)\, \psi(x)\, dx\, d\eta$$

Now $\operatorname{supp} \varphi^*(\alpha u) \subseteq \varphi^{-1}(V)$
since $\operatorname{supp} \varphi^*(\alpha u) \subseteq \varphi^{-1}(\operatorname{supp} \alpha u)$.
so 1st int. can have $f(x)$ inserted
($f + (1-f)$ but $(1-f) = 0$ on $\operatorname{supp} \varphi^*(\alpha u)$).
We have (finally), by int. by parts,

$$\iint e^{i\langle\varphi(x),\eta\rangle} g(\eta) ({}^tM)^k (f(x)\psi(x))\, dx\, d\eta \quad \forall k$$

since $Me^{i\langle\varphi(x),\eta\rangle} = e^{i\langle\varphi(x),\eta\rangle}$

on $\operatorname{supp} f \times \operatorname{supp} g$.

$\square$

Def Let $u \in \mathcal{D}'_\Gamma(Y)$; let $\alpha \in C_0^\infty(V)$
Then $(\varphi^*(\alpha u))(\psi) :=$

$$\iint e^{i\langle\varphi(x),\eta\rangle} (\widehat{\alpha u})(\eta) g(\eta) ({}^tM)^k (f(x)\psi(x))\, dx\, d\eta$$

$$+ \iint e^{i\langle\varphi(x),\eta\rangle} (\widehat{\alpha u})(\eta)(1-g(\eta)) f(x)\psi(x)\, dx\, d\eta$$

for k suff. large (depending on the order of αu). $k > m + \operatorname{order}(\alpha u)$

In order to prove that the defⁿ is independent of the choice of f, g, f_1, g_1, etc. we just prove that if $u_\varepsilon \in C^\infty(Y)$, $u_\varepsilon \to u$ in $\mathcal{D}'_\Gamma(Y)$, then
$\varphi^*(\alpha u_\varepsilon) \to \varphi^*(\alpha u)$ in $\mathcal{D}'(X)$
$\forall\ \alpha \in C_0^\infty(V)$.

Let $\psi \in C_c^\infty(X)$.

$(\varphi^*(\alpha u_\varepsilon)(\psi) - \varphi^*(\alpha u)(\psi))$

$$= \iint e^{i\langle\varphi(x),\eta\rangle} (\alpha(u_\varepsilon - u))^\wedge(\eta)\, g(\eta)({}^tM)^k(\psi(x) f(x))\, dx\, d\eta$$

$$+ \iint e^{i\langle\varphi(x),\eta\rangle} (\alpha(u_\varepsilon - u))^\wedge(\eta)(1-g(\eta)) f(x)\psi(x)\, dx\, d\eta .$$

Write $\iint_{|\eta|\le R} + \iint_{|\eta|>R}$,

R suff. large, to be chosen.

Both integrads are decaying as fast as we want for k large enough.

So $\iint_{|\eta|\geq R}$ can be made as small as we want by R suff large.

i.e. $\iint_{|\eta|\geq R} (1+|\eta|)^{-1}(1+|\eta|) \longmapsto$

$$\leq \frac{1}{1+R} \iint (1+|\eta|) \longmapsto$$

Now show $(\alpha(u_\varepsilon - u))^\wedge(\eta) \to 0$ as $\varepsilon \to 0$ uniformly for $|\eta| \le R$.

But

$(\alpha(u_\varepsilon - u))^\wedge(\eta) = \alpha(u_\varepsilon - u)(e^{-i\langle\cdot,\eta\rangle})$

We have $\to 0$ uniformly in compact sets of η Arzela-Ascoli (equicts family since

$\frac{\partial}{\partial\eta_j}(\alpha(u_\varepsilon - u)(e^{-i\langle\cdot,\eta\rangle}))$

$= \alpha(u_\varepsilon - u)(\frac{\partial}{\partial\eta_j} e^{-i\langle\cdot,\eta\rangle})$

$= -i x_j \alpha(u_\varepsilon - u)(e^{-i\langle \cdot, \eta\rangle})$

bndd since α comp suptd.

Then $(\alpha(u_\varepsilon - u))^\wedge$ equi-cts.

Now, let $u \in \mathcal{D}'_\Gamma(Y)$, to define $\varphi^* u$, take a covering of Y by V_{y_i}, $\{\alpha_i\}$ part^n of unity subordinate to covering,

$$\varphi^*(u)(\psi) = \sum_i \varphi^*(\alpha_i u)(\psi)$$

Check Σ is finite, since

$$\operatorname{supp} \varphi^*(\alpha_i u) \subseteq \varphi^{-1}(V_i)$$

& then supp ψ only intersects a finite number of V_i.

(10/14) $\varphi : X \to Y$,

$N_\varphi = \{(y, \eta) \in Y \times \mathbb{R}^n \setminus \{0\},\ y = \varphi(x),\ {}^t d\varphi_x(\eta) = 0\}$.

Γ closed conic nbhd of WFu.

$\Gamma \cap N_\varphi = \emptyset$

$\varphi^* u \in \mathcal{D}'(X)$ well defined

$u_\varepsilon \in C^\infty(Y)$, $u_\varepsilon \to u$ in $\mathcal{D}'(Y)$

$\varphi^* u_\varepsilon \to \varphi^* u$ in $\mathcal{D}'(X)$.

$$\mathcal{D}'(Y) \longrightarrow \mathcal{D}'(X)$$
$$u \longmapsto \varphi^* u \quad \text{cts.}$$

φ^* has a unique cts ext^n $\varphi^* : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'_{\tilde{\Gamma}}(X)$

w/ $\tilde{\Gamma} = \{(x, \xi) \in X \times \mathbb{R}^n \setminus \{0\},\ (\varphi(x), y) \in \Gamma,\ {}^t d\varphi_x \eta = \xi\}$.

<u>Prop</u>[n] $\operatorname{supp} \varphi^* u \subseteq \varphi^{-1}(\operatorname{supp} u)$ $(WF u \subseteq \Gamma)$.

Proof. $u_\varepsilon \in C^\infty(Y)$, $u_\varepsilon \to u$, in $\mathcal{D}'(Y)$,

$\operatorname{supp} u_\varepsilon \searrow \operatorname{supp} u$.

$\operatorname{supp} \varphi^* u_\varepsilon \subseteq \varphi^{-1}(\operatorname{supp} u_\varepsilon)$ for $u_\varepsilon \in C^\infty(Y)$

...

$\blacksquare$

<u>Next</u>: Calc. $WF(\varphi^* u)$.

Let $G(\varphi^* u) = (WF(\varphi^* u))^c$

$(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$

$(x_0, \xi_0) \in G(\varphi^* u)$.

We want

$$(\psi \varphi^* u)^\wedge(t\xi) = (\psi \varphi^* u)\left(e^{-i\langle \cdot, t\xi\rangle}\right) = O(t^{-N})$$

$\xi \in$ nbhd of ξ_0

$\operatorname{supp} \psi \subseteq$ nbhd of x_0.

$$(\psi \varphi^* u)^\wedge(t\xi) = \varphi^* u\left(e^{-i\langle \cdot, t\xi\rangle} \psi\right)$$

= finite sum of terms of the form (p. of u).

$$\iint e^{i(\langle \varphi(x), \eta\rangle - \langle x, t\xi\rangle)} \psi(x) g(\eta) f(x) \widehat{\alpha u}(\eta)\, d\eta\, dx$$

$$+ \iint e^{i\langle \varphi(x), \eta\rangle} \widehat{\alpha u}(\eta)(1-g)(\eta)({}^tM)^k\left(e^{-i\langle x, t\xi\rangle} f(x)\psi(x)\right) d\eta\, dx$$

for k suff. large.

$$({}^tM)^k\left(e^{-i\langle x, t\xi\rangle} f(x)\psi(x)\right)$$

is a sum of terms of the form

$$e^{-i\langle x, t\xi\rangle} (t\xi)^j F(x) \tilde{\psi}(x), \quad 0 \le j \le k.$$

so,

$$\iint e^{i(\langle\varphi(x),\eta\rangle - \langle x, t\xi\rangle)} \alpha\hat{u}(\eta)(t\xi)^{i}(1-g)(\eta)F(x)\tilde{\psi}(x)\,dx\,d\eta$$

change $\eta \to t\eta$.

to get

$$t^{J}\iint e^{it(\langle\varphi(x),\eta\rangle - \langle x,\xi\rangle)} g(t\eta)\alpha\hat{u}(t\eta)F(x)\tilde{\psi}(x)\,dx\,d\eta$$

& $(1-g)(t\eta)$ . . .

Near $G(\alpha u)$, $\alpha\hat{u}(t\eta) = O(t^{-N})$ & we have decay in t of all orders.

We have to concentrate in directions η near $WF(\alpha u)$. ~~We ne~~ It suffices to find a diff. op.

$$L = \sum_{j=1}^{n} L_j(x,\eta)\frac{\partial}{\partial x_j}$$

$$L^{k} e^{it(\langle\varphi(x),\eta\rangle - \langle x,\xi\rangle)} = (it)^{k} e^{it(\langle\varphi(x),\eta\rangle - \langle x,\xi\rangle)}$$

$$\frac{\partial}{\partial x_j} e^{i(\langle\varphi(x),\eta\rangle - \langle x,\xi\rangle)} = i\left(\left(\sum_{k=1}^{n}\frac{\partial\varphi_k}{\partial x_j}\eta_k\right) - \xi_j\right)e^{i(\langle\varphi(x),\eta\rangle - \langle x,\xi\rangle)}.$$

Cond^{n} will be

$$\xi_j \neq \sum_{k=1}^{n}\frac{\partial\varphi_k}{\partial x_j}\eta_k \qquad \text{for all } j \text{ near } WF(\alpha u).$$

Proposition.
Let $u \in \mathcal{D}'_\Gamma(Y) = \{u \in \mathcal{D}'(Y);\ WF u \subseteq \Gamma\}$
such that
$$\Gamma \cap N_\varphi = \emptyset$$
Then $WF(\varphi^* u) \subseteq \{(x,\xi) \in X \times \mathbb{R}^n - 0;\ (\varphi(x), \eta) \in WF u$ and ${}^t d\varphi_x \eta = \xi\}$
$:= \tilde{\Gamma}$

Proof (as above). ☐

Applications.

(a) Restriction. $i : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$
$i(x') \longmapsto (x', 0)$.

We can define $i^* u$ if
$WF u \subseteq \Gamma$, Γ closed a conic
$\Gamma \cap \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n - 0;\ x = (x', 0),\ \xi = (0, \xi_n), \xi_n \neq 0\}$
$= \phi$.
And $Ru := i^* u$.
Exercise: Calculate $WF(i^* u)$
More generally, $i : X \to Y$ inclusion.
Define $i^* u$.

(b) Multiplication.
$f, g \in C^\infty(X)$.

$$f \cdot g = \Delta^*(f \otimes g), \quad (f \otimes g) \in C^\infty(X \times X)$$
$$(f \otimes g)(x,y) = f(x) g(y).$$

$$\Delta : X \to X \times X$$
$$(x) \longmapsto (x, x).$$

$$N_\Delta = \{((x,\xi),(y,\eta)) \in (X \times X) \times (\mathbb{R}^{2n} \setminus 0\}; \; x = y, \; \xi = -\eta \}$$

$$= \{((x,\xi),(x,-\xi)); \; (x,\xi) \in X \times \mathbb{R}^n - 0\}.$$

Suppose $u_1, u_2 \in \mathcal{D}'(X)$, $WF u_i \subseteq \Gamma_i$,
Γ_i closed cone in $X \times \mathbb{R}^n \setminus 0$.

We can define

$$\Delta^*(u_1 \otimes u_2)$$

as long as $WF(u_1 \otimes u_2) \cap N_\Delta = \emptyset$.

$S \in \mathcal{D}'(X)$, $T \in \mathcal{D}'(Y)$

$$(S \otimes T)(\varphi \otimes \psi) = S(\varphi) T(\psi), \quad \varphi \in C_0^\infty(X)$$
$$\psi \in C_0^\infty(Y).$$

$$((\varphi \otimes \psi)(S \otimes T))^\wedge (t\xi)$$
$$= (\varphi \otimes \psi)(S \otimes T)(e^{-i\langle \cdot, (\xi_x, \xi_y) \rangle})$$
$$\xi \in \mathbb{R}^n_x \times \mathbb{R}^n_y \times 0$$

(*) $$= (\varphi S)(e^{-i\langle \cdot, t\xi_x \rangle})(\psi T)(e^{-i\langle \cdot, t\xi_y \rangle}).$$

<u>Prop^n</u> $WF(S \otimes T) \subset \{((x,y),(\xi_x,\xi_y)) \in X \times Y \times (\mathbb{R}^n_x \times \mathbb{R}^n_y - 0);$
$(x,\xi_x) \in WF S, \; \& \; \xi_x \neq 0\}$
$\cup \{\cdots; (y,\xi_y) \in WF T, \; \xi_y \neq 0\}$
$\cup \{(x,y),(0,\xi_y); \; (y,\xi_y) \in WF T\}$
$\cup \{(x,y),(\xi_x,0); \; (x,\xi_x) \in WF S\}.$

(10/25)

If $u, v \in \mathcal{D}'(X)$,

$(u \otimes v) \in \mathcal{D}'(X \times X)$

& $(u \otimes v)(\varphi \otimes \psi) := u(\varphi)\, v(\psi) \quad \varphi, \psi \in C_0^\infty(X)$.

<u>Prop^n</u> $WF(u \otimes v) \subseteq$

$$\{(x, y; \xi, \eta) \in (X \times X) \times \mathbb{R}^{2n} - \{0\},\ (x, \xi) \in WF u,\ (y, \eta) \in WF v\}$$

$$\cup \{(x, \xi) \in WF u,\ \eta = 0\}$$

$$\cup \{(y, \eta) \in WF v,\ \xi = 0\}$$

Proof (look @ (*) & just require that some part is rapidly decreasing). $\square$

To define uv we first compute N_Δ

$$N_\Delta = \{(x, y; \xi, \eta) \in (X \times X) \times \mathbb{R}^{2n} \setminus \{0\},\ \ {}^t d\Delta_{(x,x)}(\xi, \eta) = 0,\ x = y\}.$$

$$
\begin{aligned}
{}^t d\Delta_{(x,x)}(\xi, \eta) &= (\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n) \\
&= 0 \\
\iff \quad \xi &= -\eta.
\end{aligned}
$$

<u>Prop^n</u> $N_\Delta = \{(x, x; \xi, -\xi);\ x \in X,\ \xi \in \mathbb{R}^n \setminus \{0\}\}$.

$u \in \mathcal{D}'_{\Gamma_1}(X)$, Γ_1 a closed cone in $X \times \mathbb{R}^n \setminus \{0\}$.

$v \in \mathcal{D}'_{\Gamma_2}(X)$, Γ_2 a closed cone in $X \times \mathbb{R}^n \setminus \{0\}$.

To define $uv = \Delta^*(u \otimes v)$, it is sufficient that

$$N_\Delta \cap WF(u\otimes v) = \phi.$$

But we know $WF(u\otimes v) \subseteq (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times 0) \cup (0 \times \Gamma_2)$

$\{(y,0), y \in X\}$
i.e. $\xi = 0$.

It suffices that:

$$N_\Delta \cap (\Gamma_1 \times \Gamma_2) = \phi$$

__Prop__^{n} $u, v \in \mathcal{D}'(X)$, $X \subseteq \mathbb{R}^n$ open, $u \in \mathcal{D}'_{\Gamma_1}(X)$, $v \in \mathcal{D}'_{\Gamma_2}(X)$
Γ_1, Γ_2 closed conic sets in $X \times \mathbb{R}^n \setminus \{0\}$.
Assume $\Gamma_1 \cap (-\Gamma_2) = \phi$

where $(-\Gamma_2) = \{(x,\xi);\ (x,-\xi) \in \Gamma_2\}$.
Then

$$uv = \Delta^*(u\otimes v)$$

is well defined; moreover, if

$u_\varepsilon \in C^\infty(X)$ $\quad v_\varepsilon \in C^\infty(X)$
$u_\varepsilon \to u$ in $\mathcal{D}'(X)$ $\quad v_\varepsilon \to v$ in $\mathcal{D}'(X)$

then

$$u_\varepsilon v_\varepsilon \to uv \text{ in } \mathcal{D}'(X).$$

$\blacksquare$

__Example__ $WF(\delta_{x_1}) = \{(0, x_2; \xi_1, 0); \xi_1 \neq 0\} = \Gamma_1$
$(n=2)$ $WF(\delta_{x_2}) = \{(x_1, 0; 0, \xi_2); \xi_2 \neq 0\} = \Gamma_2$.
$\Gamma_1 \cap (-\Gamma_2) = \phi$.

Now compute $WF(uv)$:

$$uv = \Delta^*(u \otimes v)$$

$$WF(uv) \subseteq \{(x,\tau) \in X \times \mathbb{R}^n \setminus \{0\} \,;\, {}^t d\Delta_{(x,x)}\tilde{\eta} = \tau, \quad (x,x,\tilde{\eta}) \in \underbrace{WF(u \otimes v)}_{\subseteq (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times 0) \cup (0 \times \Gamma_2)} \}$$

where $\tilde{\eta} \in \mathbb{R}^{2n} \setminus \{0\}$
$\tilde{\eta} = (\xi, \eta)$.

(x,x,ξ,η) $(x,x,\xi,0)$ $(x,x,0,\eta)$

$${}^t d\Delta_{(x,x)}\tilde{\eta} = (\xi_1 + \eta_1, \dots, \xi_n + \eta_n)$$

so...

(*) <u>Theorem</u> <u>(Multiplication)</u>

Let $\Gamma_i \subseteq X \times \mathbb{R}^n \setminus \{0\}$ be closed, conic, $i = 1, 2$.

$$C^\infty(X) \times C^\infty(X) \longrightarrow C^\infty(X)$$
$$(u, v) \longmapsto u \cdot v$$

has a unique cts ext^{n} to

$$\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(X) \longrightarrow \mathcal{D}'_{\tilde{\Gamma}}(X)$$

if

$$\Gamma_1 \cap (-\Gamma_2) = \emptyset$$

where

$(\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$

$$\tilde{\Gamma} = (\Gamma_1 + \Gamma_2) \cup (\Gamma_1 \times 0) \cup (0 \times \Gamma_2)$$

<u>Excercise</u>

$\tilde{\Gamma}$ is closed & conic in $X \times \mathbb{R}^n \setminus \{0\}$.

<u>COMPOSITION OF OPERATORS</u>

General situation: $X \subseteq \mathbb{R}^n$ open
$Y \subseteq \mathbb{R}^m$ open
$Z \subseteq \mathbb{R}^p$ open

$$\left.\begin{array}{l} A : C_0^\infty(Y) \to \mathcal{D}'(X) \\ B : C_0^\infty(X) \to \mathcal{D}'(Z) \end{array}\right\} \text{ linear \& cts.}$$

<u>Question</u>: can $B \circ A$ make sense?

It is enough to define $K_{B\circ A}$, the Schwarz kernel of $B\circ A$.
$K_{B\circ A} \in \mathcal{D}'(Y\times Z)$.

Suppose $A: C_0^\infty(Y) \to C_0^\infty(X)$ } linear ops
$B: C_0^\infty(X) \to C_0^\infty(Z)$
$B\circ A: C_0^\infty(Y) \to C_0^\infty(Z)$

$$K_{B\circ A}(y,z) = \int K_A(y,x) K_B(x,z)\,dx$$

(formally at least!)

Rewrite this:

$$K_{B\circ A}(y,z) = \int \Delta^*(K_A \otimes K_B)(y,x,z)\,dx$$

$$K_A \otimes K_B \in C^\infty(Y\times X\times X\times Z)$$

$$\Delta: Y\times X\times Z \to Y\times X\times X\times Z$$
$$(y,x,z) \longmapsto (y,x,x,z).$$

We will rewrite

$$K_{B\circ A} = \pi_* \Delta^*(K_A\otimes K_B).$$

w/ $\pi: Y\times X\times Z \to Y\times Z$ is projection
$(y,x,z) \mapsto (y,z)$.

(10/28)

$\varphi: X\to Y$ C^∞ Define $\varphi_* u$, $u\in\mathcal{D}'(X)$.

Moral: $A: C_0^\infty(Y) \to \mathcal{D}'(X)$

$B: C_0^\infty(Z) \to \mathcal{D}'(Y)$

what is $A\circ B$?

$K_{A\circ B}$ = Schwarz kernel of $A\circ B$

Formally, $K_{A\circ B}(x,z) = \int K_A(x,y)\, K_B(y,z)\, dy$

$$= \pi_* \Delta^* (K_A \otimes K_B)(x,z)$$

$$\Delta : X\times Y\times Z \longrightarrow X\times Y\times Y\times Z$$
$$(x,y,z) \longmapsto (x,y,y,z)$$
$$\pi : X\times Y\times Z \to X\times Z \quad \text{projection:}$$
$$(x,y,z) \longmapsto (x,z).$$

$\varphi : X \to Y$, notice $\varphi^* : C^\infty(Y) \to C^\infty(X)$ is always defined on smooth fc^{n}s, not always on dist^{n}s.

How about push-forwards?

$$\varphi_*(u)(\psi) \triangleq u(\varphi^*\psi) \qquad \psi \in C_0^\infty(Y)$$
$$= u(\psi\circ\varphi)$$

Problem: $\psi\circ\varphi \in C^\infty$ but not nec. compactly supp'd.

<u>Def^n</u> Let $u \in \mathcal{D}'(X)$, $\varphi : X \to Y$ C^∞ such that $\varphi|\mathrm{supp}\, u$ is proper.

Then
$$(\varphi_* u)(\psi) \triangleq u(\varphi^*\psi) = u(\psi\circ\varphi).$$

<u>Prop^n</u> $\varphi_* : \mathcal{D}'(X) \to \mathcal{D}'(Y)$ linear, cts.

Example $X \subseteq \mathbb{R}^n$ open, $Y \subseteq \mathbb{R}^M$ open

$$\pi * X \times Y \to X$$
$$(x,y) \mapsto x.$$

let $u \in C_0^\infty(X \times Y)$.

$$(\pi_* u)(\psi) = u(\psi \circ \pi) = \int u(x,y)(\psi \circ \pi)(x,y)\,dy\,dx$$
$$= \int u(x,y)\psi(x)\,dx\,dy$$
$$= \int \left(\int u(x,y)\,dy\right)\psi(x)\,dx.$$

so, $\pi_* u = \int u(x,y)\,dy$

"Integration over the fiber"

——— " ———

Calculate $WF(\varphi_* u)$: (φ as in def^{n}).

$WF(\varphi_* u) \subseteq Y \times \mathbb{R}^M \setminus \{0\}$.

$$(\varphi_* u)(\psi) = u(\psi \circ \varphi)$$

First, y must be: $y = \varphi(x)$ for some $x \in X$

Notice $\varphi^{-1}(\{y\}) \cap \operatorname{supp} u$ is compact.

Let $y_0 \in \varphi(x_0)$, $x_0 \in X$.

$$(\varphi_* u)(\psi e^{-i\langle\cdot, t\eta\rangle}) \qquad \psi \text{ supptd near } y_0$$

Let $\alpha \in C_0^\infty(X)$ be supptd near $\varphi^{-1}(\{y_0\})$. Consider

$$\varphi_*(\alpha u)(\psi e^{-i\langle\cdot, t\eta\rangle})$$

$$= (\alpha u)((\psi \circ \varphi)(e^{-i\langle\cdot, t\eta\rangle} \circ \varphi))$$

(use $u(f) = \int \hat{u}(\xi)\check{f}(\xi)\,d\xi$.)

$(\xi \to t\xi)$

$$= t^{n}\int \widehat{\chi u}(t\xi)\int e^{it(\langle x,\xi\rangle - \langle \varphi(x),\eta\rangle)}\psi(\varphi(x))\,dx\,d\xi$$

We ask: where does

$$d_x(\langle x,\xi\rangle - \langle\varphi(x),\eta\rangle) = 0\ ?$$

Ans: when ${}^t d\varphi_x \eta = \xi$.

If $\xi \neq {}^t d\varphi_x \eta$, we can find a diff. operator so that

$$L^M(e^{it(\langle x,\xi\rangle - \langle\varphi(x),\eta\rangle)}) = \frac{e^{it(\langle x,\xi\rangle - \langle\varphi(x),\eta\rangle)}}{(it)^M}$$

So...

<u>Theorem</u> (Push-forward WF)

$X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, open.

$\varphi : X \to Y$ C^∞,

$u \in \mathcal{D}'(X)$: $\varphi|\mathrm{supp}\,u$ is proper

Then $\varphi_* u \in \mathcal{D}'(Y)$ and

$$WF(\varphi_* u) \subseteq \{(y,\eta) \in Y \times \mathbb{R}^m \setminus \{0\} \mid y = \varphi(x)\ \exists x \in X \text{ and } {}^t d_x\varphi(\eta) = \xi \text{ and } (x,\xi) \in WFu\}.$$

<u>Example</u> $\pi : X \times Y \to Y$ project^n.

$u \in \mathcal{D}'(X \times Y)$

$\pi|\mathrm{supp}\,u$ proper.

$$WF(\pi_* u) \subseteq \{(y,\eta) \in Y \times \mathbb{R}^m \setminus \{0\} \mid \text{if } (x,y;\xi,\eta) \in X \times Y \times \mathbb{R}^{n+m} \setminus \{0\},\ (x,y,0,\eta) \in WFu\}$$

since ... ${}^t d\pi_{(x,y)}(\xi) = (\tilde{\xi}, \tilde{\eta})$

$$\underset{=0}{\uparrow}$$

— u —

Suppose $A: C_0^\infty(Y) \to \mathcal{D}'(X)$ linear, cts.

Def^n $WF A \triangleq WF(K_A) \subseteq X \times Y \times \mathbb{R}^{n+m} \setminus \{0\}$.

Example: $Af(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)d\xi$, $f \in C_0^\infty(X)$

$$a \in S^m(X \times \mathbb{R}^n)$$

$K_A \in \mathcal{D}'(X \times Y)$ & as an oscillatory int.,

$$K_A = \int e^{i\langle x-y, \tilde{\xi}\rangle} a(x,\tilde{\xi})d\tilde{\xi}.$$

So $WF(K_A) \subseteq \{(x, x; \xi, \eta)\}$

Now $(\psi K_A)(e^{-i\langle\cdot, t(\tilde{\xi},\tilde{\eta})\rangle})$ $\psi \in C_0^\infty(X \times Y)$
spt'd near diagonal

$= K_A(\psi e^{-i\langle\cdot, t(\tilde{\xi},\tilde{\eta})\rangle})$

$$= t^{-n}\int e^{it(\langle x, \xi-\tilde{\xi}\rangle - \langle y, \xi+\tilde{\eta}\rangle)} a(x,t\xi)\psi(x,y)\,d\xi\,dx\,dy$$

$d_{(x,y)}(\langle x, \xi-\tilde{\xi}\rangle - \langle y, \xi+\tilde{\eta}\rangle)$

$d_x\langle x, \xi-\tilde{\xi}\rangle = 0$ when $\xi = \tilde{\xi}$ $\quad \tilde{\xi} = -\tilde{\eta}$
$d_y\langle y, \xi+\tilde{\eta}\rangle = 0$ when $\xi = -\tilde{\eta}$.

so $WF(K_A) \subseteq \{(x, x; \xi, -\xi),\ \xi \in \mathbb{R}^n \setminus \{0\}\}$

(10/30)

<u>Def^n</u> $WF'(A) \triangleq \{(x,y,\xi,-\eta) \,;\, (x,y,\xi,\eta) \in WF\,A\}$.

<u>Prop^n</u> Let $A \in \Psi^m(X)$.
Then $WF(A) \subseteq \{(x,x,\xi,-\xi) \,;\, a(x,\xi)$ is not rapidly decreasing in $\xi.\}$.

Proof//
(Exercise: Prove equality!).

$$K_A = \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\,d\xi$$

$$(\alpha K_A)(e^{-i\langle\cdot,\, t(\tilde\xi,\tilde\eta)\rangle}) \qquad \alpha \in C_0^\infty(X\times Y)$$

$$= K_A(\alpha e^{-i\langle\cdot,\, t(\tilde\xi,\tilde\eta)\rangle})$$

$$= \int e^{i(\langle x,\xi - t\tilde\xi\rangle + \langle y,-\xi - t\tilde\eta\rangle)} a(x,\xi)\,\alpha(x,y)\,d\xi\,dx\,dy$$

$$\xi \to t\xi$$

$$= t^n \int e^{it(\langle x,\xi-\tilde\xi\rangle + \langle y,-\xi-\tilde\eta\rangle)} a(x,t\xi)\,\alpha(x,y)\,dx\,dy\,d\xi$$

$$\xi = \tilde\xi \ \ \& \ \ \xi = -\tilde\eta$$

are where $d_{(x,y)}(\langle x,\xi-\tilde\xi\rangle + \langle y,-\xi-\tilde\eta\rangle) = 0$

$\square$

Back to composition...

$$A : C_0^\infty(Y) \to \mathcal{D}'(X), \quad B : C_0^\infty(Z) \to \mathcal{D}'(Y)$$

$$K_{A\circ B} \triangleq \pi_* \Delta^*(K_A \otimes K_B).$$

Under what conditions can we define
$\Delta^*(K_A \otimes K_B)$?

$$N_\Delta \cap WF(K_A \otimes K_B) = \phi :$$

$$WF(K_A \otimes K_B) \subseteq (WF(K_A) \times WF(K_B)) \cup (WF(K_A) \times 0) \cup (0 \times WF(K_B))$$

$$N_\Delta = \{(x, y, \tilde{y}, z; \xi, \eta, \tilde{\eta}, \zeta) \in X \times Y \times Y \times Z \times \mathbb{R}^{n+2m+p} - \{0\};$$
($y = \tilde{y}$ for image of Δ)
$$ {}^t d\Delta_{(x,y,y,z)}(\xi, \eta, \tilde{\eta}, \zeta) = 0 \}.$$

$$= \{(x, y, y, z; \xi, \eta, \tilde{\eta}, \zeta); \xi = \zeta = 0, \eta = -\tilde{\eta}, \eta \neq 0\}.$$

$$= \{(x, y, y, z; 0, \eta, -\eta, 0); \eta \neq 0\}.$$

<u>Def</u>

__Def^n__ $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ linear, cts.

$WF_X(A) \triangleq \{(x, \xi) \in X \times \mathbb{R}^n \setminus \{0\}; (x, y; \xi, 0) \in WF A\}$ for some $y \in Y$.

″ $WF_Y(A) \triangleq \{(y, \eta) \in Y \times \mathbb{R}^m \setminus \{0\}; (x, y, 0, \eta) \in WF A \ \exists x\}$

__Prop^n__ $N_\Delta \cap WF(K_A \otimes K_B) = \phi$

if

$$WF_Y'(A) \cap WF_Y'(B) = \phi$$

— n —

$\pi_* \Delta^*(K_A \otimes K_B)$ is well defined if

$$\pi|_{\operatorname{supp} \Delta^*(K_A \otimes K_B)}$$

is proper.

<u>Def^n</u> $C_1 \subseteq \{(x,y,\xi,\eta) \in X \times Y \times \mathbb{R}^{n+m} \setminus \{0\}\}$ relation
$C_2 \subseteq \{(y,z,\eta,\zeta) \in Y \times Z \times \mathbb{R}^{m+p} \setminus \{0\}\}$ "

$$C_1 \circ C_2 \triangleq \{(x,z,\xi,\zeta) \in X \times Z \times \mathbb{R}^{n+p} \setminus \{0\} \,;\, \exists (y,\eta) \in Y \times \mathbb{R}^m \setminus \{0\} \text{ w/ } (x,y,\xi,\eta) \in C_1 \text{ \& } (y,z,\eta,\zeta) \in C_2\}.$$

<u>Theorem</u> (COMPOSITION).
Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, $B : C_0^\infty(Z) \to \mathcal{D}'(Y)$ linear cts.
Assume $WF'_Y(A) \cap WF'_Y(B) = \emptyset$ & $\pi|_{\operatorname{supp} \Delta^*(K_A \otimes K_B)}$ proper;
then $A \circ B : C_0^\infty(Z) \to \mathcal{D}'(X)$ linear cts
with Schwarz kernel given by
$$K_{A\circ B} = \pi_* \Delta^* (K_A \otimes K_B).$$
and
$$WF'(A \circ B) \subseteq WF'(A) \circ WF'(B) \cup (0 \times WF'_Z(B)) \cup (WF'_X(A) \times 0)$$
$WF'(A\circ B) \subseteq X \times Z \times (\mathbb{R}^{n+p} \setminus \{0\})$

— " —

<u>Corollary</u> If in addition, $WF'_Z(B) = \emptyset$, $WF'_X(A) = \emptyset$ then
$$WF'(A \circ B) \subseteq WF'(A) \circ WF'(B).$$

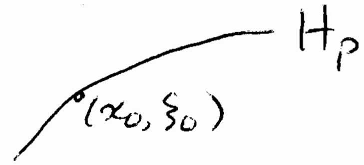
PROPOGATION OF SINGULARITIES

P "hyperbolic equation" $\left(\begin{array}{l} P = \partial_t^2 - c^2(x)\Delta \\ P = \partial_t^2 - \Delta_g \end{array} \right)$

$Pu = 0$

Suppose $(x_0, \xi_0) \in WF\,u$; where does the sing. propogate?

Prove: the bicharacteristic curve through (x_0, ξ_0) $\subseteq WF u$.



bicharacteristics = integral curves of H_p where $p(x, \xi)$ = princ. symbol of P.

<u>Theorem.</u> (Another characterizat^{n} of $WF u$).

Let $u \in \mathcal{D}'(X)$.

$$WF u = \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ \text{s.t. } Au \in C^\infty(X)}} \text{charact. } A$$

recall: $\text{charact } A = \{ (x, \xi) \in X \times \mathbb{R}^n \setminus \{0\} ;\ \sigma_0(A)(x,\xi) = 0 \}$.

Let $A \in \Psi^0_{cl}(X)$, $(x_0, \xi_0) \notin$ charact A

so $\sigma_0(A)(x_0, \xi_0) \neq 0$.

We can construct a properly supported $B \in \Psi^0(X)$ so that $(BA - I)f = \int e^{i\langle x, \xi \rangle} r(x,\xi) \hat{f}(\xi)\, d\xi$

with r rapidly decreasing in a conic nbhd of (x_0, ξ_0).

$\Rightarrow (x_0, \xi_0) \notin WF((BA - I)u)$

Assume that $Au \in C^\infty(X)$; then $BAu \in C^\infty(X)$

$\Rightarrow (x_0, \xi_0) \notin WFu$.

Thus

$$\bigcap_{\substack{A \in \Psi^0_{cl}(X), \\ Au \in C^\infty}} \operatorname{Char} A \supseteq WFu.$$

(11/1)

(*) Def^n $A \in \Psi^m_{cl}(X)$, classical ΨDO of order m

$$A f(x) = \int e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\, d\xi, \quad f \in C_0^\infty(X)$$

$a \in S^m_{cl}(X \times \mathbb{R}^n)$ i.e.

$$a \sim \sum_{j \le m} a_j(x,\xi), \quad a_j \in S^j(X \times \mathbb{R}^n)$$

hom. of deg. j, $|\xi|$ large.

Prop^n $A \in \Psi^0_{cl}(X)$, $WFu \subseteq WF(Au) \cup \operatorname{Char} A$

Proof// Let $(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$, $(x_0, \xi_0) \notin (WFAu \cup \operatorname{Char} A)$

Then $\sigma_0(A)(x_0, \xi_0) \neq 0$; find microlocal inverse at (x_0, ξ_0) i.e. $\exists\, B \in \Psi^0_{cl}(X)$ properly sppt'd, s.t.

$(x_0, \xi_0) \notin WF(BAu - u) \quad \forall u \in \mathcal{D}'(X)$

$(x_0, \xi_0) \notin WF\,Au \Rightarrow (x_0, \xi_0) \notin WF(BAu)$

$\Rightarrow (x_0, \xi_0) \notin WFu$. ∎

Prop$^{\underline{n}}$ Let $A \in \Psi^m_{cl}(X)$; then
$$WF\, Au \subseteq WF\, u \quad \forall\, u \in \mathcal{D}'(X)$$
(generaliz$^{\underline{n}}$ of pseudolocal property:
$$\text{singsupp } Au \subseteq \text{singsupp } u \text{)}.$$
Proof:
$$Au = \int e^{ix\cdot\xi} a(x,\xi)\,\hat{u}(\xi)\,d\xi$$
$$(\alpha Au)^\wedge(t\tilde{\xi}) = (Au)(\alpha e^{-i\langle\cdot, t\tilde{\xi}\rangle})$$
$$= u(A^t(\alpha e^{-i\langle\cdot, t\tilde{\xi}\rangle}))$$

WLOG, take A properly supported

$$= \beta u(A^t(\alpha e^{-i\langle\cdot, t\tilde{\xi}\rangle}) \qquad \beta \equiv 1 \text{ near spt } \alpha$$
$$\beta \in C_0^\infty(X)$$
Now
$$A^t(\alpha e^{-i\langle\cdot, t\tilde{\xi}\rangle}) = \int e^{i\langle x-y,\xi\rangle}\, a^t(y,\xi)\,\alpha(y)\, e^{-i\langle y, t\tilde{\xi}\rangle}\,dy\,d\xi$$
$$a^t(y,\xi) \sim \sum_{|\alpha|} \frac{i^\alpha}{\alpha!} \partial_\xi^\alpha a(x,\xi)\Big|_{y=x}$$

Spose $(x_0,\xi_0) \notin WF\, u$; then we know
$$(\beta u)(e^{-i\langle\cdot, t\tilde{\xi}\rangle})) = O(t^{-N}) \quad (\alpha(x_0) \neq 0 \dots)$$
$$Au = \int e^{ix\cdot\xi} a(x,\xi)\,\hat{u}(\xi)\,d\xi\,;$$
$$A(\beta u) = \int e^{ix\cdot\xi} a(x,\xi)\,\widehat{\beta u}(\xi)\,d\xi$$
(... come back to this later ...)

Alternately... $A: C_0^\infty(Y) \to \mathcal{D}'(X)$
$B: C_0^\infty(Z) \to \mathcal{D}'(Y)$.

$A \circ B$: $WF_Y'(A) \cap WF_Y'(B) = \emptyset$

$$WF(A\circ B) \subseteq WF'(A) \cup WF'(B) \cup (O \times WF_Y'(B)) \cup \cdots$$

Th$^{\underline{m}}$ on comp$^{\underline{n}}$ can be extended to X, Y, Z C^∞ manifolds.
Let $u \in \mathcal{D}'(Y)$; $A: C_0^\infty(Y) \to \mathcal{D}'(X)$ linear, cts.
Take $Z = \{pt\}$. Then

$$A \circ B\varphi = Au \qquad (\text{take } \varphi = 1\ ?)$$

Then

$$WF(Au) \subseteq \underbrace{WF'(A) \circ WF u}_{= WF u} \qquad \text{since } \Psi DO\text{'s}.$$

Back to the theorem...

<u>Theorem</u> $\displaystyle WF u = \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ Au \in C^\infty(X)}} \operatorname{Char} A.$

Pf// So far, $WF u \subseteq WF Au \cup \operatorname{Char} A$. $(*)$

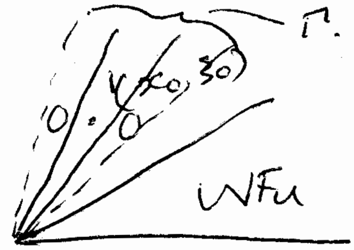
Take

$$(x_0, \xi_0) \notin \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ Au \in C^\infty(X)}} \operatorname{Char} A$$

So $\exists\, A \in \Psi^0_{cl}(X)$ with $\sigma_0(A)(x, \xi_0) \neq 0$, $Au \in C^\infty(X)$
$\Rightarrow (x_0, \xi_0) \notin WF u$ from $(*)$;

i.e. $\displaystyle WF u \subseteq \bigcap_{\cdots} \operatorname{Char} A.$

Now take $(x_0,\xi_0) \notin WFu$



Take $a_0(x,\xi)$ homo. of degree 0 large $|\xi|$,
$a_0(x,\xi) \equiv 1$ in a conic nbhd of (x_0,ξ_0)
$a_0(x,\xi) = 0$ in Γ^c where Γ is a larger conic nbhd of (x_0,ξ_0) & $\Gamma \cap WFu = \emptyset$

Define $A_0 f = \int e^{ix\cdot\xi} a_0(x,\xi) \hat{f}(\xi)\, d\xi$.

Then $A_0 u$ is smooth, i.e. $(x_0,\xi_0) \notin \bigcap\limits_{\dots} \operatorname{Char} A$.

$\square$

Note// $\bigcap\limits_{\dots} \operatorname{Char} A \supseteq WFu$ is independent of the statement $(WF(Au) \subseteq WFu)$.

$(x_0,\xi_0) \notin \bigcap\limits_{\dots} \operatorname{Char} A \Rightarrow \exists\, A$ w/ $\sigma_0(A)(x_0,\xi_0) \neq 0$ & Au is smooth.

Now construct $B \in \Psi^0_{cl}(X)$ s.t. properly spted & $(x_0,\xi_0) \notin WF(BAu - u)$

$\Rightarrow (x_0,\xi_0) \notin WFu$

Now since we have $WFu = \bigcap\limits_{\dots} \operatorname{char} A$,

<u>Corollary</u> Let $\tilde{A} \in \Psi^0_{cl}(X)$. Then
$WF\tilde{A}u \subseteq WFu \quad \forall\, u \in \mathcal{D}'(X)$.

Proof// $$WF(\tilde{A}u) = \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ A(\tilde{A}u) \in C^\infty}} \operatorname{char} A \subseteq \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ Au \in C^\infty(X)}} \operatorname{char} A = WFu$$

since: if $A\tilde{A}u \in C^\infty$, $Au \in C^\infty$

(11/4)

<u>Remark</u> $(x_0, \xi_0) \notin WFu \quad (x_0,\xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$

$\iff \exists\ A \in \Psi^0_{cl}(X),\ \sigma_0(A)(x_0,\xi_0) \neq 0$

s.t. Au smooth.

<u>Remark</u> $B \in \Psi^0_{cl}(X)$, properly supported.

$$Bf(x) = \int e^{ix\cdot\xi} b(x,\xi) \hat{f}(\xi)\, d\xi, \qquad b \in S^0_{cl}(X \times \mathbb{R}^n)$$

Let $u \in \mathcal{D}'(X)$, $(x_0,\xi_0) \notin WFu$

If $b \in S^{-\infty}(X \times \mathbb{R}^n)$ in a conic nbhd of WFu,

then $Bu \in C^\infty(X)$

<u>Prop^n</u>: (Microlocal Pseudolocality)

Let $A \in \Psi^m_{cl}(X)$. Then

$$WF(Au) \subseteq WFu \qquad \forall u \in \mathcal{D}'(X)$$

Proof//

Suppose $(x_0,\xi_0) \notin WFu$. Choose $\chi(x,D) \in \Psi^0_{cl}(X)$

s.t. $\chi(x,D) f(x) \int e^{ix\cdot\xi} \chi(x,\xi) \hat{f}(\xi)\, d\xi$

$\chi(x_0,\xi_0) \neq 0$ & $\chi \equiv 0$ in a conic nbhd of WFu

Then $\chi A \in \Psi^m_{cl}(X)$, &

$$(\chi A) f(x) = \int e^{ix\cdot\xi} b(x,\xi) \hat{f}(\xi)\, d\xi$$

modulo smoothing

$$b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \chi\, \partial_x^\alpha a$$

so b vanishes in a conic nbhd of WFu.

Thus $(\chi A) u \in C^\infty(X)$.

But $WFu = \bigcap \operatorname{char} A \qquad \Rightarrow (x_0,\xi_0) \notin WF(Au)$

$A \in \Psi^0_{cl}(X),\ Au \in C^\infty(X)$

$\square$

Summary

① $WF\,u = \bigcap_{\substack{A \in \Psi^0_{cl}(X) \\ Au \in C^\infty(X)}} \mathrm{Char}\, A.$

② $WF(Au) \subseteq WF\,u \quad \forall\ A \in \Psi^m_{cl}(X),\ \forall\ u \in \mathcal{D}'(X)$

③ (Microlocal elliptic regularity)
$WF\,u \subseteq WF(Au) \cup \mathrm{Char}(A)$

Remark $A : C_0^\infty(Y) \to \mathcal{D}'(X)$
$B : C_0^\infty(Z) \to \mathcal{D}'(Y).$

$A \circ B,\quad WF'_Y(A) \cap WF'_Y(B) = \phi.$

Let $u \in \mathcal{D}'(Y)$ & take $Z = \{pt\}$.

$B(\{pt\})(\varphi) := u(\varphi).$

Prop^n Let $u \in \mathcal{E}'(Y)$, $u \in \mathcal{D}'_\Gamma(Y)$, $\Gamma \cap WF'_Y(A) = \phi$.
Then $A : \mathcal{E}'(Y) \cap \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'(X)$ cts,
$WF(Au) \subseteq (WF'(A) \circ WF\,u) \cup WF'_X A.$

Proof// follows from th$^{\underline{m}}$ about $A \circ B$. (See insert 24a)

Corollary If $A \in \Psi^m_{cl}(X)$, $WF'A \subseteq \{(x, x; \xi, \xi),\ x \in X, \xi \in \mathbb{R}^n \setminus \{0\}\}$.
Then $WF'_Y(A) = 0$, so
$WF(A) \subseteq WF'(A) \circ WF u. = WF u.$

If $Z = \{pt\}$, $B : C_0^\infty(\{pt\} = Z) \to \mathcal{D}'(Y)$
given by
$$B(\{pt\})(\varphi) = u(\varphi).$$
Then (formally)
$$(Bv)(\varphi) = \int \left[\int K_B(y, pt.)\, v(pt.)\, dz \right] \varphi(y)\, dy$$
$$= \int K_B(y, pt.)\, \varphi(y)\, dy$$
$$\Rightarrow K_B(y, pt.) = u$$
$$\Rightarrow \quad WF(B) = \{(y, pt.; \eta, 0) : (y, \eta) \in WF\, u\}$$
$$\equiv WF\, u$$
$$\Rightarrow WF'(A \circ B) \subseteq (WF'(A) \circ WF\, u) \cup WF'_X A$$
since $WF_Z B = \emptyset$.

Propogation of Singularities.

Let $p : X \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, real valued.

Think of $p(x, \xi)$ as the principal symbol of

$$P(t,y,\partial_t,\partial_y) = \partial_t^2 - \sum_{i,j=1}^{n} g^{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \text{First order}$$

$g = (g_{ij}(x))$ Riemannian metric.

eg $g = c^2(y)\,\delta_{ij}(y)$, $c \in C^\infty(X)$, $c \geq \varepsilon > 0$.

$$p(t,y,\tau,\eta) = -\left(\tau^2 - \sum g^{ij}(y)\eta_i \eta_j\right)$$

Let $u \in \mathcal{D}'(X)$, $P(x, D_x)u \in C^\infty(X)$.

Let $(x_0, \xi_0) \in X \times \mathbb{R}^n \setminus \{0\}$, ~~$(x_0,\xi_0) \in WFu$~~.

If $\sigma_m(p)(x_0, \xi_0) \neq 0$, then $(x_0, \xi_0) \notin WFu$.

(since $WFu \subseteq WF(Pu) \cup \mathrm{Char}(P)$.)

i.e.:

Prop^n $Pu \in C^\infty$ & $\sigma_m(P)(x_0, \xi_0) \neq 0$

Then $(x_0, \xi_0) \notin WFu$.

Def^n $$H_p \triangleq \sum_{i=1}^{n} \left(\frac{\partial p}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i}\frac{\partial}{\partial \xi_i} \right)$$

Remark $\omega = \sum d\xi_i \wedge dx_i$ on $X \times \mathbb{R}^n \setminus \{0\}$, non-deg. 2-form,

let $p : X \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, C^∞; H_p is s.t.

$$\omega(H_p, T) = dp(T),$$

Exercise $H_p p = 0$

$\underline{\text{Def}^n}$ Bicharacteristic curve through (x_0, ξ_0) is the integral curve of H_p through (x_0, ξ_0).

$\underline{\text{Def}^n}$ Null-Bicharacteristics are those contained in $p = 0$.

<u>Exercise</u> $p(t, y, \tau, \xi) = -(\tau^2 - \sum g^{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \text{lower order})$.
Bicharacteristic curves
$(t(s), y(s), \tau(s), \xi(s))$.
$\pi : \mathbb{R} \times Y \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to Y$
Then $\pi(\cdots) = y(s)$ = geodesics of g.

<u>Theorem.</u> (Version 1: Prop^n of Singularities).
$p : X \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ C^∞.
$P(x, D)$ diff-operator of order m with principal symbol $p(x, \xi)$.
Let $u \in \mathcal{D}'(X)$ s.t. $P(x, D) u \in C^\infty(X)$
Let $(x_0, \xi_0) \in WFu$, then the whole connected comp. of the (null) bicharacteristic curve is in WFu if $\partial_\xi p(x, \xi) \neq 0$ on bicharacteristic in $X \times \mathbb{R}^n \setminus \{0\}$.

(11/6)
Notice since $d_\xi p \neq 0$, $\{p = 0\}$ is a nice manifold.

Proof,, (Hörmander).
WLOG we may assume $P \in \Psi^1_{cl}(X)$ satisfying the same cond^{n}s, $\sigma_1(P)(x_0, \xi_0) = 0$, $d_\xi \sigma_1(P) \neq 0$ on bichar. curve thru' (x_0, ξ_0).
Since: if P a diff. op. of order m, then

$(\sqrt{I-\Delta})^{-m+1} P \in \Psi^1_{cl}(X)$ & satisfies the same cond^{n}s. Also notice

$$H_{(1+|\xi|^2)^{-\frac{m+1}{2}} p(x,\xi)} = (1+|\xi|^2)^{-\frac{m+1}{2}} H_p \text{ on } p=0$$

Not^n: $\sigma_1(P)(x,\xi) = p(x,\xi)$

Example: $P = (D_t - \sqrt{-\Delta})$

Idea: Assume $(x_0, \xi_0) \in \Gamma$

Find $B \in \Psi^0_{cl}(X)$ so that $Bu \in C^\infty(X)$

$\sigma_0(B) \neq 0$ on the bichar. curve thru' (x_0,ξ_0)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\overset{\shortparallel}{\Gamma}$

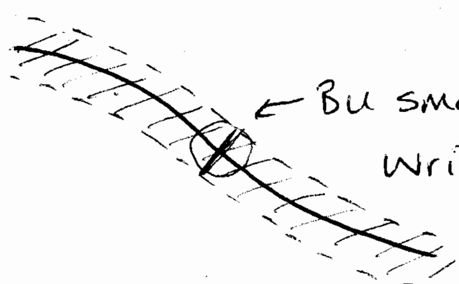
(then this $\Rightarrow (x_0,\xi_0) \notin WFu$).

For any B, $BPu \in C^\infty(X)$

But $\underbrace{BPu}_{C^\infty} = PBu + \underbrace{[B,P]u}_{\substack{\text{try to make this} \\ \text{smoothing}.\ \in \Psi^{-\infty}(X)}}$

Then $PBu \in C^\infty(X)$.

We will construct so that $Bu \in C^\infty$ near x_0



$\leftarrow$ Bu smooth here

Write P as a hyperbolic operator & then conclude $Bu \in C^\infty$ in tube

Remark H_p is homog. of deg. 0

$$H_p(f_1(x,\lambda\xi)) = H_p(f(x,\xi)).$$

Step 1 Construct^n of B, $\sigma_0(B) \neq 0$ on Γ & $[B,P] \in \Psi^{-\infty}(X)$; B supported in as small a nbhd of Γ as we wish.

$$\sigma_0([B,P]) = -iH_p\,\sigma_0(B) = -i\sum\left(\frac{\partial p}{\partial \xi_j}\frac{\partial \sigma_0(B)}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial \sigma_0(B)}{\partial \xi_j}\right)$$

$H_p\,\sigma_0(B) = 0$:

So take $\sigma_0(B) = 1$ on Γ, $\sigma_0(B)$ hom. deg 0

$\sigma_0(B)$ supp. in a conic nbhd of Γ.

Let $B_0 \in \Psi^0_{cl}(X)$ with $\sigma_0(B_0)$ as above.

Now,

$$\underbrace{[B_0, P]}_{= R_{-1}} \in \Psi^{-1}_{cl}(X)$$

Choose B_{-1} so that $[B_0 + B_{-1}, P] \in \Psi^{-2}_{cl}(X)$

So we need

$$\sigma_{-1}([B_{-1}, P] + R_{-1}) = 0$$

But LHS $= H_p\,\sigma_{-1}(B_{-1}) + \sigma_{-1}(R_{-1})$

So choose $H_p\,\sigma_{-1}(B_{-1}) = -\sigma_{-1}(R_{-1})$

We can solve this for a sol^{n} hom. of deg -1

$\sigma_{-1}(B_{-1}) \neq 0$ on a transversal hypersurface (thru (x_0, ξ_0)) to H_p, with any initial cond^{n} we may choose.

So we choose B_{-1} satisfying the D.E. above.

$\sigma_{-1}(B_{-1})$ supported in a conic nbhd of Γ.

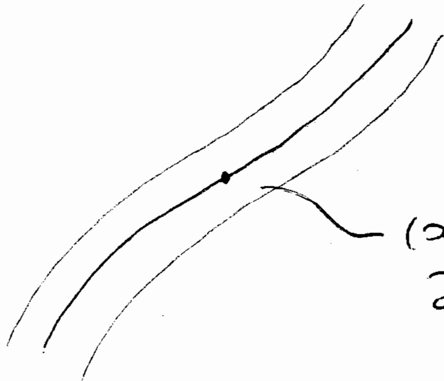
...continue...

Thus we get $B \in \Psi^0_{cl}(X)$, $\ni$

$$b(x,\xi) \sim \sum_{j\le 0} b_j(x,\xi), \quad b_j \text{ hom. of deg } j. \text{ large } |\xi|.$$

and $[B, P] \in \Psi^{-\infty}(X)$.

So we now have PBu is in $C^\infty(X)$.
We need to prove $Bu \in C^\infty(X)$.



$(x_0, \xi_0) \notin WFu$

$\partial_\xi P(x_0, \xi_0) \neq 0$, $\quad p : X \times \mathbb{R}^n - \{0\} \xrightarrow{C^\infty} X$

so WLOG, $\partial_{\xi_n} p(x_0, \xi_0) \neq 0$

So Imp. fe$^{\underline{n}}$ th$^{\underline{m}}$ $\Rightarrow$ we can write (locally)

$$p(x,\xi) = (\xi_n - \lambda(x,\xi'))\, a(x,\xi)$$

$\uparrow$ principal symbol!

$a(x_0, \xi_0) \neq 0$

$\xi = (\xi', \xi_n)$

a hom. of deg. 0, C^∞

<u>Idea</u>:

$$P = A(x, D_x)(D_{x_n} - \lambda(x, D_{x'}))$$

modulo smoothing, with (dep. sm. on x_n)

$\lambda \in \Psi^1_{cl}(\mathbb{R}^{n-1})$, $\sigma_1(\lambda)$ $\mathbb{R}$-valued

$A \in \Psi^0_{cl}(X)$ elliptic $(\sigma_0(A)(x_0,\xi_0) \neq 0)$.

$$A(x, D_x)(D_{x_n} - \lambda(x, D_{x'}))Bu \in C^\infty(\mathcal{U})$$

$\mathcal{U}$ nbhd of x_0

Thus $(D_{x_n} - \lambda(x, D_{x'}))Bu \in C^\infty(\mathcal{U})$

Since $(x_0, \xi_0) \notin WFu$, b is spt away from WFu

& so $Bu|_{x_n = x_0^{(n)}} \in C^\infty$

~~Thus~~

(11/8) We have the full symbol of
$p(x,D_x) - (D_{x_n} - \lambda(x, D_{x'}))A(x, D_x)$
in $S^{-\infty}$ in a conic nbhd of (x_0, ξ_0)
since ...
full sym: $\lambda(x, D_{x'}) \sim \lambda_1(x,\xi') + \lambda_0(x,\xi') + \cdots + \lambda_{-j} + \cdots$
λ_j hom. of deg $j(\xi')$.
& for A, $\sim a_0(x,\xi) + a_{-1}(x,\xi) + \cdots + a_{-j} + \cdots$

We have $p(x,\xi) = (\xi_n - \lambda_1(x,\xi'))a_0(x,\xi)$.
Now compare terms hom. of deg. 0:

$$p_0(x,\xi) = \frac{1}{i}\partial_{x_n} a_0(x,\xi) + (\xi_n - \lambda_1(x,\xi'))a_{-1}(x,\xi)$$
(*)
$$+ \frac{1}{i}\sum_{j=1}^{n-1} \frac{\partial \lambda_1}{\partial \xi_j}\frac{\partial a_0}{\partial x_j} - \lambda_0(x,\xi')a_0(x,\xi)$$

We must determine λ_0 and a_{-1}.
Now on $\xi_n = \lambda_1(x,\xi')$, choose

$$\lambda_0(x,\xi') = \frac{1}{a_0(x,\xi)}\left(\frac{1}{i}\sum \partial_{\xi_j}\lambda_1 \partial_{x_j} a_0 + \frac{1}{i}\partial_{x_n} a_0 - p_0(x,\xi)\right)\Big|_{\xi_n = \lambda_1}$$

If $C(x,\xi) = $ LHS − RHS of (*),
$C(x,\xi) = (\xi_n - \lambda_1(x,\xi'))a_{-1}(x,\xi)$
for some $a_{-1}(x,\xi)$, by Taylor's Thm
so take this $a_{-1}(x,\xi)$. (continue inductively).

So, $P(x,D_x) = (D_{x_n} - \lambda_1(x, D_{x'}))A(x,D_x)$
modulo $\Psi^{-\infty}$ in a conic nbhd of (x_0, ξ_0).

We choose B so that $(x,\xi) \notin WF(Bu)$ if (x,ξ) near (x_0,ξ_0)

Let $Q = \{(x,\xi) \in \Gamma : (x,\xi) \notin WF u\}$

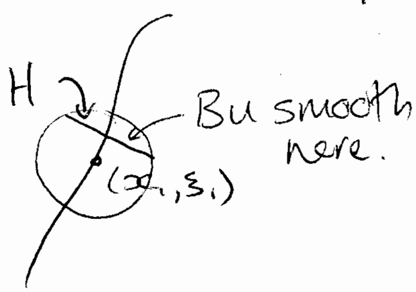
Show Q is both open & closed.

Let $(x_1,\xi_1) \in \overline{Q}$. WLOG. $\partial_{\xi_n} P(x_1,\xi_1) \neq 0$

From above,

$$P(x,D_x) = (D_{x_n} - \lambda(x,D_{x'}))A(x,D_x) \bmod \Psi^{-\infty}$$

in a conic nbhd of (x_1,ξ_1).



We know that Bu is smooth in a transversal hypersurface to Γ near H (say), with $H \cap \Gamma \in Q$.

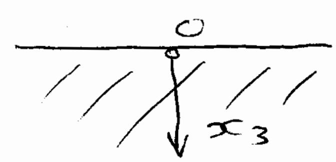
By regularity th$^{\underline{m}}$ for hyperbolic eq$^{\underline{n}}$, $Bu \in C^\infty$ in a nbhd of (x_1,ξ_1) i.e. $(x_1,\xi_1) \notin WF u$.

(11/13)

Seismic Migration Problem

$$\overline{\mathbb{R}^3_+} = \{x = (x_1,x_2,x_3),\ x_3 \geq 0\}$$

$$c \in C^\infty(\overline{\mathbb{R}^3_+})$$



Acoustic equation

$$\left(\frac{1}{c^2(x)}\partial_t^2 - \Delta\right)u(x,t) = \delta_0 \text{ in } \mathbb{R}^3$$

$$u = 0 \quad \text{for } t \ll 0.$$

Inverse Seismic Problem

$$c(x) \xrightarrow{F} u|_{x_3=0}(t, x; 0).$$

Q: Is F invertible? It is not even known that F is injective, even if c is close to 1.

This problem is formally determined: $u|_{x_3=0}$ is a fc^{n} of 3 variables & $c(x)$ is also.

Not^{n} $\quad m(x) = \dfrac{1}{c^2(x)}, \quad c(x) > 0.$

We formally linearize the problem: if δ_m is a perturbation, consider

$$m + \delta_m$$

$\delta_m \in \mathcal{D}'(\mathbb{R}^3)$,
$\operatorname{supp} \delta_m \subseteq \{x_3 > 0\}$.

Solve

$$(m + \delta_m)\partial_t^2(u + \delta u) - \Delta(u + \delta u) = \delta_0 \text{ in } \mathbb{R}^3$$
$$u + \delta u = 0 \text{ for } t \ll 0.$$

and u solves

$$\left.\begin{aligned} m\partial_t^2 u - \Delta u &= \delta_0 \\ u &= 0 \quad t \ll 0 \end{aligned}\right\} (**)$$

We get:

$$\left.\begin{aligned} (m\partial_t^2 - \Delta)\delta u &= -\partial_t^2 u \cdot \delta_m \\ \delta u &= 0 \text{ for } t \ll 0 \end{aligned}\right\} (*)$$

(neglecting $\delta_m \partial_t^2 \delta u$).

Seismic migrat^{n} problem:

$$\delta_m \xrightarrow{dF(m)} \delta u|_{x_3=0}.$$

Not known if $dF(m)$ is invertible!
We will show however that $WF(\delta u|_{x_3=0})$ determines in some sense $WF(\delta m)$
Later: we will find that $dF(m)$ is a Fourier integral operator which is elliptic in some regions of (x,ξ) space

<u>Theorem</u> (Rakesh)
(a) No null-bicharacteristics starting at $\{x_3=0\}$ come back to $\{x_3=0\}$
(b) No null-bicharactristics stay on $\{x_3=0\}$.
Let $(t,x',1,\xi') \in WF(\delta u|_{x_3=0})$, $(t,x') \neq (0,0)$
Then $\exists$ null-bicharacteristics

$$\gamma_{in}, \quad \gamma_{ref} \quad \& \quad t>0, \quad t \in [0, t_{ref}]$$

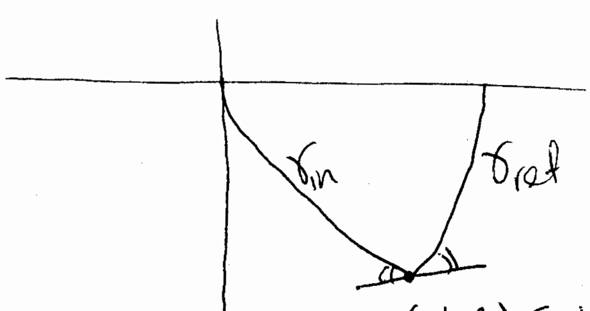
so that

$$\gamma_{ref}(t) = (t, x', 0, 1, \xi', \xi_3),$$
$$-m(x',0) + |\xi'|^2 + \xi_3^2 = 0.$$
$$\gamma_{in}(0) = (0,0,1,\xi_0) \text{ for some } \xi_0 \text{ s.t.}$$
$$m(0) - |\xi_0|^2 = 0.$$

and

$$\gamma_{in}(t_{ref}) \in \gamma_{ref}(t_{ref}) + WF(\delta_m)$$

in freq. space



$(y,\eta) \in WF(\delta_m)$

Proof (Calculus of WF sets plus prop^n of sing's).

~~Restriction makes sense:~~

$m\,\partial_t^2 \delta_u - \Delta \delta_u = -\partial_t^2 u\, \delta_m$

so,

(1) $WF(\delta u) \subseteq WF(\partial_t^2 u\, \delta_m) \cup$ null-bichar.'s thru $WF(\partial_t^2 u\, \delta m)$

(prop^n of sing's)

(2) $WF(\partial_t^2 u\, \delta_m) \subseteq WF(\partial_t^2 u) \cup WF(\delta_m) \cup \left(WF(\partial_t^2 u) + WF(\delta_m)\right)$

(3) note, $WF(\partial_t^2 u) \subseteq WF u$ (diff-op.)

(4) $WF(\delta_m) \subseteq \{(t, x, \tau, \xi);\ x_3 > 0,\ \tau = 0\}$

$\left(\varphi\, \delta_m \left(e^{-i\lambda(t\tau + x\cdot\xi)}\right)\right)$

↑ f^n of t & x ↑ f^n of x alone

$\underbrace{\varphi(t,x)\left(e^{-i\lambda(t\tau + x\xi)}\right)}_{\text{this is rapidly dec. if } \tau \neq 0.}\delta_m\left(e^{-i\lambda(x\cdot\xi)}\right)$

(5) $WF u \subseteq WF(\delta_0) \cup$ null-bichar.'s thru $WF(\delta_0)$.

Mult^n: $\partial_t^2 u\, \delta_m$ is well defined since

$WF(\partial_t^2 u) \cap -WF(\delta_m) = \emptyset$

since by (4) $WF(\delta_m) \subseteq \{\tau = 0\}$

If $(t, x, \tau, \xi) \in WF(\partial_t^2 u) \cap -WF(\delta_m)$,

then $\tau = 0$ & $x_3 > 0$, but by (3) & (5)

$m(x)\,\tau^2 = |\xi|^2$ on null bich.

$\Rightarrow \xi = 0$ $\otimes$

(11/15)

<u>Not^n</u> $\Box_m \delta u = (m\partial_t^2 - \Delta)\delta u$

(1) $WF(\partial_t^2 u) \subseteq WF(u)$

(2) $WF(\delta_m) \subseteq \{(t,x,\tau,\xi);\ \tau = 0,\ x_3 > 0\}$

(3) $WF(\partial_t^2 u\, \delta_m) \subseteq WF(\partial_t^2 u) \cup WF(\delta_m) \cup (WF(\partial_t^2 u) + WF(\delta_m))$

(4) $WF(\delta u) \subseteq WF(\partial_t^2 u\, \delta_m) \cup Char(\Box_m)$

(5) $WF(u) \subseteq WF(\delta_0) \cup Char(\Box_m)$

+ propogat^n of singularities.

<u>Mult^n</u> well defined: $\partial_t^2 u\, \delta_m$.

We show $WF(\partial_t^2 u) \cap -WF(\delta_m) = \emptyset$

From (1) we just check $WF(u) \cap -WF(\delta_m) = \emptyset$

From (5), we just check $WF(\delta_0) \cap -WF(\delta_m) = \emptyset$ (a)

and $Char(\Box_m) \cap -WF(\delta_m) = \emptyset$. (b)

(a) holds since by (2) & $WF(\delta_0) = \{(0,\tau,\xi)\}$

(b) holds since

$char(\Box_m):\ m\tau^2 - |\xi|^2 = 0$

by (2), $\tau = 0$, then $|\xi| = 0$ $\otimes$

<u>Rest^n</u>: define $\delta u|_{x_3=0} = 0$

It suffices to show

$\emptyset = WF(\delta u) \cap \underbrace{\{(t,x,\tau,\xi): x_3 = 0,\ \tau = 0,\ \xi' = 0\}}_{N}$

It suffices by (4) to show

(a) $\quad \mathrm{Char}(\Box_m) \cap N = \phi$

$$\underset{m\tau^2 = |\xi|^2}{\uparrow} \qquad \underset{\tau = \xi' = 0}{\nwarrow} \qquad \checkmark$$

and (b) $\mathrm{WF}(\partial_t^2 u\, \delta_m) \cap N = \phi$

so show (by (3))

(i) $\quad \mathrm{WF}(u) \cap N = \phi$

(ii) $\quad \mathrm{WF}(\delta_m) \cap N = \phi$

(iii) $\quad (\mathrm{WF}(\partial_t^2 u) + \mathrm{WF}(\delta_m)) \cap N = \phi$

For (i) use (5):

- $\mathrm{Char}(\Box_m) \cap N = \phi$ from above
- $\mathrm{WF}(\delta_0) \cap N = \phi$:

  $LHS = \{(\underset{t}{0}, \underset{x}{0}; \underset{\tau}{0}, \underset{\xi'}{0}, \xi_3)\} = \phi$ if $(t, x') \neq 0$.

  We only need to define $\delta u|_{x_3=0}$ away from $(t, x) = (0,0)$.

For (ii) $\mathrm{WF}(\delta_m)$ has $x_3 > 0$, N has $x_3 = 0$. ✓

For (iii) for $\mathrm{WF}(\delta_m)$, $x_3 > 0$, & N has $x_3 = 0$ again ✓

<u>Conclusion</u> $\delta u|_{x_3=0} \in \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^2_{x'})$ away from $(t, x') = (0,0)$.

$$\mathrm{WF}(\delta u|_{x_3=0}) \subseteq \left\{ (t, x', \tau, \xi') \;\middle|\; \begin{array}{l} \exists\, \xi_3 \text{ with} \\ (t, x', 0; \tau, \xi', \xi_3) \in \mathrm{WF}(\delta u) \\ \& \ (\tau, \xi') \neq (0,0) \end{array} \right\}$$

We use (1), (3), (4):

$$\mathrm{WF}(\delta u) \subseteq (\mathrm{WF}(u) + \mathrm{WF}(\delta_m)) \cup (\mathrm{WF}\, u) \cup \mathrm{WF}(\delta_m) \cup \mathrm{Char}(\Box_m).$$

Proof !!

Let $(t,x',\tau,\xi') \in WF(\dot{\delta} u|_{x_3=0})$, $(t,x') \neq (0,0)$

Then $(t,x',0;\tau,\xi',\xi_3) \in WF(\dot{\delta} u)$ from above, $\exists\, \xi_3$.

But

$$\Box_m \delta u = (\partial_t^2 u)\,\delta m$$
$$\delta u = 0 \text{ for } t \ll 0$$

Let $z_0 = (t,x',0;\tau,\xi',\xi_3) \in WF(\delta u) \setminus WF(\partial_t^2 u\,\delta m)$

Claim: Null-bich. thru' z_0 intersects $WF(\partial_t^2 u\,\delta m)$ for some $x_3 > 0$.

If not, then the null-bich. will be in $WF(\dot{\delta} u)$ for all t, a contradiction since for $t \ll 0$, $\delta u = 0$ i.e. "$WF(\delta u) = \emptyset$" for $t \ll 0$.

Conclusion: $\exists\, z_1 \in WF(\partial_t^2 u\,\delta m)$

$$\subseteq WFu \cup WF(\delta m) \cup (WFu + WF(\delta m)) \quad \text{(by (3))}$$

Now $z_1 \notin WF(\delta m)$ since there $\tau = 0$, and @ z_1, $m\tau^2 - |\xi|^2 = 0$, so $\tau = 0 \Rightarrow |\xi| = 0$

Further $z_1 \notin WFu$ since if so, the whole null bich. thru $z_1 \in WFu$, but $u = 0$ $t \ll 0$, ↯ (unless the bichar comes back to the surface, which cannot happen).

Thus $z_1 \in WFu + WF(\delta m)$

z_1 is in the null-bich. coming from z_0.

(11/18) Write $z_1 = z_2 + z_3 \in WFu + WF(\delta m)$.

$z_2 = (t_2, x_2, \tau_2, \xi_2)$

$z_3 = (t_2, x_2, \tau_3, \xi_3) \rightarrow \tau_3 = 0$, & $(x_2)_3 > 0$

$z_1 = (t_2, x_2, \tau_1, \xi_1) \Rightarrow \tau_1 = \tau_2$ & $\xi_1 = \xi_2 + \xi_3$

We know $z_2 \notin WF\,\delta_0$. By prop^{n} of sing's, the whole null-bichar. thru' z_1 must be contained in WFu as long as the null-bichar. does not intersect $WF\delta_0$

<u>Claim</u>: The null-bich. thru z_2 must go through $(0,0)$.

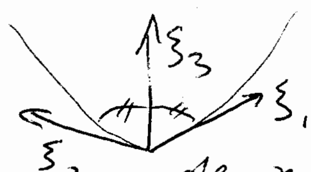
If not $\Box_m u = 0$ on null-bich. $\forall\, t$, but $u=0$ $t \ll 0$ $\otimes$.

<u>Conclusion</u>: $\gamma_{ref} \in \gamma_{inc} + WF(\delta_m)$.

From Ham vec-field eq^{n},

$$\frac{dx_j}{dt} = \frac{\partial p}{\partial \xi_j} = -2\xi_j \qquad (p = m\tau^2 - |\xi|^2)$$

i.e. $\frac{dx}{dt} \parallel$ to ξ.



angle in = angle out. $\square$

<u>Question</u>: Can we find $WF(\delta_m)$ from $\delta u|_{x_3=0}$, $x \neq 0$? Can we find δ_m itself?

In certain regions of phase space prove

$$\delta_m \to \delta u|_{x_3=0}$$

is invertible modulo a smoothing operator.

(11/18)

LAX-PARAMETRIX CONSTRUCTION

We want to solve

$$\Box_m u = \delta_0$$
$$u = 0 \quad t \ll 0$$

&

$$\Box_m \delta u = -\partial_t^2 u \, \delta m$$
$$\delta u = 0 \text{ for } t \ll 0.$$

as explicitly as possible

<u>Example</u> $\Box u + qu = (\partial_t^2 - \Delta)u + qu, \quad q \in C_0^\infty(\mathbb{R}^n)$

Solve $(\Box + q)u = 0$ in $\mathbb{R}_t \times \mathbb{R}^n_x$.

$$u|_{t=0} = 0$$
$$\partial_t u|_{t=0} = f.$$

If $q = 0$, then

$$u(t,x) = \int e^{i(\langle x,\xi\rangle + t|\xi|)} \frac{\hat{f}(\xi)}{2i|\xi|} d\xi - \int e^{i(\langle x,\xi\rangle - t|\xi|)} \frac{\hat{f}(\xi)}{2i|\xi|} d\xi$$

Try

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle + t|\xi|)} a_+(t,x,\xi) \hat{f}(\xi) d\xi + \int e^{i(\langle x,\xi\rangle - t|\xi|)} a_-(t,x,\xi) \hat{f}(\xi) d\xi.$$

$a_\pm \sim \sum_{j \le -1} a_\pm^j \quad a_\pm^{(j)}$ hom. of deg j in ξ, large $|\xi|$.

$$(\Box + q)u = \int e^{i(\langle x,\xi\rangle + t|\xi|)} (\Box + q) a_+ \hat{f}(\xi) d\xi + \int e^{i(\langle x,\xi\rangle - t|\xi|)} (\Box + q) a_- \hat{f}(\xi) d\xi$$

(cont over)

$$+\int e^{i(\langle x,\xi\rangle + t|\xi|)} (2i|\xi|\partial_t a_+ + 2i \sum \xi_j \partial_{x_j} a_+) \hat{f}(\xi) d\xi$$

$$+\int e^{i(\langle x,\xi\rangle - t|\xi|)} (-2i|\xi|\partial_t a_- + 2i \sum \xi_j \partial_{x_j} a_-) \hat{f}(\xi) d\xi.$$

Terms of hom. of deg 0:

$$\int e^{i(\langle x,\xi\rangle + t|\xi|)} 2i\left(|\xi|\partial_t a_+^{(-1)} + \sum_j \xi_j \partial_j a_+^{(-1)}\right) \hat{f}(\xi) d\xi$$

$$+\int e^{i(\langle x,\xi\rangle - t|\xi|)} 2i\left(-|\xi|\partial_t a_-^{(-1)} + \sum_j \xi_j \partial_j a_-^{(-1)}\right) \hat{f}(\xi) d\xi$$

So, $\pm|\xi|\partial_t a_\pm^{(-1)} + \sum \xi_j \partial_j a_\pm^{(-1)} = 0$

and $\left(a_+^{(-1)} = -a_-^{(-1)}\right)\Big|_{t=0}$

and $\left(i|\xi|a_+^{(-1)} - i|\xi|a_-^{(-1)}\right)\Big|_{t=0} = \frac{1}{(2\pi)^n}$

Thus $\boxed{a_\pm^{(-1)} = \frac{1}{(2\pi)^n}\frac{\pm 1}{2i|\xi|}}$

<u>Remark</u>:

$(\pm|\xi|\partial_t b + \sum \xi_j \partial_j b) = R.$

1^{st} order linear PDE. (vector field).

$$\left.\begin{aligned} \frac{dt^\pm}{ds} &= \pm|\xi| \\ \frac{dx_j^\pm}{ds} &= \xi_j \end{aligned}\right\} \text{Char's are straight lines.}$$

i.e. $\frac{d}{ds} b(t(s), x(s)) = R(t(s), x(s))$

(11/20)

Next step: Term hom. of deg -1

$$2i\left(\pm|\xi|\partial_t a_\pm^{(-2)} + \sum \xi_j \partial_j a_\pm^{(-2)}\right) = \underbrace{-(\Box+q)a_\pm^{(-1)}}_{\text{hom. of deg } -1}$$

Choose $a_\pm^{(-2)}$ hom. of deg -2.

Solve

$$\pm\partial_t a_\pm^{(-2)} + \sum \frac{\xi_j}{|\xi|}\partial_j a_\pm^{(-2)} = -\frac{(\Box+q)a_\pm^{(-1)}}{2i}$$

by integrating along characteristics

$$\frac{dt^\pm}{ds} = \pm 1$$

$$\frac{dx_j^\pm}{ds} = \frac{\xi_j}{|\xi|}.$$

Initial cond$^{\text{ns}}$: $\left(a_+^{(-2)} = -a_-^{(-2)}\right)\Big|_{t=0}$.

& $\left(i|\xi|a_+^{(-2)} + \partial_t a_+^{(-1)}\right)\Big|_{t=0} = \left(i|\xi|a_-^{(-2)} - \partial_t a_-^{(-1)}\right)\Big|_{t=0}$

Inductive Step (Terms of hom. j).

$$2i\left(\pm|\xi|\partial_t a_\pm^{(j-1)} + \sum \xi_k \partial_k a_\pm^{(j-1)}\right) = -(\Box+q)a_\pm^{(j)}$$

Init. Conds $\left(a_+^{(j-1)} = -a_-^{(j-1)}\right)\Big|_{t=0}$

& $\left(i|\xi|a_+^{(j-1)} + \partial_t a_+^{(j)}\right)\Big|_{t=0} = -\left(-i|\xi|a_-^{(j-1)} + \partial_t a_-^{(j)}\right)\Big|_{t=0}$

By Borel's Lemma, $\exists\, a_\pm \in S_{cl}^{-1}(\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ s.t.

$$a_\pm \sim \sum_{j\le -1} a_\pm^{(j)}$$

We take

$$u(t,x) = \int e^{i(\langle x,\xi\rangle + t|\xi|)} a_+(t,x,\xi)\hat{f}(\xi)d\xi$$

(*)

$$+ \int e^{i(\langle x,\xi\rangle - t|\xi|)} a_-(t,x,\xi)\hat{f}(\xi)d\xi.$$

<u>Propos^n</u> Let $f \in C_0^\infty(\mathbb{R}^n)$. Let u be as above (*).

Then $(\Box + q)u = \int e^{i(\langle x,\xi\rangle + t|\xi|)} b_+(t,x,\xi)\hat{f}(\xi)d\xi$

$$+ \int e^{i(\langle x,\xi\rangle - t|\xi|)} b_-(t,x,\xi)\hat{f}(\xi)d\xi$$

with $b_\pm \in S^{-\infty}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

$u|_{t=0} = 0$

$\partial_t u|_{t=0} = f + \int e^{i\langle x,\xi\rangle} r(x,\xi)\hat{f}(\xi)d\xi$

with $r \in S^{-\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

<u>Example</u> $\left(\frac{1}{c^2(x)}\partial_t^2 - \Delta\right)u = 0$ in $\mathbb{R}_t \times \mathbb{R}^n_x$

$\Box_c$

$u|_{t=0} = 0$

$\partial_t u|_{t=0} = f.$

$c(x) > 0$, $c \in C^\infty(\mathbb{R}^n_x)$.

Here, $\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\left(e^{i(\langle x,\xi\rangle + t|\xi|)}\right) \neq 0.$

Find $\varphi(t,x,\xi)$ hom. of deg. 1 in ξ. s.t.

$$\varphi(0,x,\xi) = \langle x,\xi\rangle$$

$$\triangle \quad \left(\frac{1}{c^2}\partial_t^2 - \Delta\right) e^{i\varphi(t,x,\xi)} = 0$$

Then $\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\int e^{i\varphi(t,x,\xi)} \frac{1}{(2\pi)^n}\hat{f}(\xi)\,d\xi = 0$

This does not satisfy the initial cond$^{\Delta}$s.

$$\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)e^{i\varphi(t,x,\xi)}$$

$$= -\left(\frac{1}{c^2}\partial_t^2\varphi - \Delta\varphi\right) = 0$$

This doesn't help.

So now we try...

$$u(t,x) = \int e^{i\varphi_+(t,x,\xi)} a_+(t,x,\xi)\hat{f}(\xi)\,d\xi$$

$$+ \int e^{i\varphi_-(t,x,\xi)} a_-(t,x,\xi)\hat{f}(\xi)\,d\xi$$

$$a_\pm \sim \sum_{j\le -1} a_\pm^{(j)}$$

Then

$$\partial_t u = \int (i\partial_t\varphi_+)e^{i\varphi_+}a_+\hat{f} + \int e^{i\varphi_+}(\partial_t a_+)\hat{f} \quad \& \ a_- \text{ etc}$$

$$\partial_t^2 u = \int (i\partial_t\varphi_+)^2 e^{i\varphi_+}a_+\hat{f} + \int (i\partial_t^2\varphi_+)e^{i\varphi_+}a_+\hat{f}$$

$$+ \int e^{i\varphi_+}(i\partial_t\varphi_+)(\partial_t a_+)\hat{f}$$

$$+ \int e^{i\varphi_+}(\partial_t^2 a_+)\hat{f} \qquad \& \ a_- \text{ etc.}$$

Similarly for $\partial_j^2 u$ etc.

Highest homogeneity terms:

$$\left(\frac{1}{c^2}\partial_t^2 u - \Delta u\right) = \int e^{i\varphi_+}\frac{1}{c^2}(i\partial_t\varphi_+)^2 a_+ \hat{f}$$

$$+ \int e^{i\varphi_-}\frac{1}{c^2}(i\partial_t\varphi_-)^2 a_- \hat{f}$$

$$- \int e^{i\varphi_+}\sum(i\partial_k\varphi_+)^2 a_+\hat{f} - \int e^{i\varphi_-}\sum(i\partial_k\varphi_-)^2 a_-\hat{f}$$

+ terms of lower order homogeneity.

So we want

$$-\frac{1}{c^2}\left(\frac{\partial\varphi_\pm}{\partial t}\right)^2 + \sum\left(\frac{\partial\varphi_\pm}{\partial x_k}\right)^2 = 0.$$

$$\varphi_\pm(0,x,\xi) = \langle x,\xi\rangle.$$

<u>Eikonal Eq^n</u>

$$\boxed{\begin{aligned}\left(\frac{\partial\varphi_\pm}{\partial t}\right)^2 &= c^2(x)\,|\nabla\varphi_\pm|^2\\ \varphi_\pm(0,x,\xi) &= \langle x,\xi\rangle\end{aligned}}$$

Non-linear, first order eq^n.

(11/22)

<u>Remark</u> $\Box_c e^{i\varphi} = \Big(\underbrace{-\Box_c\varphi}_{\text{hom. of degree 1.}} - \underbrace{\left(\tfrac{1}{c^2(x)}(\partial_t\varphi)^2 - |\nabla\varphi|^2\right)}_{\text{hom. of deg. 2}}\Big)e^{i\varphi}$

so need each part to be zero.

<u>Idea</u>. Try to kill $\Box_c\varphi$ by introducing the amplitudes a_+ & a_-.

i.e. $(\Box_c\varphi_\pm)a_\pm$ using

$$\frac{2i}{c^2(x)} \partial_t \varphi_+ \partial_t a_+ - 2i \sum \partial_j \varphi_+ \partial_j a_+ - (\Box_c \varphi_+) a_+ = 0$$

——— n ———

Goal: Solve $(\partial_t \varphi_\pm)^2 = c^2(x) |\nabla \varphi_\pm|^2$

$$\varphi_\pm (0, x, \xi) = \langle x, \xi \rangle .$$

Example $\left((\partial_t^2 - \Delta_g) + N(x, D_x) \right) u = 0$

$$u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0.$$

- g a Riem. metric
- $N(x, D_x)$ diff-operator order ≤ 1.

Try the same construction,

$$u(t,x) = \int e^{i\varphi_+(t,x,\xi)} a_+(t,x,\xi) \hat{f}(\xi) d\xi + \int e^{i\varphi_-(t,x,\xi)} a_-(t,x,\xi) \hat{f}(\xi) d\xi$$

& we get ...

$$(\partial_t \varphi_\pm)^2 = \sum g^{ij}(x) \partial_i \varphi_\pm \partial_j \varphi_\pm$$

$$\varphi_\pm (0, x, \xi) = \langle x, \xi \rangle$$

We know if $g_{ij} = \delta_{ij}$, we know

$$\varphi_\pm (t, x, \xi) = \langle x, \xi \rangle \pm t|\xi|$$

Now $\partial_t \varphi_\pm = \pm \sqrt{\sum g^{ij}(x) \partial_i \varphi_\pm \partial_j \varphi_\pm}$

$$\varphi_\pm (0, x, \xi) = \langle x, \xi \rangle, \quad (\xi \neq 0)$$

General sitn Let $\omega = \frac{\xi}{|\xi|}$

$p : T^* X \to \mathbb{R}$ (eg $p = \tau + \sqrt{\sum g^{ij} \xi_i \xi_j}$, $X = \mathbb{R}_t \times \mathbb{R}^n_x$)

Find $\varphi(t,x,\omega)$ so that

$$\rho(t,x,d_{t,x}\varphi) = 0$$

$$\varphi(0,x,\omega) = \langle x,\omega\rangle$$

& $\quad d_\xi \rho \neq 0 \quad$ on $\rho = 0$.

<u>In general,</u> $\left\{\begin{array}{l} \rho : T^*X \setminus \{0\} \to \mathbb{R} \\ d_\xi \rho \neq 0 \text{ on } \rho = 0 \\ \rho(x, d_x\varphi) = 0 \\ \varphi \text{ is given on some hypersurface} \\ \quad \text{transversal to } H_\rho. \end{array}\right.$

<u>Hamilton-Jacobi Theory</u>

$T^*X \setminus \{0\}$, $\quad \rho = 0$.

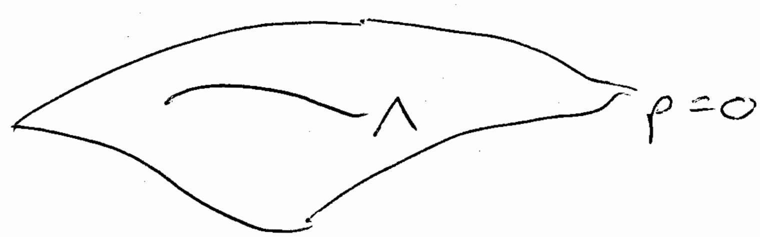
Find submanifold Λ of $T^*X \setminus \{0\}$ of dim. n,

$\Lambda \subseteq \rho^{-1}(0)$

with

$\Lambda = \{(x, d_x\varphi)\}$

for some $\varphi \in C^\infty$.



For then,

$$\rho(x, d_x\varphi) = 0$$

Such a submanifold is called a Lagrangian subman.

$X \subset \mathbb{R}^n$, open. $\quad T^*X \quad (x,\xi)$.

$$\sigma = \sum^n d\xi_i \wedge dx_i \quad \text{on } T^*X$$

Notice that $\sigma = d\alpha$, $\quad \alpha = \sum \xi_i dx_i \quad$ (<u>contact</u> form)

<u>Defn</u> A Lagrangian manifold Λ is a smooth submanifold of dimension n so that $(\Lambda \subseteq T^*X \setminus \{0\})$, σ vanishes on Λ, i.e.

$$\sigma(u,v) = 0 \quad \forall \; u,v \in T(\Lambda).$$

<u>Examples</u> $\varphi : X \to \mathbb{R}$ smooth. $(d_x\varphi \neq 0)$

$\Lambda = \{(x, d_x\varphi)\}$ is Lagrangian:

Check: $\xi_i = \frac{\partial \varphi}{dx_i}$

$$\sigma = \sum d\left(\frac{\partial \varphi}{\partial x_i}\right) \wedge dx_i$$

$$= \sum \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right) dx_j \wedge dx_i$$

$$= 0.$$

<u>Example</u> $\delta_{x_1=0}(\varphi) = \int \varphi(0,x')dx'$, $\varphi \in C_0^\infty(\mathbb{R}^n)$.

$WF(\delta_{x_1=0}) = \{(0,x';\xi_1,0)\} = N^*\{x_1=0\} = \Lambda$

$\dim \Lambda = n$.

Here, $\sigma((a_x, b_\xi),(a_x', b_\xi'))$

$$= b'(a) - b(a')$$

$$= 0$$

<u>Exercise</u> Let $Y \subseteq X$ a submanifold. Then

- the conormal bundle

$$N^*Y = \{(y,\xi) : \xi \perp T_y(Y) \; \forall \, y \in Y\}$$

is a Lagrangian submanifold.

Prop^{n} Let $\Lambda \subseteq T^*X \setminus \{0\}$ be Lagrangian.

Let $(x_0, \xi_0) \in \Lambda$,

$$\pi_* = d(\pi)_{(x_0,\xi_0)} : T(\Lambda)_{(x_0,\xi_0)} \to T_{x_0}(X)$$

an isomorphism.

Then $\exists\, \varphi : X \to \mathbb{R}$, $\varphi \in C^\infty$ so that near (x_0, ξ_0),

$$\Lambda = \{(x, d_x\varphi)\}$$

Proof// By Imp. F. Th^{m}.

$$\Lambda = \{(x, \xi(x))\} \text{ locally near } (x_0, \xi_0).$$

Since Λ Lang.,

$$\sum_i d\xi_i \wedge dx_i = 0 \quad \text{on } \Lambda$$

$$d\xi_i = \sum \frac{\partial \xi_i}{\partial x_j} dx_j$$

$(*)$ means $\dfrac{\partial \xi_i}{\partial x_j} = \dfrac{\partial \xi_j}{\partial x_i}$

By Poincaré (closed, locally exact)

$$\xi_i = \frac{\partial \varphi}{\partial x_i}$$

locally for some φ. ∎

So now, to solve $p(x, d_x\varphi) = 0$ the goal is to construct Lagrangian submanifolds $\Lambda \subset p^{-1}\{0\}$ (& worry about init. cond^{n}s).

(11/25) From last lecture...

$p : T^*X \setminus \{0\} \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$ open

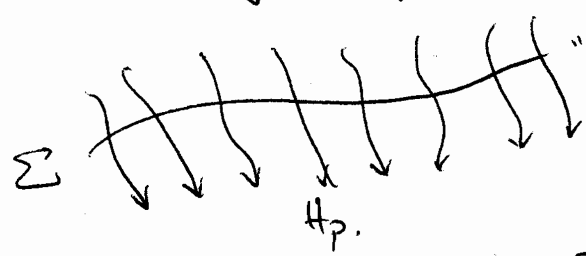
$p(x,\xi)$ homog. of deg. 1 in ξ.

To find φ s.t. $p(x, d_x\varphi) = 0$ locally (near (x_0, ξ_0)) it is enough to find a Lagrangian submanifold $\Lambda \subseteq T^*X \setminus \{0\}$ s.t.

$$\pi : T_{(x_0,\xi_0)}\Lambda \to T_{x_0}\Lambda$$

is an isomorphism.

In the cotangent space,



"initial" manifold dim $n-1$
σ vanishes on Σ
H_p transversal to Σ.

In our example, $p = \tau - \sqrt{\sum g^{ij}(x)\xi_i\xi_j}$, $X = \mathbb{R}\times\mathbb{R}^n$.

$$\Sigma = \{(t=0, x;\ d_t\varphi(0,x,\omega),\ d_x(0,x,\omega)\}$$

$$= \left\{(0,x;\ \sqrt{\textstyle\sum g^{ij}(x)\omega_i\omega_j},\ \omega)\right\}$$

$$\left(\text{from } \begin{cases} \dfrac{\partial\varphi_\pm}{\partial t} = \pm\sqrt{\sum g^{ij}(x)\dfrac{\partial\varphi_\pm}{\partial x_i}\dfrac{\partial\varphi_\pm}{\partial x_j}} \\ \varphi_\pm(0,x,\omega) = \langle x,\omega\rangle \quad \omega\in S^{n-1} \end{cases} \right)$$

$\dim \Sigma = n$.

Check: σ vanishes on Σ:

$\sigma = dt\wedge d\tau + \sum d\xi_j\wedge dx_j$. $\qquad \xi_j = \omega_j$ on Σ

$= 0$. $\qquad t = 0$ on Σ

<u>Def^{n}</u> Let $X \subseteq \mathbb{R}^m$, open
$\Sigma \subset T^*X$ a submanifold.
Σ is called <u>isotropic</u> if σ vanishes on Σ.

<u>Def^{n}</u> as above, Σ is called <u>involutive</u> if
$T(\Sigma)^{\sigma} \subseteq T(\Sigma)$
orthogonal wrt σ.

Now define (locally) Λ as the flow out from Σ by H_p; Λ is an $(n+1)$-dim^{n}al manifold since H_p is transversal to Σ,

$$\frac{\partial p}{\partial \tau} = 1,$$

$$H_p = \frac{\partial p}{\partial \tau}\frac{\partial}{\partial t} + \sum \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial t}\frac{\partial}{\partial \tau} - \sum \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}$$

which is thus transversal to $\{t=0\}$.

Also, $\pi : \Lambda \to X$ is locally a diffeo.

We know $p(t,x,d_{t,x}\varphi) = 0$ locally,
& $\varphi(0,x,\omega) = \langle x, \omega \rangle$.

In general, for $p(x, d_x\varphi) = 0$ if $\partial_\xi p \neq 0$ on $p=0$ find coordinates so that

$$p(x,\xi) = a(x,\xi)(\xi_n - \lambda(x', x_n, \xi'))$$
$$a(x_0,\xi_0) \neq 0.$$

$$\frac{\partial \varphi}{\partial x_n} = \lambda\left(x', x_n, \frac{\partial \varphi}{\partial x'}\right) \qquad \lambda \text{ hom. of deg } 1 \text{ in } \xi'.$$

$$\varphi\big|_{x_n=0} = \langle x', \xi' \rangle$$

Now proceed as before ....

<u>Lax-Parametrix Construction</u>

$$\left(\frac{1}{c^2(x)}\partial_t^2 - \Delta\right)u = 0, \quad u\big|_{t=0} = 0,$$
$$\partial_t u\big|_{t=0} = f \in C_c^\infty(\mathbb{R}^n)$$

$$u(t,x) = \int e^{i\varphi_+(t,x,\xi)} a_+(t,x,\xi)\, \hat{f}(\xi)\, d\xi + \int e^{i\varphi_-(t,x,\xi)} a_-(t,x,\xi)\, \hat{f}(\xi)\, d\xi$$

$$\begin{cases} \partial_t \varphi_\pm = \pm c(x)\,|\nabla \varphi_\pm| \\ \varphi_\pm(0,x,\xi) = \langle x, \xi \rangle \end{cases}$$

Find $a_\pm$:

$$(-\Box_c \varphi_\pm)\, a_\pm + 2i\left(\frac{\partial \varphi_\pm}{\partial t}\frac{\partial a_\pm}{\partial t} - \sum \frac{\partial \varphi_\pm}{\partial x_j}\frac{\partial a_\pm}{\partial x_j}\right) = 0$$

$$a_+\big|_{t=0} = -a_-\big|_{t=0}$$

$$\left(i\frac{\partial \varphi_+}{\partial t} a_+ + \frac{\partial a_+}{\partial t} + i\frac{\partial \varphi_-}{\partial t} a_- + \frac{\partial a_-}{\partial t}\right)\Bigg|_{t=0} = \frac{1}{(2\pi)^n}$$

But $\dfrac{\partial \varphi_\pm}{\partial t} = \pm c(x)\,|\nabla \varphi_\pm|$;

$$\frac{\partial \varphi_\pm}{\partial t}\Big|_{t=0} = \pm c(x)\,|\xi|.$$

$$\Rightarrow \left(i c(x)|\xi| a_+ + \partial_t a_+ - i\, c(x)|\xi| a_- + \partial_t a_- \right)\Big|_{t=0} = \left(\frac{1}{2\pi}\right)^n$$

Rewrite... $(V_\pm + \alpha_\pm) a_\pm = 0$

$$V_\pm = \left(\partial_t \varphi_\pm \partial_t - \sum \partial_{x_j} \varphi_\pm \partial_{x_j} \right)$$

$$\alpha_\pm = -\frac{\Box_c}{2i} \varphi_\pm$$

Take Characteristics of $V_\pm$:

$$\frac{dt_\pm}{ds} = \frac{\partial \varphi_\pm}{\partial t} = \tau(s) \qquad \text{from}$$

$$\left(\frac{dx_j}{ds}\right)_\pm = -\frac{\partial \varphi_\pm}{\partial x_j} = \xi(s) \qquad p(t, x, d_{t,x}\varphi_\pm) = 0$$

Write $a_\pm \sim \sum_{j \le -1} a_\pm^{(j)}$ as usual.

$$\Rightarrow V_\pm a_\pm^{(-1)} + \alpha_\pm a_\pm^{(-1)} = 0$$

$$\left(a_+^{(-1)} = -a_-^{(-1)}\right)\Big|_{t=0}$$

$$i\left(c(x)|\xi| a_+^{(-1)} - c(x)|\xi| a_-^{(-1)}\right)\Big|_{t=0} = \frac{1}{(2\pi)^n}.$$

$$\Rightarrow \quad 2i c(x)|\xi| a_+^{(-1)} = \frac{1}{(2\pi)^n} \quad @\ t=0$$

i.e. $\quad a_+^{(-1)} = \dfrac{1}{2i(2\pi)^n c(x)|\xi|}$ at $t=0$.

continue inductively.

<u>Proposition.</u> Let $c(x) > 0$, $c \in C^\infty(\mathbb{R}^n)$

Let $f \in C_0^\infty(\mathbb{R}^n)$. We can find

$$a_\pm \in S^{-1}_{cl}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$$

$$\varphi_\pm \text{ hom. of deg. } 1 \text{ in } \xi$$

so that

$$u(t,x) = \int e^{i\varphi_+} a_+ \hat{f} + \int e^{i\varphi_-} a_- \hat{f}$$

solves $\left(\frac{1}{c^2(x)}\partial_t^2 - \Delta\right)u = \int e^{i\varphi_+} b_+ \hat{f} + \int e^{i\varphi_-} b_- \hat{f}$

with $b_\pm \in S^{-\infty}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$

$$u|_{t=0} = 0$$

$$\partial_t u|_{t=0} = f + \int e^{i\langle x,\xi\rangle} r(x,\xi)\hat{f}(\xi)\,d\xi$$

w/ $r \in S^{-\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

(11/27) <u>Oscillatory Integrals & Generalized Huyghen's Principle.</u>

<u>Goal</u>: Extend Lax-parametrix construction to $f \in \mathcal{E}'(\mathbb{R}^n)$

eg. $f = \delta_0$.

Compute $WF\,u$ in terms of $WF\,f$.

We want to define

$$\int e^{i(\varphi_\pm(t,x,\xi) - \langle y,\xi\rangle)} a_\pm(t,x,\xi)\,d\xi$$

$$\in \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y)$$

More generally, oscillatory integrals

$$I_{a,\varphi} = \int e^{i\varphi(x,\theta)} a(x,\theta)\,d\theta$$

with $x \in X \subseteq \mathbb{R}^m$, open

& $\theta \in \mathbb{R}^N$

(a) φ is a phase function:

- $\varphi \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$
- φ is homogeneous of degree 1 in θ.
- $d\varphi \neq 0$ in $X \times (\mathbb{R}^N \setminus \{0\})$
- φ is real valued.

(b) $a \in S^k(X \times \mathbb{R}^N)$, $k \in \mathbb{R}$

Under these cond^{n}s, we will define $I_{a,\varphi} \in \mathcal{D}'(X)$.

$$I_{a,\varphi}(\psi) = \int e^{i\varphi(x,\theta)} a(x,\theta)\psi(x)\,dx\,d\theta, \quad \psi \in C_0^\infty(\mathbb{R}^n)$$

Hörmander's trick: Find

$$L = \sum_{j=1}^{m} b_j(x,\theta)\frac{\partial}{\partial x_j} + \sum_{j=1}^{N} c_j(x,\theta)\frac{\partial}{\partial \theta_j}$$

with $b_j \in S^{-1}(X \times \mathbb{R}^N)$ $c_j \in S^0(X \times \mathbb{R}^N)$

s.t.

$$Le^{i\varphi} = e^{i\varphi} \quad \text{on } X \times (\mathbb{R}^N \setminus \{0\})$$

Let $\chi \in C^\infty(\mathbb{R}^N)$, $\chi(\theta) = 0$ for $|\theta| \leq 1$
$\chi(\theta) = 1$ for $|\theta| \geq 2$.

$$I_{a,\varphi}(\psi) = \int e^{i\varphi(x,\theta)} a(x,\theta)\chi(\theta)\psi(x)\,dx\,d\theta$$

$$+ \int e^{i\varphi(x,\theta)} a(x,\theta)(1-\chi(\theta))\psi(x)\,dx\,d\theta$$

$= I_1 + I_2$; I_2 is clearly well defined.

Now
$$I_{\chi a,\varphi}(\psi) = \int L^M(e^{i\varphi})\, a(x,\theta)\,\chi(\theta)\,\psi(x)\,dx\,d\theta$$
$$= \int e^{i\varphi} (L^M)^t \big(a(x,\theta)\,\chi(\theta)\,\psi(x)\big)\,dx\,d\theta$$

Claim: $(L^M)^t\big(a(x,\theta)\chi(\theta)\psi(x)\big) \in S^{k-M}(X\times\mathbb{R}^N)$
For M such that $k-M < -(N+1)$.

<u>Def^n</u> Let φ, a satisfy cond^ns above. Then
$$I_{a,\varphi}(\psi) \triangleq \int e^{i\varphi} (L^M)^t a(x,\theta)\,\chi(\theta)\,\psi(x)\,dx\,d\theta$$
$$+ \int e^{i\varphi} a(x,\theta)\,(1-\chi(\theta))\,\psi(x)\,dx\,d\theta$$

with $M: k-M < -(N+1)$.

<u>Exercises</u> $I_{a,\varphi} \in \mathcal{D}'(X)$; independent of χ; independent of $M' \geq M$; if $a \in S^k(X\times\mathbb{R}^N)$ with $k < -(N+1)$ then
$$I_{a,\varphi} = \int e^{i\varphi(x,\theta)} a(x,\theta)\,d\theta\,;$$
independent of choice of L.

<u>Choice of L</u>

$$L = \frac{\frac{1}{i}\left(\sum_{j=1}^{m} \frac{\partial\varphi}{\partial x_j}\frac{\partial}{\partial x_j} + |\theta|^2 \sum_{j=1}^{N} \frac{\partial\varphi}{\partial\theta_j}\frac{\partial}{\partial\theta_j}\right)}{\sum\left(\frac{\partial\varphi}{\partial x_j}\right)^2 + |\theta|^2 \sum\left(\frac{\partial\varphi}{\partial\theta_j}\right)^2}$$

so $b_j = \dfrac{\frac{1}{i}\frac{\partial\varphi}{\partial x_j}}{(\sim)}$ & $c_j = \dfrac{\frac{1}{i}|\theta|^2\frac{\partial\varphi}{\partial\theta_j}}{(\sim)}$.

<u>Proposition.</u> Let $f \in \mathcal{E}'(\mathbb{R}^n)$.

Let
$$E_\pm f = \int e^{i(\varphi_\pm(t,x,\xi) - \langle y,\xi\rangle)} a_\pm(t,x,\xi) f(y)\, dy\, d\xi$$

i.e. $$K_{E_\pm} = \int e^{i(\varphi_\pm(t,x,\xi) - \langle y,\xi\rangle)} a_\pm(t,x,\xi)\, d\xi.$$

Then $\Box_c(E_+f + E_-f) \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ locally.

$$(E_+f + E_-f)\big|_{t=0} = 0$$

$$\partial_t(E_+f + E_-f)\big|_{t=0} = f + C^\infty(\mathbb{R}^n_x).$$

<u>Proof</u>

$$u(t,x) = \int e^{i\varphi_+} a_+ \hat{f}(\xi)\, d\xi + \int e^{i\varphi_-} a_- \hat{f}(\xi)\, d\xi$$

$$\Box_c u = \int e^{i\varphi_+} b_+ \hat{f}(\xi)\, d\xi + \int e^{i\varphi_-} b_- \hat{f}(\xi)\, d\xi$$

w/ $b_\pm \in S^{-\infty}$.

$\square$

<u>Corollary</u> Locally near $(0, x_0)$, <u>the</u> sol^n to

$$\Box_c v = 0$$
$$v|_{t=0} = 0$$
$$\partial_t v|_{t=0} = f \in \mathcal{E}'(\mathbb{R}^n)$$

is given by $v = (E_+f + E_-f) + g$

where $g \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$

Therefore,
$$WF\, v = WF(E_+f + E_-f)$$

<u>Not^n</u> $E_+ + E_- : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^n_x)$ is called a <u>parametrix for the Cauchy Problem</u>.

(12/2)

Now calculate $WF(I_{a,\varphi})$.

For example
$$K_{E_\pm} = \int e^{i(\varphi_\pm(t,x,\theta) - \langle y,\theta\rangle)} a_\pm(t,x,\theta)\,d\theta.$$

$$p_\pm(t,x,d_{t,x}\varphi) = 0$$
$$\varphi|_{t=0} = \langle x,\theta\rangle$$

eg. $p_\pm(t,x,\tau,\xi) = \tau \pm \sqrt{\sum g^{ij}(x)\xi_i\xi_j}$

Calc: $I_{a,\varphi}(e^{-it\langle\cdot,\xi\rangle}\psi)$ $\quad \psi \in C_0^\infty(X)$.

$$= \int e^{i(\varphi(x,\theta) - t\langle x,\xi\rangle)} a(x,\theta)\,\psi(x)\,d\theta\,dx$$

(defined as above as an osc. integral).

$\theta \to t\theta$

$$= t^{-N}\int e^{it(\varphi(x,\theta) - \langle x,\xi\rangle)} a(x,t\theta)\,\psi(x)\,d\theta\,dx$$

Where is this rapidly decreasing?

Near pts (x,θ) : $d_\theta\varphi(x,\theta) \neq 0$
$$I_{a,\varphi}(e^{-it\langle\cdot,\xi\rangle}\psi) = O(t^{-M}) \quad \forall M.$$

How about
$$d_x(\varphi(x,\theta) - \langle x,\xi\rangle) = d_x\varphi(x,\theta) - \xi$$

So for (x,θ) : $d_\theta\varphi(x,\theta) = 0$ & $\xi \neq d_x\varphi(x,\theta)$

then $$I_{a,\varphi}(e^{-it\langle\cdot,\xi\rangle}\psi) = O(t^{-M}) \quad \forall M.$$

Theorem.

Let φ be a phase function, $a \in S^k(X \times \mathbb{R}^N \setminus \{0\})$

$I_{a,\varphi}$ an oscillatory integral

$$I_{a,\varphi} = \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta \in \mathcal{D}'(X).$$

Then $\boxed{WF(I_{a,\varphi}) \subseteq \{(x,\xi) \in T^*X \setminus \{0\} \,/\, \xi = d_x\varphi(x,\theta) \wedge d_\theta\varphi(x,\theta) = 0\}}$

Now we want to compute $WF(E_\pm)$.

$$K_{E_\pm} = \int e^{i(\varphi_\pm(t,x,\theta) - \langle y,\theta\rangle)} a_\pm(t,x,\theta)\, d\theta$$

Special case: $c(x) = 1 \quad \forall x$.

Then $\tilde{\varphi}_\pm(t,x,\theta) = \langle x,\theta\rangle \pm t|\theta|$

here $\varphi_\pm(t,x,y,\theta) = \langle x,\theta\rangle \pm t|\theta| - \langle y,\theta\rangle$

$$= \langle x-y,\theta\rangle \pm t|\theta|$$

$$d_\theta\varphi_\pm = x - y \pm t\frac{\theta}{|\theta|} = 0$$

$$x - y = \mp t\frac{\theta}{|\theta|}, \quad |x-y| = |t|$$

x light cone centered @ y.

Proposition. $c(x) \equiv 1$

$$WF'(E_\pm) \subseteq \{(t,x,y;\tau,\xi,\eta) : x - y \pm t\frac{\theta}{|\theta|} = 0 \text{ for some } \theta \neq 0$$

$$\tau = |\theta|,\ \xi = \theta,\ \eta = \theta \}$$

$A_\pm$ say

For $t \neq 0$, $\Lambda_\pm$ be the conormal bundle to
$|x-y| = |t|$
This remains true even @ $t=0$.

<u>The General Case</u> $c(x) \in C^\infty(\mathbb{R}^n)$

$$
\begin{aligned}
WF'(E_\pm) \subseteq \{ (t,x,y;\tau,\xi,\eta) ;\ & d_\theta \varphi_\pm(t,x,y,\theta) = 0 \\
& \tau = d_t \varphi(t,x,y,\theta) = d_t \tilde{\varphi}_\pm \\
& \xi = d_x \varphi(t,x,y,\theta) = d_x \tilde{\varphi}_\pm \\
& -\eta = d_y \varphi(t,x,y,\theta) = -\theta \ \}
\end{aligned}
$$

Let $C_\pm = \{ (t,x,y,\theta) / \underbrace{d_\theta \varphi_\pm(t,x,y,\theta)}_{\shortparallel} = 0$ for some $\theta \neq 0 \}$

$$d_\theta \tilde{\varphi}_\pm(t,x,\theta) - y = 0$$

i.e. $y = d_\theta \tilde{\varphi}_\pm(t,x,\theta)$.

<u>Recall</u> $p(t,x,d_{t,x}\tilde{\varphi}) = 0 \qquad (\tilde{\varphi} = \tilde{\varphi}_+ \text{ or } \tilde{\varphi}_-)$.

$\uparrow$ this set $= \Sigma$

Let $(t(s), x(s), \tau(s), \xi(s))$ be null-bichar.'s of H_p.

<u>Prop^n</u> $\tilde{\varphi}(t(s), x(s), \theta) = \tilde{\varphi}(t(0), x(0), \theta) \quad \forall s$.

Proof//

$$
\begin{aligned}
\frac{d}{ds}\tilde{\varphi}(t(s),x(s),\theta) &= \frac{\partial \tilde{\varphi}}{\partial t}\frac{dt}{ds} + \sum \frac{\partial \tilde{\varphi}}{\partial x_i}\frac{dx_i}{ds} \\
&= \tau(s)\frac{\partial p}{\partial \tau} + \xi(s)\cdot\frac{\partial p}{\partial \xi} \\
&= 0
\end{aligned}
$$

by Euler's identity:
s'pose $f(x,t\theta) = t^m f(x,\theta)$. Then

$$\sum \frac{\partial f}{\partial \theta_i} \theta_i = m f(x,\theta) ;$$

here, $\sum \frac{\partial p}{\partial \theta_i} \theta_i = p(x,\theta) = 0$

since we are on $p = 0$.

<u>Corollary</u> $d_\theta \tilde{\varphi}$ is constant along char's.

$$d_\theta \tilde{\varphi}(t(s), x(s), \theta) = d_\theta \varphi(t(0), x(0), \theta) \ \forall s.$$

$$C_{\tilde{\varphi}_\pm} = \{ y = d_\theta \tilde{\varphi}_\pm(t,x,\theta) \}$$

So if $d_\theta \tilde{\varphi}_\pm(t,x,\theta) = y$

then

$$d_\theta \tilde{\varphi}_\pm(t(s), x(s), \theta) = y$$

where $(t(s), x(s))$ - proj. of bichar. thru' (t,x).

(12/4) <u>Not^n</u> $\phi = \tilde{\varphi}$ above.

<u>Prop^n</u> $p_\pm(t, x, d_{t,x}\phi_\pm) = 0$, $\phi_\pm |_{t=0} = \langle x, \theta \rangle$

We have $d_\theta \phi_\pm(t(s), x(s), \theta) = d_\theta \phi_\pm(t(0), x(0), \theta)$

$(t(s), x(s))$ called the characteristic curves are the proj^{n}s of the null-bich.'s.

In fact

$$\phi_\pm(t(s), x(s), \theta) = \phi_\pm(t(0), x(0), \theta(0)).$$

Recall $\Box_c u = 0, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = f \in \mathcal{E}'(\mathbb{R}^n)$

We have

$$u = (E_+ f + E_- f) + v \quad (\text{locally}), \quad v \in C^\infty(\mathbb{R}^{n+1})$$

$$WF\, u \subseteq WF(E_+ f) \cup WF(E_- f).$$

Prop^n Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, linear, cts.

Assume $WF_Y(A) = \{(y,\eta) : \exists\, x \text{ w/ } (x,y,0,\eta) \in WF(A)\}$
$= \phi.$

$WF_X(A) = \phi$

Then A extends ctsly $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$

and

$$WF(Au) \subseteq WF'(A) \circ WF\, u.$$

Proof// Exercise

—— // ——

$$WF'(E_\pm) \subseteq \{(t,x,y;\tau,\xi,\eta) : d_\theta \phi_\pm(t,x,\theta) = y, \ \tau = d_t\phi_\pm,\ \xi = d_x \phi_\pm,\ \eta = \theta \}.$$

In our case, $Y = \mathbb{R}^n$, $X = \mathbb{R}_t \times \mathbb{R}^n_x$.

$WF_Y(E_\pm) = \{(y,\eta) : (t,x,y;0,0,\eta) \in WF\, E_\pm\}$

$= \phi$ since $d_x\phi_\pm \neq 0 \Rightarrow \xi \neq 0.$

/// $WF_X(E_\pm) = \phi.$

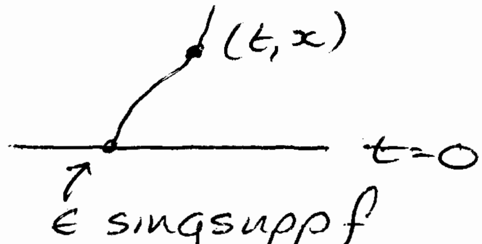
(*) So, $WF(E_\pm f) \subseteq WF'(E_\pm) \circ WF(f).$

Corollary (Generalized Huyghen's Principle)

Let $(t,x) \in \operatorname{sing\,supp} u$. Then $\exists$ characteristic curve through (t,x) (i.e. proj^n of a null-bicharacteristic) such that at time s_0,

$(t(s_0), x(s_0))$ is such that
$$t(s_0) = 0, \quad x(s_0) \in \operatorname{sing\,supp} f$$

i.e.



Proof $WF u \subseteq WF(E_+ f) \cup WF(E_- f)$

$$WF(E_\pm f) \subseteq \{(t,x;\tau,\xi) : \exists (y,\eta) \text{ s.t. } (t,x,y;\tau,\xi,\eta) \in WF'(E_\pm) \text{ \& } (y,\eta) \in WF(f)\}$$

from (*)

$\Rightarrow (y,\eta) \in WF(f) \Rightarrow y \in \operatorname{sing\,supp} f$

but

$$d_\theta \phi_\pm(t,x,\theta) = y$$

& this holds true along whole null-bichar. thru (t,x). $\blacksquare$

—— u ——

The set of all characteristics through $(0,y)$ is the <u>light-cone</u> at y

Local Theory of Fourier Integral Distributions
(Lagrangian dist^{n}s).

$$I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\,d\theta$$

$$WF(I_{a,\phi}) \subseteq \{(x,\xi);\ d_\theta \phi(x,\theta) = 0,\ \xi = d_x\phi(x,\theta)\}.$$
$$= \Lambda_\phi$$

Def^n $C_\phi \triangleq \{(x,\theta) \in X \times \mathbb{R}^n \setminus \{0\};\ d_\theta \phi(x,\theta) = 0\}.$

So $\Lambda_\phi = F(C_\phi)$ where $F(x,\theta) = (x, d_x\phi(x,\theta))$.

Def^n Let ϕ be a phase fc^n ($\mathbb{R}$-valued, $d\phi \neq 0$, ϕ hom. of degree 1).

Then ϕ is called <u>non-degenerate at a cone</u> $\Gamma \subset X \times \mathbb{R}^n - \{0\}$, $\Gamma \subset C_\phi$ if

$d_{x,\theta}\left(\frac{\partial \phi}{\partial \theta_j}\right)$ are linearly independent on Γ.

And ϕ is simply called <u>non-degenerate</u> if $\Gamma = C_\phi$.

<u>Example</u>.

(a) $A \in \Psi^k(X)$

$$K_A = \int e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\,d\xi.$$

$$\phi(x,y,\xi) = \langle x-y,\xi\rangle.$$

$C_\phi = \{(x,y,\xi);\ x = y\}$ & (check) ϕ is non-degenerate.

(b) $K_{E_\pm} = \int e^{i(\phi_\pm(t,x,\xi) - \langle y,\xi\rangle)} a_\pm(t,x,\xi)\, d\xi$

check: $\phi_\pm(t,x,\xi) - \langle y,\xi\rangle$ is non-degenerate for $t \neq 0$.

<u>Theorem</u>. Let ϕ be a non-degenerate phase function. Then

$$F : C_\phi \to \Lambda_\phi \subset X \times \mathbb{R}^n \setminus \{0\}.$$
$$(x,\theta) \mapsto (x, d_x\phi(x,\theta)) = (x,\xi) \text{ say.}$$

is an immersion;

Moreover, Λ_ϕ is a conic Lagrangian manifold.

(12/6)

Proof, C_ϕ is an m-dim^{n}al submanifold of $X \times \mathbb{R}^N \setminus \{0\}$.

Check:

$$dF_{(x,\theta)} : T_{(x,\theta)}(C_\phi) \to T_{F(x,\theta)}(\Lambda_\phi)$$

is injective.

Let $t \in T_{(x,\theta)}(C_\phi)$

$$t = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial \theta_i}$$

$dF_{(x,\theta)}(t) = 0$, show $t = 0$.

$$dF_{(x,\theta)}\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{m} \frac{\partial(x_j \circ F)}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{m} \frac{\partial(\xi_j \circ F)}{\partial x_i} \frac{\partial}{\partial \xi_j}$$
$$= \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial}{\partial \xi_j}$$

$$dF_{(x,\theta)}\left(\frac{\partial}{\partial \theta_i}\right) = \sum_j \frac{\partial(\theta_j \circ F)}{\partial \theta_i} \frac{\partial}{\partial x_j} + \sum_j \frac{\partial(\xi_j \circ F)}{\partial \theta_i} \frac{\partial}{\partial \xi_j}$$

$$= \sum_{j=1}^{m} \frac{\partial^2 \phi}{\partial x_j \partial \theta_i} \frac{\partial}{\partial \xi_j}$$

Then $dF_{(x,\theta)}(t) = \sum_{i=1}^{m} a_i \left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial}{\partial \xi_j} \right)$

$$+ \sum_{i=1}^{N} b_i \left(\sum_{j=1}^{m} \frac{\partial^2 \phi}{\partial x_j \partial \theta_i} \frac{\partial}{\partial \xi_j} \right) = 0$$

$$= \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \left[\sum_{i=1}^{m} a_i \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial^2 \phi}{\partial x_j \partial \theta_i} \right] \frac{\partial}{\partial \xi_j} = 0$$

$\Rightarrow \; a_i = 0, \; i = 1, \dots, m.$

Now since $t \in T_{(x,\theta)}(C_\phi)$,

$$t\left(\frac{\partial \phi}{\partial \theta_j}\right) = 0. \quad j = 1, \dots, N.$$

$$\Rightarrow \sum_{i=1}^{N} b_i \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = 0$$

also, from above $\sum_{i=1}^{N} b_i \frac{\partial^2 \phi}{\partial x_j \partial \theta_i} = 0 \quad j = 1, \dots, m$

But $d_{(x,\theta)}\left(\frac{\partial \phi}{\partial \theta_j}\right)$ are linearly independent

so $b_i = 0, \; i = 1, \dots, N.$

To prove that Λ_ϕ is Lagrangian, we need
(a) $\dim(\Lambda_\phi) = m$ ✓
(b) $\omega = \sum_{i=1}^{m} d\xi_i \wedge dx_i$ vanishes on Λ_ϕ.

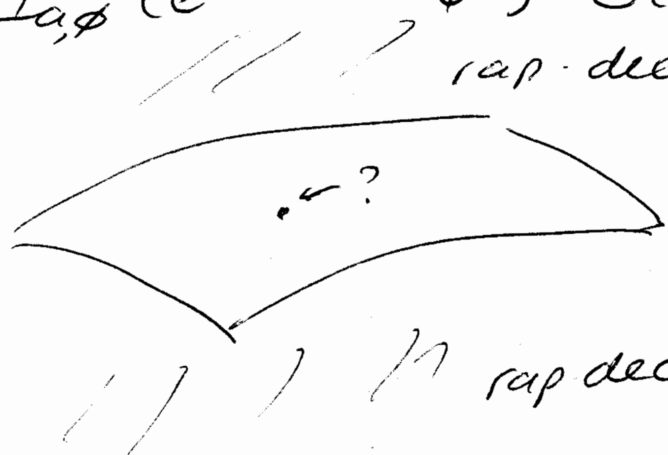
This reduces to showing

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} \quad \forall i, j \; ✓.$$

Exercise Λ is a conic Lagrangian manifold
$\Leftrightarrow$ (a) $\dim \Lambda = n$
(b) $\alpha = \sum \xi_i dx_i$ vanishes on Λ.

Summary... $I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$
ϕ non-deg. phase fc^{n}.
$a \in S^k(X \times \mathbb{R}^N)$
$I_{a,\phi} \longrightarrow \Lambda_\phi$, $\quad WF(I_{a,\phi}) \subseteq \Lambda_\phi$.

We know
$I_{a,\phi}(e^{-it\langle \cdot, \xi \rangle} \psi) = O(t^{-N})$ if $(x,\xi) \notin \Lambda_\phi$
rap. dec. here. $\psi(x) \neq 0$.



?

Λ_ϕ

rap dec. here

Question. Let $(x_0, \xi_0) \in \Lambda_\phi$.
Take $\psi \in C_0^\infty(X)$, $\psi(x_0) \neq 0$, $\xi \in$ nbhd of ξ_0
& consider
$$f_t(\xi) = I_{a,\phi}(e^{-it\langle \cdot, \xi \rangle} \psi)$$

What is the behaviour in t? Also, how does it depend on $a(x,\theta)$?

Let A be a pseudodiff-operator.

$$Af(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)d\xi \qquad a \in S^k_{cl}$$

$$\lim_{t\to\infty} \frac{e^{it\langle\cdot,\xi\rangle} A(e^{-it\langle\cdot,\xi\rangle}\psi)(x)}{t^k}$$

$$= \sigma_k(A)(x,\xi)\psi(x).$$

What is Λ_ϕ for $A \in \Psi^k(X)$

$$K_A = \int e^{i\langle x-y,\xi\rangle} a(x,\xi)d\xi$$

$$C_\phi = \{(x,x,\xi)\}$$

$$\Lambda_\phi = \{(x,x;\xi,-\xi)\}$$

$$\Lambda'_\phi = \text{diagonal}.$$

METHOD OF STATIONARY PHASE

Example. $$g(t) = \int_{\mathbb{R}^n} e^{it\phi(x)} a(x)dx$$ ϕ $\mathbb{R}$-valued, C^∞. $a \in C_0^\infty(\mathbb{R}^n)$, $d\phi \neq 0$

then... $$g(t) = O(t^{-N}) \ \forall N$$

Def^n^ A critical point of $\phi(x) \in C^\infty(X)$, is x_0 such that

$$d\phi(x_0) = 0.$$

Example $n=1$, $\phi(x) = x^2$, crit. pt. @ $x=0$.
$\phi(x) = x^3$ " " "

Def^n $\phi \in C^\infty(X)$, $\mathbb{R}$-valued, x_0 a critical pt.
x_0 is a non-degenerate point if
$\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0)\right)$ non-singular.

First goal Analyse
$$g(t) = \int e^{it\langle Qx, x\rangle} a(x,t)\,dx$$
$\operatorname{supp} a(\cdot, t) \subseteq K \subset\subset X$, $\forall t$.
Q symmetric non-singular.
Assume some cond^ns on a.

(12/9)

Example $\phi(x) = \langle \frac{Qx}{2}, x\rangle$, Q invert. & symmetric.

$x=0$ non-degenerate critical point.

$$g(t) = \int e^{it\langle \frac{Qx}{2}, x\rangle} a(x)\,dx$$
$$= \int \left(e^{it\langle \frac{Q\cdot}{2}, \cdot\rangle}\right)^\wedge(\xi)\, \check{a}(\xi)\,d\xi.$$

Proposition. Let $f(x) = e^{i\langle \frac{Qx}{2}, x\rangle}$

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q} e^{-i\langle \frac{Q^{-1}\xi}{2}, \xi\rangle}$$

Proof// WLOG we can assume

$$Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

So $f(x) = \prod_{j=1}^{n} e^{i\lambda_j \frac{x_j^2}{2}}$

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx = \prod_{j=1}^{n} \int e^{-ix_j\xi_j} e^{i\lambda_j \frac{x_j^2}{2}}\,dx_j$$

$$= \prod_{j=1}^{n} \hat{f}_j(\xi_j) \qquad f_j(x) = e^{i\lambda_j \frac{x_j^2}{2}}$$

(all in the sense of dist^{n}s).

So let

$$h(x) = e^{ia\frac{x^2}{2}} \qquad a \in \mathbb{R}\setminus\{0\}.$$

$h \in \mathcal{S}'(\mathbb{R})$, $\hat{h} \in \mathcal{S}'(\mathbb{R})$.

Recall:

$$\left(e^{-\alpha\frac{x^2}{2}}\right)^{\wedge}(\xi) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sqrt{\alpha}} e^{-\xi^2/2\alpha}, \quad \alpha > 0.$$

Proof: let $v(x) = e^{-x^2/2}$;

$$\frac{d}{dx} v + x v = 0$$

F.T. $\Rightarrow$ same eq^{n}. $\frac{d}{d\xi}\hat{v} + \xi\hat{v} = 0$

$\therefore$ same up to a constant

$$v(x) = c\,\hat{v}(x).$$

$$1 = c\,\hat{v}(0)$$

$$\Rightarrow \sqrt{2\pi} = \int e^{-x^2/2}\,dx = \hat{v}(0) = \frac{1}{2}$$

//

Define $v_z(x) = e^{-zx^2/2}$, $z \in \mathbb{C}$, $\operatorname{Re} z > 0$

$\in \mathcal{S}(\mathbb{R})$

$$z \in \mathbb{R}^+ \Rightarrow \quad \hat{V}_z(\xi) = \frac{1}{(2\pi)^{1/2}} \frac{e^{-\xi^2/2z}}{z^{1/2}}$$

This extends analytically to $\operatorname{Re} z > 0$.

Also $\hat{V}_z(\xi) = \int e^{-ix\cdot\xi} V_z(x)\,dx = \int e^{-ix\xi} e^{-\frac{zx^2}{2}}\,dx$

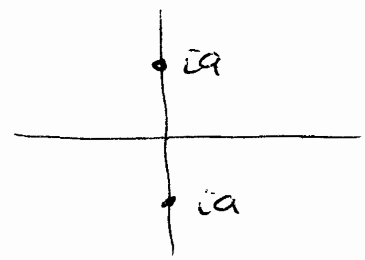
is analytic for $\operatorname{Re} z > 0$

These coincide for $z \in \mathbb{R}^+$, & so

$$\hat{V}_z(\xi) = \frac{1}{(2\pi)^{1/2}} \frac{e^{-\xi^2/2z}}{z^{1/2}} \quad \text{for } \operatorname{Re} z > 0.$$

Now

$$h(x) = V_{-ia}(x).$$



Claim $\hat{h}(\xi) = \frac{1}{(2\pi)^{1/2}} \frac{e^{\frac{\pi i}{4}\operatorname{sgn} a}}{|a|^{1/2}} e^{-\frac{i\xi^2}{2a}}$

This follows since $\hat{V}_z \to \hat{V}_{ia}$ as $\underset{\operatorname{Re} z > 0}{z \to -ia}$ in $\mathcal{S}'(\mathbb{R})$.

Corollary $\left(e^{it\frac{\langle Qx, x\rangle}{2}}\right)^\wedge(\xi)$

$$= \frac{1}{(2\pi)^{n/2}} t^{-n/2} e^{-i\frac{\langle Q^{-1}\xi, \xi\rangle}{2t}} |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}$$

Proof c.o.v. $\sqrt{t}x \mapsto y$.

$$e^{it\frac{\langle Qx, x\rangle}{2}} = e^{i\frac{\langle Qy, y\rangle}{2}}$$

Recall: $f_\lambda(x) = f(\lambda x)$; $\hat{f}_\lambda(\xi) = \frac{1}{|\lambda|^n} \hat{f}\left(\frac{\xi}{\lambda}\right)$ $\lambda > 0$.

$$g(t) = \int e^{it\frac{\langle Qx, x\rangle}{2}} u(x)\,dx \qquad u \in C_0^\infty(X)$$

$$= \int \left(e^{it\langle \frac{Q\cdot}{2}, \cdot\rangle}\right)^\wedge(\xi)\, \check{u}(\xi)\,d\xi$$

$$= \left(\frac{1}{(2\pi)^{n/2}} t^{-n/2} |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}\right) \times$$

$$\int e^{-i\frac{\langle Q^{-1}\xi, \xi\rangle}{2t}} \check{u}(\xi)\,d\xi.$$

$$e^{ix} = \sum_{1}^{N} \frac{(ix)^j}{j!} + R_N(x)$$

w/ $R_N(x) = O(|x|^N)$

$$\int e^{-i\frac{\langle Q^{-1}\xi, \xi\rangle}{2t}} \check{u}(\xi)\,d\xi$$

$$= \sum_{j=0}^{N} \frac{t^{-j}}{j!} \int \left(-i\left\langle \frac{Q^{-1}\xi}{2}, \xi\right\rangle\right)^j \check{u}(\xi)\,d\xi$$

$$+ \int R_N\left(-i\frac{\langle Q^{-1}\xi, \xi\rangle}{2t}\right) \check{u}(\xi)\,d\xi.$$

$j=0$: $\int \check{u}(\xi)\,d\xi = u(0)\,(2\pi)^{n/2}$.

<u>Notice</u>: $g(t) = |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}\, u(0)\, t^{-n/2}$

$$+ O(t^{-n/2-1}).$$

$j=1$: $\displaystyle\int \frac{-i\langle Q^{-1}\xi,\xi\rangle}{2} \check{u}(\xi)\,d\xi.$

$$n=1,\quad -i\int \frac{a\xi^2}{2}\check{u}(\xi)\,d\xi = +\frac{ia}{2}\int \check{u}''(\xi)\,d\xi$$

$$= \frac{ia}{2}u''(0)(2\pi)^{n/2}$$

so $$\int \frac{-i\langle Q^{-1}\xi,\xi\rangle}{2}\check{u}(\xi)\,d\xi$$

$$= -\frac{i}{2}(2\pi)^{n/2}\left(Q^{-1}(D)\right)u(0)$$

where $\sigma(Q^{-1}(D)) = \langle Q^{-1}\xi,\xi\rangle.$

In general, $\displaystyle\int\left(\frac{-i\langle Q^{-1}\xi,\xi\rangle}{2}\right)^j \check{u}(\xi)\,d\xi$ order $2j$ diff. operator.

$$= \frac{(-i)^j}{2^j}\left((Q^{-1}(D))^j u\right)(0)\cdot(2\pi)^{n/2}.$$

So
$$g(t) = |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}\, t^{-n/2}\left\{u(0) + \frac{-i}{2}Q^{-1}(D)u(0)t^{-1} + O(t^{-2})\right\}$$

Moreover:
$$g(t) = C(Q)t^{-n/2}\left\{u(0) + a_1 Q^{-1}(D)u(0)\,t^{-1} + \cdots + a_N(Q^{-1}(D))^N u(0)\,t^{-N} + O(t^{-N-1})\right\}.$$

(errors!! see later)

(12/11)

Theorem

$$g(t) = \sum_{j=0}^{N} \frac{(2\pi)^{n/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}}{j!\,|\det Q|^{1/2}\, t^{n/2+j}} \left(\frac{Q^{-1}(D)}{2i}\right)^{j} u(0)$$

$$+ R_N(t,u)$$

with $|R_N| \leq \dfrac{C_Q}{(N+1)!\, t^{n/2+N+1}} \|u\|_{H^{2(N+1)+n/2+\varepsilon}}$ with $\varepsilon > 0$.

Proof// Recall

$$\left(e^{it\frac{\langle Qx,x\rangle}{2}}\right)^{\wedge}(\xi) = \frac{e^{\frac{\pi i}{4}\operatorname{sgn} Q}}{(2\pi)^{n/2} t^{n/2} |\det Q|^{1/2}} e^{-i\frac{\langle Q^{-1}\xi,\xi\rangle}{2t}}$$

We must estimate

$$\int |\langle Q^{-1}\xi,\xi\rangle|^{N+1} |\check{u}(\xi)|\, d\xi$$

$$= \int (1+|\xi|^2)^{\frac{-n-\varepsilon}{4}} (1+|\xi|^2)^{\frac{n+\varepsilon}{4}} |\langle Q^{-1}\xi,\xi\rangle|^{N+1} |\hat{u}(\xi)|\, d\xi$$

$$\leq \tilde{C}\left(\int (1+|\xi|^2)^{\frac{-n-\varepsilon}{2}} d\xi\right)^{1/2} \left(\int (1+|\xi|^2)^{\frac{n+\varepsilon}{2}} (1+|\xi|^2)^{2(N+1)} |\hat{u}(\xi)|^2 d\xi\right)^{1/2}$$

$\Longrightarrow$ result.

To compute asymptotic expansion of

$$\int e^{it\phi(x)} u(x)\,dx, \qquad u \in C_0^\infty(X),$$

$x_0 \in X$

ϕ real-valued, $\phi \in C^\infty(X)$

$d\phi(x_0) = 0 \quad d\phi(x) \neq 0$ for $x \neq x_0$, $x \in X$

$$\text{Hess}\,\phi(x_0) = \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(x_0)\right) \text{ non-singular.}$$

<u>Lemma</u> (Morse Lemma) " ... – – – ... " Lemma

Let $x_0 \in X$, $d\phi(x_0) = 0$, $\text{Hess}\,\phi(x_0)$ non-singular. Then $\exists$ $\psi : U(x_0) \leftrightarrow V(0)$

U open nbhd of x_0

V open nbhd of 0

s.t.

$$\phi(x) = \phi(x_0) + \tfrac{1}{2}\langle Q(x_0)\psi(x), \psi(x)\rangle$$

$$Q(x) = \text{Hess}\,\phi(x_0)$$

Moreover

$$\psi(x) = x - x_0 + O(|x - x_0|^2).$$

<u>Proof of Lemma</u>:

Taylor exp$^{\underline{n}}$: $\phi(x) = \phi(x_0) + \frac{1}{2}\langle B(x)(x-x_0), x - x_0\rangle$

with $B(x_0) = Q(x_0)$

So we want

$$\langle B(x)(x-x_0), x-x_0\rangle = \langle Q(x_0)\psi(x), \psi(x)\rangle.$$

Idea: write

$$\psi(x) = R(x)(x-x_0),$$ $R(x)$ $n\times n$ matrix smooth in x.

So,

$$\langle Q(x_0)R(x)(x-x_0), R(x)(x-x_0)\rangle$$
$$= \langle R^t(x)Q(x_0)R(x)(x-x_0), (x-x_0)\rangle$$
$$\text{want} = \langle B(x)(x-x_0), (x-x_0)\rangle.$$

To find R, solve

$$R^t(x)\,Q(x_0)R(x) = B(x).$$

& $$R(x_0) = I$$

Let $\Gamma : (n\times n)$ $\mathbb{R}$valued matrices $\longrightarrow$ $(n\times n)$ $\mathbb{R}$-symmetric matrices.

$$R(x) \longmapsto R^t(x)\,Q(x_0)\,R(x).$$

What is $d\Gamma\big|_{R=I}(N)$?

$$d\Gamma\big|_{R=I}(N) = N^t(x)Q(x_0) + Q(x_0)N(x)$$

Claim: $d\Gamma\big|_{R=I}$ is surjective.

Let S be a symmetric matrix. Find N so that

$$N^t Q(x_0) + Q(x_0)N = S.$$

This is solved by

$$N = \tfrac{1}{2}Q^{-1}(x_0)S.$$

So, by the inverse function theorem, (with smooth dependence on x) $\exists$

$$\tilde{\Gamma} : (n\times n)\ \mathbb{R}\text{-symmetric} \to (n\times n)\ \mathbb{R}$$

such that

$$\Gamma\circ\tilde{\Gamma} = I.$$

Thus we can solve

$$R^t(x)\,Q(x_0)\,R(x) = B(x)$$

& $\quad \psi(x) = R(x)(x-x_0),$

$\quad R(x_0) = I.$

$$\int e^{it\phi(x)} u(x)\,dx$$

Morse = Lemma
$$\int e^{it(\phi(x_0) + \frac{1}{2}\langle Q(x_0)\psi(x), \psi(x)\rangle)} u(x)\,dx + O(t^{-M}) \quad \forall M$$

away from $x = x_0$ = crit. pt., we have decay

$$= e^{it\phi(x_0)} \int e^{it\frac{1}{2}\langle Q(x_0)\psi(x), \psi(x)\rangle} u(x)\,dx + O(t^{-M}) \quad \forall M$$

$(y = \psi(x))$ $\quad \psi(x) = x - x_0 + O(|x - x_0|^2)$
$= R(x)(x - x_0)$.

$$= e^{it\phi(x_0)} \int e^{i\frac{t}{2}\langle Q(x_0)y, y\rangle} \underbrace{u \circ \psi^{-1}(y) \left|\frac{dx}{dy}\right| dy}_{\text{call this } v(y) = u\circ\psi^{-1}(y)\left|\frac{dx}{dy}(\psi^{-1}(y))\right|} + O(t^{-M}) \quad \forall M$$

$$g(t) = \int e^{i\frac{t}{2}\langle Q(x_0)y, y\rangle} v(y)\,dy$$

By theorem,
$$g(t) = \sum_{j=0}^{N} \frac{(2\pi)^{n/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q(x_0)}}{j!\,|\det Q(x_0)|^{1/2}\, t^{n/2+j}} \left(\frac{Q^{-1}(x_0)(D)}{2i}\right)^j v(0)$$
$$+ R_N(t, v).$$

We have that
$$\cancel{Q^{-1}(x_0)(D)v(0) = Q^{-1}(x_0)(D)u(x_0)}$$

~~since~~ $v(0) = u(x_0)$
so when $j = 0$, $\left(Q^{-1}(x_0)(D)(v)\right)^j(0) = v(0) = u(x_0)$

$$\frac{\partial}{\partial y_k}\left(u\circ\psi^{-1}\left|\frac{dx}{dy}\right|(\psi^{-1}y)\right)(0) = \frac{\partial}{\partial x_k} u(x_0) \;??$$
Not necessarily (depends on the ψ).

Notice that the higher order terms $(j \geq 1)$ depend on derivatives of ψ at x_0.

(1/6) Review...

Oscillatory integrals: $X \subset \mathbb{R}^n$, open.

$$I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta.$$

ϕ : phase funct^{n} : $\phi \in C^\infty(X \times \mathbb{R}^N - 0)$
$\mathbb{R}$-valued, hom. of deg. 1 in θ.
$d\phi \neq 0$ in $X \times \mathbb{R}^N \setminus \{0\}$

a: amplitude: $a \in S^m(X \times \mathbb{R}^N)$

i.e. $\sup_{x \in K \subset\subset X} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta,K}(1+|\xi|)^{m-|\beta|}$.

<u>Prop^n</u>: $I_{a,\phi} \in \mathcal{D}'(X)$ that coincides with $I_{a,\phi}$ as a function if $m < -(N+1)$.

<u>Prop^n</u>: $WF(I_{a,\phi}) \subseteq \{(x,\xi) \in X \times \mathbb{R}^n \setminus \{0\},\ \exists\, \theta \in \mathbb{R}^N \setminus \{0\},$
$\xi = d_x\phi(x,\theta)$
$d_\theta \phi(x,\theta) = 0\ \}$
$= N_\phi$.

<u>Def^n</u> ϕ : phase fc^n, let $\Gamma \subset X \times \mathbb{R}^N \setminus \{0\}$ be a cone. Then ϕ is called <u>non-degenerate</u> in Γ if

$$d_{x,\theta}\left(\frac{\partial \phi}{\partial \theta_j}\right) \text{ are lin. ind., } j = 1, \cdots, N$$

on $C_\phi \cap \Gamma$, with $C_\phi = \{(x,\theta) \in X \times \mathbb{R}^N \setminus \{0\};$
$d_\theta \phi(x,\theta) = 0\}$.

<u>Propn</u>: Let $\Gamma \subset X \times \mathbb{R}^N \setminus \{0\}$, ϕ non-deg. on Γ. Then $C_\phi \cap \Gamma$ is a submanifold (conic) of dimension n in $X \times \mathbb{R}^N \setminus \{0\}$, and moreover,

$$\begin{aligned} C_\phi \cap \Gamma &\xrightarrow{F} \Lambda_\phi \\ (x,\theta) &\longmapsto (x, d_x\phi(x,\theta)) \end{aligned}$$

is an immersion, & Λ_ϕ is a conic Lagrangian manifold of $X \times \mathbb{R}^n \setminus \{0\}$.

If $I_{a,\phi}$ an oscillatory integral, then
$$WF(I_{a,\phi}) \subseteq \Lambda_\phi$$
i.e.
$$I_{a,\phi}(e^{-i\langle \cdot, t\xi_0\rangle}\psi) = O(t^{-M}) \quad \psi(x_0) \neq 0$$
$$\text{if } (x_0,\xi_0) \notin \Lambda_\phi$$

<u>Question</u>: What happens if $(x_0,\xi_0) \in \Lambda_\phi$?

Compute...
$$I_{a,\phi}(e^{-i\langle \cdot, t\xi_0\rangle}\psi)$$
$$= \int e^{i[\phi(x,\theta) - \langle x, t\xi_0\rangle]} a(x,\theta)\psi(x)\,dx\,d\theta$$
$$= t^{-N}\int e^{it[\phi(x,\theta) - \langle x,\xi_0\rangle]} a(x,t\theta)\psi(x)\,dx\,d\theta$$
$$= t^{-N}\int e^{itf(x,\theta,\xi_0)} g(x,t,\theta)\,dx\,d\theta$$

A simpler situation:

$$h(t) = \int e^{itf(x)} g(x)\,dx,$$

$f \in C^\infty(X)$, $\mathbb{R}$-valued.
$g \in C_c^\infty(X)$.
$df(x_0) = 0$, $x_0 \in X$
$\operatorname{Hess} f(x_0)$ non-singular.
$df(x) \neq 0$ $x \neq x_0$,
$x \in \operatorname{supp} g$.

<u>Morse Lemma</u> $\exists\ y : \mathcal{U} \to \mathbb{R}^n$ diffeo. s.t.
$\mathcal{U}$ nbhd of x_0

$$y(x_0) = 0$$
$$y(x) = x - x_0 + O(|x - x_0|^2)$$

so that

$$f(x) = f(x_0) + \left\langle \frac{Q\,y(x), y(x)}{2} \right\rangle$$

$$Q = \operatorname{Hess} f(x_0).$$

So we get

$$h(t) = e^{itf(x_0)} \int e^{it\langle \frac{Qy(x), y(x)}{2}\rangle} g(x)\,dx.$$

writing $x = \psi^{-1}(y)$.

$$h(t) = e^{itf(x_0)} \int e^{it\langle \frac{Qy, y}{2}\rangle} \underbrace{g(\psi^{-1}(y)) \left|\frac{dx}{dy}\right|}_{k(y) \text{ say.}} dy$$

$$= e^{itf(x_0)} \int e^{it\langle \frac{Qy, y}{2}\rangle} k(y)\,dy$$

$k \in C_0^\infty(\mathbb{R}^n)$.

So

$$e^{-itf(x_0)} h(t) = \int \left(e^{it\frac{\langle Qy,y\rangle}{2}}\right)^{\wedge}(\xi)\,\check{k}(\xi)\,d\xi$$

$$= \int \left(\frac{2\pi}{t}\right)^{n/2} e^{-i\frac{\langle Q^{-1}\xi,\xi\rangle}{2t}} \check{k}(\xi) |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q}\, d\xi$$

$$= \left(\frac{2\pi}{t}\right)^{n/2} |\det Q|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q} \sum_{j=0}^{\infty} \frac{1}{j!} \int \left(\frac{\langle Q^{-1}\xi,\xi\rangle}{2it}\right)^{j} \check{k}(\xi)\,d\xi$$

$$= \left(\frac{2\pi}{t}\right)^{n/2} |\det Q|^{-1/2} e^{\pi i/4\operatorname{sgn} Q} \sum_{j=0}^{\infty} \left(\frac{\langle Q^{-1}\partial_y, \partial_y\rangle}{2i}\right)^{k} k(0) \cdot \frac{t^{-j}}{j!}$$

$Q = \operatorname{Hess} f(x_0)$.

<u>Remark</u>: $k(y) = g(\psi^{-1}(y)) \left|\frac{dx}{dy}\right|(y)$.

$$k(0) = g(x_0)$$

but

$$\partial_{y_j} k(y) = \partial_{y_j}\, g\circ\psi^{-1}(y) \left|\frac{dx}{dy}\right|(y) + g\circ\psi^{-1}(y) \frac{\partial}{\partial y_j} \left|\frac{dx}{dy}\right|(y).$$

@ 0,

$$\partial_{y_j} k(0) = \sum_{\ell=1}^{n} a_\ell(x_0) \frac{\partial g}{\partial x_\ell}(x_0) + g(x_0)\cdot b(x_0).$$

<u>Theorem</u>. (Stationary Phase).

Let $f \in C^\infty(X)$, $df(x_0)=0$, $\operatorname{Hess} f(x_0)$ non-singular, $g \in C_0^\infty(X)$, $df(x) \neq 0$, $x \neq x_0$, $x \in \operatorname{supp} g$.

Then

$$e^{-itf(x_0)} \int e^{itf(x)} g(x)\,dx$$

$$\sim \left(\frac{2\pi}{t}\right)^{n/2} |\det Q|^{-1/2} e^{\frac{\pi i}{4} \operatorname{sgn} Q} \sum_{j=0}^{\infty} \frac{(\tilde{Q})^j g(x_0)}{j!} t^{-j}$$

where $\tilde{Q}$ is a diff-operator of order 2, $(\tilde{Q})^0 = \text{identity}$.

<u>Application</u>. Composition of pseudo-diff. operators.

$$Af(x) = \int e^{i\langle x-y,\xi\rangle} a(x,y,\xi) f(y)\,dy\,d\xi.$$

$$Bg(y) = \int e^{i\langle y-z,\eta\rangle} b(y,z,\eta) g(z)\,dz\,d\eta.$$

Then $$(ABf)(x) = \int e^{i[\langle x-y,\xi\rangle + \langle y-z,\eta\rangle]} a(x,y,\xi) b(y,z,\eta) f(z)\,dy\,d\eta\,d\xi\,dz$$

$$\underline{\text{we want}} = \int e^{i\langle x-z,\theta\rangle} c(x,z,\theta) f(z)\,dz\,d\theta.$$

$a \in S^m(X \times X \times \mathbb{R}^n)$
$b \in S^{\tilde{m}}(X \times X \times \mathbb{R}^n)$.

(Assume B is properly supported).

(1/8)

Formally,

$$\int e^{i\langle x,\xi\rangle}\left(\int e^{\overbrace{i(\langle y,\eta-\xi\rangle-\langle z,\eta\rangle)}^{\phi(y,z,\eta,\xi)}}\,a\cdot b\cdot dy\,d\eta\right)f(z)\,dz\,d\xi$$

treated as parameters.

Look @ critical points of ϕ wrt y & η.

$$d_y\phi=0 \Rightarrow \eta=\xi$$
$$d_\eta\phi=0 \Rightarrow y=z$$

Non degeneracy in y,η. $\mathrm{Hess}_{y,\eta}$ (critical pts) non-sing.

$$\mathrm{Hess}_{y,\eta}=\begin{pmatrix}\partial_y^2 & \partial_{y,\eta}^2\\ \partial_{y,\eta}^2 & \partial_\eta^2\end{pmatrix}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}=Q.$$

We want to understand this for large ξ.

Recall $\int e^{itf(x)}g(x)\,dx \sim e^{itf(x_0)}|\det\cdots|e^{\frac{\pi i}{4}\cdots}t^{\cdots}\sum\cdots$

at critical point we get

$$\sum_{j=0}^{\infty}\int e^{i\langle x-y,\xi\rangle}\frac{1}{j!}\left(\left\langle Q^{-1}\frac{\partial_{y,\eta}}{2i},\partial_{y,\eta}\right\rangle\right)^j(ab)\Big|_{\substack{y=z\\ \eta=\xi}}f(y)\,dy\,d\xi$$

Q as above.

$$=\int e^{i\langle x-y,\xi\rangle}c(x,y,\xi)f(y)\,dy\,d\xi$$

with $c\sim\sum\frac{1}{j!}\left\langle Q^{-1}\frac{\partial_{y,\eta}}{2i},\partial_{y,\eta}\right\rangle^j(ab)\Big|_{\substack{y=z\\ \eta=\xi}}$

check: this is $c \sim \sum \frac{1}{\alpha!} D_\xi^\alpha a \, \partial_x^\alpha b$.

——— n ———

Rigorously...

Stationary Phase depending on a Parameter.

$$h(t,a) = \int e^{itf(x,a)} g(x,a,t)\,dx.$$

$f \in C^\infty(X \times A)$ $\quad X \subset \mathbb{R}^n$ open
$\quad A \subset \mathbb{R}^p$ compact (parameter).

f $\mathbb{R}$-valued

$g \in C^\infty(X \times A \times \mathbb{R}^+)$

$g(x,a,t) = 0$ whenever $x \notin K \subset\subset \mathbb{R}^n$
K compact, $\forall\, a \in A, t \in \mathbb{R}^+$.

$$\sup_{\substack{x \in K \\ a \in A}} |\partial_x^\alpha g(x,a,t)| \le C_{K,A,\alpha} t^{M + \delta\alpha} \quad \text{(for some } m)$$
with $\delta < \frac{1}{2}$.

$\forall\, a \in A$ $\exists!\, x(a)$ non-degenerate critical point of f, i.e.

$d_x f(x(a), a) = 0$

& $\partial_x^2 f(x(a), a) = Q(a)$ non-singular

& the map $x: A \to X$ is C^∞

<u>Theorem</u> With $h(t,a)$ as above, uniformly in A,

$$h(a,t) \sim |\det Q(a)|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q(a)} \left(\frac{2\pi}{t}\right)^{n/2} e^{itf(x(a),a)} \sum_{j=0}^\infty \frac{t^{-j}}{j!} \left(\frac{\langle Q^{-1}(a)\partial_x, \partial_x \rangle}{2i} \right)^j g(x(a),a,t)$$

The generic term in the asymptotic exp^n is of the form

$$k(a,t) = \langle a^{-1}(a)\partial_x, \partial_x\rangle^j g(x(a), a, t)\; t^{-j}$$

$$|k(a,t)| \leq C t^{m+\delta 2j} \cdot t^{-j} = C t^{m+j(2\delta-1)}$$

so if $\delta < \frac{1}{2}$, as $j \to \infty$, $j(2\delta-1)$ decreases.

<u>Composition of ΨDO revisited...</u>

$f \in C_0^\infty(X)$ $\quad Af(x) = \int e^{i\langle x-y,\xi\rangle} a(x,y,\xi) f(y)\,dy\,d\xi$.

$g \in C_0^\infty(X)$ $\quad Bg(x) = \int e^{i\langle y-z,\eta\rangle} b(y,z,\eta) g(z)\,dz\,d\eta$

$a \in S^m(X \times X \times \mathbb{R}^n)$
$b \in S^{\tilde{m}}(X \times X \times \mathbb{R}^n)$
B properly sptd.

$$(A \circ B)f(x) = \int e^{i(\langle x-y,\xi\rangle + \langle y-z,\eta\rangle)} ab\, f(z)\,dz\,d\eta\,d\xi\,dy.$$

defined as an oscillatory integral.

$$= \int e^{ix\cdot\xi}\left(\int e^{i(y\cdot(\eta-\xi) - z\cdot\eta)} ab\,dy\,d\eta\right) f(z)\,dz\,d\xi$$

$$= \int e^{ix\cdot\xi}\left(\int e^{i|\xi|\left(y\cdot\left(\frac{\eta-\xi}{|\xi|}\right) - z\cdot\frac{\eta}{|\xi|}\right)} ab\,dy\,d\eta\right) f(z)\,dz\,d\xi.$$

c. of v.

$$\tilde{\eta} = \frac{\eta}{|\xi|}, \quad \omega = \frac{\xi}{|\xi|}, \quad t = |\xi|.$$

So inner integral ...

$$t^n \int e^{it((y-z)\cdot\tilde{\eta} - y\cdot\omega)} a(x,y,\xi) b(y,z,t\tilde{\eta}) d\tilde{\eta}\, dy$$

$h(\omega, z, x, t)$

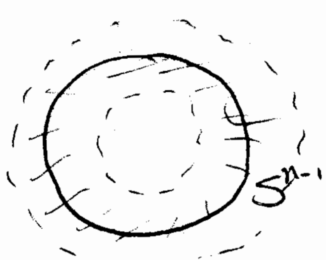
$$= \int e^{it((y-z)\cdot\tilde{\eta} - y\cdot\omega)} a(x,y,t\omega) b(y,z,t\tilde{\eta}) d\tilde{\eta}\, dy.$$

Parameter space: $A = S^{n-1} \times K \times \tilde{K} \quad K, \tilde{K} \subset\subset \mathbb{R}^n$.

(1/10) Notice that h is defined only as an oscillatory integral.

We'll show it is actually a function:

idea: $d_y(y(\tilde{\eta} - \omega) - z\tilde{\eta}) = 0 \iff \tilde{\eta} = \omega \in S^{n-1}$



Let $\chi \in C_0^\infty(\mathbb{R}^n)$

$$\chi(\xi) = \begin{cases} 1 & |\xi| \le 1 \\ 0 & |\xi| \geq 2. \end{cases}$$

Write

$$h(t,\omega,z,x) = \int e^{it[y\cdot(\tilde{\eta}-\omega) - z\cdot\tilde{\eta}]} ab\, \chi(\tilde{\eta}-\omega) dy\, d\tilde{\eta}$$

$$+ \int e^{it[y\cdot(\tilde{\eta}-\omega) - z\cdot\tilde{\eta}]} ab\, (1-\chi)(\tilde{\eta}-\omega) dy\, d\tilde{\eta}$$

$$= h_1 + h_2.$$

h_1 is a fc$^{\underline{n}}$ since χ compactly sptd.

(int. in y o.k. since b prop. supported).

On $\mathrm{supp}(1-\chi)(\tilde{\eta}-\omega)$

$$d_y\left(y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}\right)\neq 0$$

W.L.O.G., assume $\tilde{\eta}_1\neq\omega_1$ on $\mathrm{supp}\,(1-\chi)(\tilde{\eta}-\omega)$.

Write
$$e^{it[y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}]}=\frac{\partial_{y_1}^{M}e^{it[y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}]}}{\left(it\,(\tilde{\eta}_1-\omega_1)\right)^{M}}$$

Int. by parts to get

$$h_2=\frac{1}{(it)^M}\int e^{it[y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}]}\,(-\partial_{y_1})^{M}\frac{(a\,b)(1-\chi)(\tilde{\eta}-\omega)}{(\tilde{\eta}_1-\omega_1)^{M}}\,dy\,d\tilde{\eta}$$

$$h_2=O(t^{-\tilde{M}})\quad\forall\ \tilde{M}$$

We have written

$$h_1(t,\omega,z,x)=\int e^{it[y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}]}(ab)\ \chi\ (\tilde{\eta}-\omega)\,d\tilde{\eta}\,dy$$
$$=\int e^{itf(u,a)}g(u,a,t)\,du\ ,\quad u\in\mathbb{R}^{2n}.$$

<u>Excercise</u>

$f\ \&\ g$ sat. conds of stationary phase with parameters.

$$f(u,a)=y\cdot(\tilde{\eta}-\omega)-z\cdot\tilde{\eta}.$$

$$d_y f = 0 \iff \tilde{\eta} = \omega = \frac{\xi}{|\xi|}$$

$$d_{\tilde{\eta}} f = 0 \iff z = y$$

$$\text{Hess}_{y,\tilde{\eta}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{non-degenerate.}$$

$$h_1 \sim \underbrace{\left(\frac{2\pi}{t}\right)^{\frac{2n}{2}}}_{e^{-itz\cdot\omega}} \sum \frac{t^{-j}}{j!}\left(\left\langle Q^{-1}\frac{\partial_{y,\tilde{\eta}}}{2i}, \partial_{y,\tilde{\eta}}\right\rangle\right)^j \left(ab\,\chi(\tilde{\eta}-\omega)\right)\Big|_{\substack{\tilde{\eta}=\omega \\ z=y}}$$

Result:

$$(A\circ B)f(x) = \int e^{i(x\cdot\xi - z\cdot\xi)} c(x,z,\xi) f(z)\,dz\,d\xi \qquad (*)$$

where $c(x,z,\xi) \sim (2\pi)^n \sum_{j=0}^{\infty} \frac{|\xi|^{-j}}{j!}\left(\left\langle Q^{-1}\frac{\partial_{y,\tilde{\eta}}}{2i}, \partial_{y,\tilde{\eta}}\right\rangle\right)^j (ab)\Big|_{\substack{\tilde{\eta}=\omega \\ y=z}}$

We want

$$(A\circ B)f(x) = \int e^{ix\cdot\xi} d(x,\xi)\hat{f}(\xi)\,d\xi.$$

We have $(*)$; apply st. phase to $(*)$ again!

$$(*) = (A\circ B)f(x) = \int \frac{1}{(2\pi)^n} e^{i(x-z)\xi} c(x,z,\xi) \int e^{iz\eta}\hat{f}(\eta)\,d\eta\,d\xi\,dz$$

$\swarrow \varphi$ say

$$= \frac{1}{(2\pi)^n}\int e^{i[(x-z)\cdot\xi + z\cdot\eta]} c(x,z,\xi)\hat{f}(\eta)\,d\eta\,d\xi\,dz$$

St. Ph. in ξ & z

Critical pts: $d_z\varphi = \eta - \xi = 0 \iff \eta = \xi.$

$\qquad d_\xi\varphi = x - z = 0 \iff z = x.$

$\mathrm{Hess}_{z,\eta}\,\varphi$ is non-sing. $\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$

Get $\frac{1}{(2\pi)^n}\int e^{ix\cdot\eta}\, d(x,\eta)\hat{f}(\eta)\,d\eta$

with
$$d \sim \sum \frac{1}{j!}\left(\frac{\langle Q^{-1}\partial_{z,\xi},\partial_{z,\xi}\rangle}{2i}\right)^j c(x,z,\xi)\Big|_{\substack{x=z\\ \xi=\eta}}.$$

<u>Final answer</u>:

<u>Theorem</u> $A\in\Psi^m(X)$, $B\in\Psi^{\tilde m}(X)$ prop. sptd,

Then
$$A\circ B\in\Psi^{m+\tilde m}(X)$$

& $(A\circ B)f(x) = \int e^{ix\cdot\xi}\, d(x,\xi)\hat{f}(\xi)\,d\xi$

& $d(x,\xi)\sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha a(x,\xi)\,\partial_x^\alpha b(x,\xi)$

(1/13) Generalized Radon Transform (Part I)

Recall: $f \in C_0^\infty(\mathbb{R}^n)$

$$Rf(s,\omega) \triangleq \int_{x\cdot\omega=s} f(x)\,dH$$

$\omega \in S^{n-1}$, $s \in \mathbb{R}$, dH Lebesgue measure on $\{x \mid x\cdot\omega = s\}$.

$$Rf(-s,-\omega) = Rf(s,\omega)$$

$$R^t g(x) \triangleq \int_{S^{n-1}} g(x\cdot\omega,\omega)\,d\omega \qquad g \in C^\infty(\mathbb{R}\times S^{n-1})$$

Then $\langle Rf, g\rangle_{L^2(\mathbb{R}\times S^{n-1})} = \langle f, R^t g\rangle_{L^2(\mathbb{R}^n)}$

Radon Inversion Formula:

$$f = c_n R^t |\partial_s|^{n-1} Rf$$

where

$$|\partial_s|^{n-1} f(s,\omega) = \frac{1}{2\pi}\int e^{is\cdot\rho} |\rho|^{n-1} \mathcal{F}_s f(\rho,\omega)\,d\rho$$

If n is odd, then $|\partial_s|^{n-1} = \partial_s^{n-1}$

Note $\quad R\Delta f = \frac{\partial^2}{\partial s^2} Rf \quad f \in C_0^\infty(\mathbb{R}^n)$

further, $\quad R(-\Delta)^\alpha f = \left(\frac{-\partial^2}{\partial s^2}\right)^\alpha Rf \quad$ (eg. $\alpha \geq 0$).

These follow from the property:

$$\hat{f}(\rho\omega) = (\tilde{\mathcal{F}}_s Rf)(\rho,\omega) \qquad \rho\in\mathbb{R},\ \omega\in S^{n-1}$$

Then: $\tilde{\mathcal{F}}_s(R(-\Delta)^\alpha f)(\rho,\omega) = ((-\Delta)^\alpha f)^\wedge(\rho\omega)$

$$= |\rho\omega|^{2\alpha}\hat{f}(\rho\omega) = |\rho|^{2\alpha}\hat{f}(\rho\omega)$$

$$= \tilde{\mathcal{F}}_s\left(\left(\frac{-\partial^2}{\partial s^2}\right)^\alpha Rf\right)(s,\omega)$$

<u>Note</u> $\quad R^t(-\partial_s^2)^\alpha g = (-\Delta)^\alpha R^t g.$

This is just $(R(-\Delta)^\alpha)^t = (-\Delta)^\alpha R^t$
& $((-\partial_s^2)^\alpha R)^t = R^t(-\partial_s^2)^\alpha$.

Radon inversion formula may be rewritten:

$$f = C_n(-\Delta)^{\frac{n-1}{2}} R^t R f.$$

when $n=3$, $\quad f = C_n(-\Delta)R^tRf.$

In particular,
R^tR is an elliptic pseudodifferential operator
& $R^tR = \dfrac{(-\Delta)^{-(\frac{n-1}{2})}}{C_n}.$

More generally, we'll define generalized Radon transforms by integrating fc^ns over more general surfaces. We'll prove (under some restrictions) R^tR is an elliptic ΨDO.

Then we can find E so that

$$E R^t R f = f + Kf \quad \text{w/ } K \text{ smoothing.}$$

This says that we can recover singularities of f from knowledge of Rf; in fact from knowledge of the singularities of Rf.

Formally,

$$Rf(s,\omega) = \int \delta(s - \phi(x,\omega))\, a(x,\omega)\, f(x)\, dx$$

$$\omega \in S^{n-1}, \quad s \in \mathbb{R}, \quad f \in C_0^\infty(\mathbb{R}^n)$$

this generalizes

$$R_0 f(s,\omega) = \int \delta(s - x\cdot\omega) f(x)\, dx$$

Remark: Even in the case

$$\tilde{R} f(s,\omega) = \int \delta(s - x\cdot\omega)\; a(x,\omega) f(x)\, dx$$
$$a > 0$$

it is in general not known how to invert $\tilde{R}$, or whether $\tilde{R}$ is injective.

<u>Conditions on ϕ</u>

$$\phi \in C^\infty(\mathbb{R}^n \times S^{n-1})$$
$$d_x \phi(x,\omega) \neq 0 \qquad \forall\ x,\omega$$
$$\det\left(\frac{\partial^2 \phi}{\partial x \partial \omega}\right) \neq 0 \qquad \forall\ x,\omega.$$

<u>Eg</u> ϕ = inner product
or ϕ close to inner product.

$a \in C^\infty(\mathbb{R}^n \times S^{n-1})$, $a \geq \varepsilon > 0$ in $\mathbb{R}^n \times S^{n-1}$

$$Rf(s,\omega) \triangleq \int \delta(s - \phi(x,\omega))\, a(x,\omega) f(x)\, dx$$

$$= \int e^{i(s-\phi(x,\omega))\rho}\, a(x,\omega) f(x)\, dx\, d\rho .$$

$$\overset{!}{=} \int e^{i(s-\phi(x,\omega))\rho}\, a(x,\rho) f(x)\, dx\, d\rho$$

where we extend a by

$$a(x,\rho) = a(x,\omega) \quad w/\ \omega = \frac{\rho}{|\rho|} .$$

Let $A : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R} \times S^{n-1})$ of the form

$$Af(s,\omega) = \int e^{i(s-\phi(x,\omega))\rho}\, a(x,\rho) f(x)\, dx\, d\rho .$$

What kind of operator is A^* ?

Consider

$$Bf(x) = \int_{S^{n-1}} f(\underbrace{\phi(x,\omega)}_{s = \phi(x,\omega)}, \omega)\, b(x,\omega)\, d\omega$$

$b \in C^\infty(\mathbb{R}^n \times S^{n-1})$, $b \geq \varepsilon > 0$ in $\mathbb{R}^n \times S^{n-1}$.

$f \in C^\infty(\mathbb{R} \times S^{n-1})$

$B : C^\infty(\mathbb{R} \times S^{n-1}) \to C^\infty(\mathbb{R}^n)$.

Q: What is the composition BA with A, B of the forms above?

$$BAf(x) = \int_{S^{n-1}} Af(\phi(x,\omega),\omega)\, b(x,\omega)\, d\omega$$

$$= \int_{S^{n-1}} \int e^{i(\phi(x,\omega)-\phi(y,\omega))\rho} a(y,\rho)\, b(x,\omega) f(y)\, dy\, d\rho\, d\omega$$

(1/15) Extend $\phi(x,\omega)$ to $\phi(x,\xi)$ with ϕ homogeneous of degree 1 in ξ.
Similarly $a(y,\xi)$ & $b(x,\xi)$ hom of deg 1

Set $\xi = \rho\omega$, $\rho = |\xi|$, $\omega = \dfrac{\xi}{|\xi|}$, $d\xi = \rho^{n-1} d\omega\, d\rho$

Assume
$$\phi(x,-\omega) = \phi(x,\omega)$$
$$a(x,-\omega) = a(x,\omega)$$
$$b(x,-\omega) = b(x,\omega)$$

So

$$BAf(x) = 2\int_{\mathbb{R}^n_y}\int_0^\infty \int_{S^{n-1}} e^{i(\phi(x,\omega)-\phi(y,\omega))\rho} a(y,\omega)\, b(x,\omega) f(y)\, dy\, d\rho\, d\omega$$

c of v as above

$$= 2\iint e^{i(\phi(x,\xi)-\phi(y,\xi))} \chi(\xi) \frac{a(y,\xi)\, b(x,\xi)}{|\xi|^{n-1}} f(y)\, d\xi\, dy \quad + \text{Smoothing}$$

$\chi \in C^\infty$, $\chi = 0$ near 0
$\chi = 1$ $|\xi| \geq 1$ say.

Expanding $\phi(x,\xi) - \phi(y,\xi)$ in Taylor series we get...

$$BAf(x) = 2\iint e^{i\langle x-y,\, h(x,y,\xi)\rangle} \chi(\xi)\, \frac{a(y,\xi)\, b(x,\xi)}{|\xi|^{n-1}} f(y)\, dy\, d\xi$$

mod. smoothing.

Put $\eta = h(x,y,\xi)$ (note h = hom. deg 1 in ξ).

We need

$$d_x \phi(x,\omega) \neq 0 \quad \text{(a global c.of.v).}$$

$$\det \left| \frac{\partial^2 \phi}{\partial x_i \partial \xi_j} \right| \neq 0$$

since

$$h(x,y,\xi) = \nabla_x \phi(x,\tilde{y},\xi) \quad \text{(of this form).}$$

Now

$$BAf(x) = 2\iint e^{i\langle x-y,\eta\rangle} \chi(h^{-1}(\eta))\, a(y, h^{-1}(\eta))\, b(x, h^{-1}(\eta)) \cdots$$

$$\cdots \frac{1}{|h^{-1}(\eta)|^{n-1}} \left|\frac{d\xi}{d\eta}\right| f(y)\, d\eta\, dy$$

$h^{-1}(\eta)$ is defined by a partition of unity on the sphere, extended to be homogeneous of deg. 1 in ξ.

$$c(x,y,\eta) = \frac{\chi(h^{-1}(\eta))\, a(y, h^{-1}(\eta))\, b(x, h^{-1}(\eta))}{|h^{-1}(\eta)|^{n-1}} \left|\frac{d\xi}{d\eta}\right|$$

$$\in S^{-(n-1)}(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\eta)$$

and $c \neq 0$ for large η if $a \neq 0$, $b \neq 0$

Conclusion:

BA is an elliptic ΨDO; therefore $\exists\ E \in \Psi^{(n-1)}$ so that $EBA = I + K$ with K smoothing.

<u>Theorem</u> (Beylkin, CPAM, ~'82).

Let
$$Rf(s,\omega) = \int e^{i(s-\phi(x,\omega))\rho} a(x,\omega) f(x)\, dx\, d\rho$$
with ϕ $\mathbb{R}$-valued,
$$\phi \in C^\infty(\mathbb{R}^n \times S^{n-1}), \qquad f \in C_0^\infty(\mathbb{R}^n).$$
$$\phi(x,\omega) = \phi(x,-\omega).$$
$$\phi(x,\xi) = |\xi|\,\phi\left(x, \frac{\xi}{|\xi|}\right), \quad \xi \neq 0$$

$$a \in C^\infty(X \times S^{n-1})$$
$$a(x,\omega) = a(x,-\omega)$$
$$a \geq \varepsilon > 0 \text{ in } \mathbb{R}^n \times S^{n-1}$$
$$a(x,\xi) = a\left(x, \frac{\xi}{|\xi|}\right) \quad \xi \in \mathbb{R}^n,\ \xi \neq 0.$$

$$d_x \phi(x,\omega) \neq 0 \quad \forall\ (x,\omega)$$

$$\det \left| \frac{\partial^2 \phi}{\partial x_i\, \partial \xi_j}(x,\omega) \right| \neq 0 \quad \forall\ (x,\omega)$$

Let
$$R^t f(x) = \int_{S^{n-1}} f(\phi(x,\omega),\omega)\, b(x,\omega)\, d\omega$$
with
$$b \in C^\infty(\mathbb{R}^n \times S^{n-1})$$
$$b \geq \varepsilon > 0 \text{ in } \mathbb{R}^n \times S^{n-1}$$
$$b(x,\omega) = b(x,-\omega)$$
$$b(x,\xi) = b\left(x, \frac{\xi}{|\xi|}\right), \quad \xi \in \mathbb{R}^n,\ \xi \neq 0$$

Then

$$R^t R \in \Psi^{-(n-1)}(\mathbb{R}^n), \text{ is elliptic,}$$

and its amplitude can be expressed explicitly. $\square$

Local Theory of Fourier Integral Operators and Fourier Integral Distributions

Def^{n} $\quad I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$

ϕ a phase fc^{n}: $\quad \phi \in C^\infty(X \times \mathbb{R}^N \setminus \{0\})$

ϕ $\mathbb{R}$-valued

$d\phi \neq 0$ on $X \times \mathbb{R}^N \setminus \{0\}$.

$a \in S^m(X \times \mathbb{R}^N)$.

Assume ϕ is a non-degenerate phase fc^{n} on a cone $\Gamma \subset X \times \mathbb{R}^N \setminus \{0\}$ (i.e. $d\left(\frac{\partial \phi}{\partial \theta_j}\right)$ are linearly independent on $\Gamma \cap C_\phi = \Gamma \cap \{d_\theta \phi = 0\}$).

Assume $a \in S^{-\infty}$ in the complement of Γ

Then

$I_{a,\phi}$ is called a Fourier Integral Distribution $\square$

Def^{n} $\quad Y \subseteq \mathbb{R}^{n_y}$ $\quad Y$ open

$X \subseteq \mathbb{R}^{n_x}$ $\quad X$ open

Recall ϕ non-degen $\Rightarrow \Lambda_\phi$ is Lagrangian p43

$A : C_0^\infty(Y) \longrightarrow \mathcal{D}'(X)$ $\quad$ linear, cts

A is a <u>Fourier Integral Operator</u> if
$K_A \in \mathcal{D}'(X \times Y)$ is a Fourier Int. Distⁿ.

<u>Goal</u> Prove that the linearized Seismic migratⁿ map is a FIOp.

(1/17).

<u>Theorem</u> (Composition of FIOP's I)

$A_1 : C_0^\infty(Y) \to \mathcal{D}'(X)$
FIOP, $Y \subseteq \mathbb{R}^{n_Y}$ $X \subseteq \mathbb{R}^{n_X}$ open
with amplitude
$a_1 \in S^{M_1}(X \times Y \times \mathbb{R}^{N_1})$
& non-degenerate phase fcⁿ ϕ_1 on Γ
Γ conic in $X \times Y \times \mathbb{R}^{N_1} \setminus \{0\}$

$A_2 : C_0^\infty(Z) \to \mathcal{D}'(Y)$
FIOP, $Z \subseteq \mathbb{R}^{n_Z}$ open
with amplitude
$a_2 \in S^{M_2}(Y \times Z \times \mathbb{R}^{N_2})$
& non-degenerate phase fcⁿ ϕ_2 on Γ
Γ conic in $Y \times Z \times \mathbb{R}^{N_2} \setminus \{0\}$

(assume $a_i \in S^{-\infty}$ in Γ_i^c $i = 1,2$).

Assume
(a) $\Pi_{X \times Y}(\operatorname{supp} a_1) \times \Pi_{Y \times Z}(\operatorname{supp} a_2) \cap (X \times \Delta_Y \times Z) \longrightarrow X \times Z$ (x, y, y, z)
is proper
(b) $\eta \neq 0$ if $(x, y; \xi, \eta) \in \Lambda'_{\phi_1}$
or $(y, z; \eta, \zeta) \in \Lambda'_{\phi_2}$

(c) $\xi \neq 0$ or $\zeta \neq 0$ if
$(x,y;\xi,\eta) \in \Lambda'_{\phi_1}$ <u>and</u> $(y,z;\eta,\zeta) \in \Lambda'_{\phi_2}$

(d) $\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}$ intersects

$T^*(X \times \Delta_y \times Z)$ transversally

<u>Then</u> $A_1 \circ A_2 : C_0^\infty(Z) \to \mathcal{D}'(X)$
is a FIOP with amplitude
$$a \in S^{m_1+m_2-n_y}(X \times Z \times \mathbb{R}^N)$$
where
$$N = N_1 + N_2 + n_y$$
and with phase function ϕ non degenerate in a cone
$$\Gamma \subset (X \times Z \times \mathbb{R}^N \setminus \{0\}).$$
Moreover,
$$\Lambda'_\phi = \Lambda'_{\phi_1} \circ \Lambda'_{\phi_2}$$

<u>Corollary</u> $A_1 \circ A_2 \in \Psi^m(X)$ if (a), (b), (c), (d) are satisfied and further,
$$\Lambda'_{\phi_1} \circ \Lambda'_{\phi_2} = \{(x,x;\xi,\xi) \mid \xi \neq 0\}$$

Recall: $Af(x) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta) f(y)\, dy\, d\theta$
$a \in S^m(X \times Y \times \mathbb{R}^N)$
ϕ non deg. ph-fc$^{\underline{n}}$ on $\Gamma \subset \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ a cone.

Define
$$C_\phi = \{(x,y;\theta) \in X \times Y \times \mathbb{R}^N \setminus \{0\} \mid d_\theta \phi = 0\}$$

Now map

$$C_\phi \cap \Gamma \xrightarrow{F} T^*(X\times Y)\setminus\{0\}$$
$$(x,y,\theta) \longmapsto (x,y,d_{x,y}\phi(x,y,\theta))$$

Then

$$\Lambda_\phi = F(C_\phi \cap \Gamma)$$

$$\Lambda'_\phi = \{(x,y;\xi,\eta) \in T^*(X\times Y)\setminus\{0\} \text{ s.t. } (x,y,\xi,-\eta) \in \Lambda_\phi\}.$$

<u>Proof of Theorem</u>

$A_1 \circ A_2$ is well-defined:

$$K_{A_1\circ A_2}(x,z) = \int K_{A_1}(x,y)\, K_{A_2}(y,z)\,dy$$
$$= \pi_* \Delta^*(K_{A_1} \otimes K_{A_2})$$

(a) guarantees that π_* is well defined.

To define Δ^*.

(†) $WF_y'(A_1) \cap WF_y'(A_2) = \phi$

$$WF_y A = \{(y,\eta) \in Y\times\mathbb{R}^{n_y}\setminus\{0\};\ (x,y,0,\eta) \in WF(A)\}$$

So (†) is satisfied by (c).

Calc. $$K_{A_1\circ A_2}(x,z) = \int e^{i(\phi_1(x,y,\theta)+\phi_2(y,z,\sigma))} a_1(x,y,\theta)\, a_2(y,z,\sigma)\,dy\,d\theta\,d\sigma$$

Remark: If $(x,z) \in K$ compact, $|(\theta,\sigma)|=1$, then $\operatorname{supp}(a_1,a_2) \subseteq \tilde{K}$, compact in y-var from (a).

want $$= \int e^{i\phi(x,z,\tilde\theta)} a(x,z,\tilde\theta)\,d\tilde\theta,\quad a \in S^m$$

ϕ non-deg. in some cone Γ.

Naive idea: $\tilde{\theta} = (\theta, y, \sigma)$

$$\phi(x, z, \tilde{\theta}) = \phi_1(x, y, \theta) + \phi_2(y, z, \sigma)$$
$$a(x, z, \tilde{\theta}) = a_1(x, y, \theta) a_2(y, z, \sigma)$$

<u>But</u>

(1) ϕ is not hom. of deg. 1 in $\tilde{\theta}$

$$\phi(x, z, \lambda\tilde{\theta}) = \phi_1(x, \underset{\substack{\uparrow \\ \text{not hom. of deg. 1 in } y.}}{\lambda y}, \lambda\theta) + \phi_2(\lambda y, z, \lambda\sigma)$$

(2) a is not a symbol. if $a \in S^m$ then $\partial_y a \notin S^{m-1}$.

Furthermore, $\partial_\theta a$ should be an order lower in y, θ, σ!

To solve (1) set $y = \dfrac{\tilde{y}}{|(\theta, \sigma)|}$

$$dy = |(\theta, \sigma)|^{-n_y} d\tilde{y}$$

So
$$K_{A_1 \circ A_2}(x, z) = \int e^{i(\phi_1(x, \frac{\tilde{y}}{|(\theta,\sigma)|}, \theta) + \phi_2(\frac{\tilde{y}}{|(\theta,\sigma)|}, z, \sigma))} a_1(x, \tfrac{\tilde{y}}{|(\theta,\sigma)|}, \theta) a_2(\tfrac{\tilde{y}}{|\theta,\sigma|}, z, \sigma) |(\theta,\sigma)|^{-n_y} d\tilde{y}\, d\theta\, d\sigma$$

<u>Claim</u>

$$\phi(x, z, \tilde{\theta}) = \phi_1(x, \tfrac{\tilde{y}}{|(\theta,\sigma)|}, \theta) + \phi_2(\tfrac{\tilde{y}}{|\theta,\sigma|}, z, \sigma)$$

is a phase function, $\tilde{\theta} = (\tilde{y}, \theta, \sigma)$

ϕ hom. of deg 1 in $\tilde{\theta}$ is clear.

ϕ $\mathbb{R}$-valued clearly

$\phi \in C^\infty(X \times Z \times \mathbb{R}^N \setminus \{0\})$ $\quad N = N_1 + N_2 + n_y$

(1/22)

<u>Remark</u>: We can assume WLOG that $a_1 a_2 = 0$ near $(\theta, \sigma) = (0,0)$.

Now, we need an amplitude

$$a(x, z, \tilde{\theta}) = a_1\left(x, \frac{\tilde{y}}{|(\theta,\sigma)|}, \theta\right) a_2\left(\frac{\tilde{y}}{|(\theta,\sigma)|}, z, \sigma\right)$$

No problem in $\tilde{y}$ variable:

$$\partial_{\tilde{y}} a = \frac{1}{|(\theta,\sigma)|}\left(\partial_y (a_1 a_2)\right)$$

$\tilde{y} = y|(\theta,\sigma)|$ so $|\tilde{y}| \le C|(\theta,\sigma)|$ on $\operatorname{supp}(a_1 a_2)$

In the region $C_2|\theta| \le |\sigma| \le C_1|\theta|$, $C_1, C_2 > 0$

i.e. $C_2 \le \frac{|\sigma|}{|\theta|} \le C_1$

then $a(x, z, \tilde{\theta})$ is an amplitude

$$\left|\partial_\theta \left(a_1\left(x, \frac{\tilde{y}}{|\theta,\sigma|}, \theta\right) a_2\left(\frac{\tilde{y}}{|\theta,\sigma|}, z, \sigma\right)\right)\right|$$

$$= \left(\partial_y a_1 \cdot \partial_\theta \frac{1}{|\theta,\sigma|} \cdot a_2\right) + (\partial_\theta a_1) a_2 + \left(a_1 \partial_\theta \frac{1}{|\theta,\sigma|} \partial_y a_2\right)$$

Look @ $\partial_\theta a_1 \cdot a_2$

$$\sup_{(x,z)\in K \subset\subset X\times Z} |\partial_\theta a_1 a_2| \le C(1+|\theta|)^{M_1 - 1}(1+|\sigma|)^{M_2}$$

(dep. on which is pos/neg etc) $\le C \begin{cases} (1+|\theta|)^{M_1+M_2-1} \sim (1+|\theta|+|\sigma|)^{M_1+M_2-1} \\ \text{or} \\ (1+|\sigma|)^{M_1+M_2-1} \sim (1+|\theta|+|\sigma|)^{M_1+M_2-1} \end{cases}$

since $|\sigma|$ & $|\theta|$ are comparable.

The other terms are similarly decreasing in homogeneity.

Now we want

$$\sup_{(x,z)\in K} |\partial (a(x,z,\tilde{\theta}))| \le C(1+|\tilde{\theta}|)^{M_1+M_2-1}$$

$$\text{is } C(1+|\theta|+|\sigma|+|\tilde{y}|)^{M_1+M_2-1}$$

$$\tilde{y} = y|(\theta,\sigma)|$$

$$|\tilde{y}| \le C|(\theta,\sigma)| \text{ on supp } a_1 a_2.$$

We are done since

$\Leftarrow$ $$(1+|\theta|+|\sigma|)^{\alpha} \le (1+|\theta|+|\sigma|+|\tilde{y}|)^{\alpha} \quad \alpha \ge 0$$

$$(1+|\theta|+|\sigma|)^{\alpha} \le C(1+|\theta|+|\sigma|+|\tilde{y}|)^{\alpha} \quad \alpha \le 0,\ \exists C.$$

<u>Conclusion</u>

$$|(\theta,\sigma)|^{-n_y} a(x,z,\tilde{\theta}) \in S^{M_1+M_2-n_y}(X\times Z\times \mathbb{R}^{N_1+N_2+n_y})$$

in the region where

$$C_2|\theta| \le |\sigma| \le C_1|\theta|, \quad C_1, C_2 > 0.$$

$$K_{A_1\circ A_2}(x,z) = \int e^{i\phi(x,z,\tilde{\theta})} a_1\left(x,\frac{\tilde{y}}{|(\theta,\sigma)|},\theta\right) a_2\left(\frac{\tilde{y}}{|(\theta,\sigma)|},z,\sigma\right)|(\theta,\sigma)|^{-n_y} d\tilde{y}\,d\theta\,d\sigma$$

Let $\chi_1 \in C^{\infty}(\mathbb{R}^{N_1}_{\theta} \times \mathbb{R}^{N_2}_{\sigma})$

$\chi_1(\theta,\sigma)$ hom. of deg. 0 in (θ,σ)

$$\chi_1(\theta,\sigma) = \begin{cases} 1 & \text{for } |\sigma| \le C_1|\theta| \\ 0 & \text{for } |\sigma| \ge 2C_1|\theta| \end{cases}$$

C_1 to be chosen.

Use $\tilde{a}_1 = \chi_1 a_1 a_2$

What about $(1-\chi_1)a_1 a_2$?

We have: if $(x,y;\xi,\eta) \in \Lambda'_{\phi_1}$ or $(y,z;\eta,\zeta) \in \Lambda'_{\phi_2}$

then $\eta \ne 0$

<u>recall</u> $\Lambda_{\phi_1} = \{(x,y,d_{x,y}\phi_1(x,y,\theta);\ d_\theta \phi_1(x,y,\theta)=0\}$

If $d_\theta \phi_1 = 0$, then $d_y \phi_1 \neq 0$ $\quad (\theta \neq 0)$
Then $d_{(y,\theta)}(\phi_1) \neq 0$. $(\theta \neq 0)$.

<u>Claim</u>: If $|\sigma| \leq \varepsilon|\theta|$, ε suff. small,
then $d_{y,\theta}\phi \neq 0$.

<u>Proof of claim</u>:

$$d_\theta(\phi_1) \neq 0 \Rightarrow d_\theta(\phi_1+\phi_2) = d_\theta(\phi) \neq 0.$$

$$d_y(\phi_1+\phi_2) = |\theta|\, d_y\phi_1(x,y,\tfrac{\theta}{|\theta|}) + |\sigma|\, d_y\phi_1(y,z,\tfrac{\sigma}{|\sigma|})$$

$$|d_y(\phi_1+\phi_2)| \geq C|\theta| - \tilde{C}|\sigma|$$
$$\geq C|\theta| - \varepsilon|\theta| = (C-\varepsilon)|\theta| \neq 0$$

so take $C-\varepsilon > 0$, $\varepsilon < C$.

Therefore, $\int e^{i(\phi_1+\phi_2)} \chi\, a_1 a_2\, d\theta\, d\sigma\, d\tilde{y} \in C^\infty(X \times Z)$

$$\chi = \begin{cases} 1 & |\sigma| \leq \tfrac{1}{2}\varepsilon|\theta| \\ 0 & |\sigma| \geq \varepsilon|\theta| \end{cases}$$

χ hom. of deg. 0, $\chi \in C^\infty(\mathbb{R}^{N_1}_\theta \times \mathbb{R}^{N_2}_\sigma)$.

(1/24) Integrating by parts in $\tilde{y}, \theta$ variables we can lower the order as much as we wish in <u>all</u> variables since

$$(1+|\theta|)^{-N} \leq C(1+|\theta|+|\sigma|)^{-N}, \quad N>0$$

(where $|\sigma| \leq \varepsilon|\theta|$).

Thus $\int e^{i(\phi_1+\phi_2)} |(\theta,\sigma)|^{-n_y} \chi a_1 a_2 \, d\theta \, d\sigma \, d\tilde{y} \in C^\infty(X \times Z)$

We have $d_{y,\sigma}(\phi_2) \neq 0$ if $\sigma \neq 0$ by (6)
If $d_\sigma \phi_2 = 0$ then $d_y \phi_2 \neq 0$ $(\sigma \neq 0)$
Thus we can find $\delta > 0$ s.t. if

$$|\theta| \leq \delta |\sigma|$$

$$d_{y,\sigma}(\phi_1+\phi_2) \neq 0$$

So let $\tilde{\chi} \in C^\infty(\mathbb{R}^{N_1}_\theta \times \mathbb{R}^{N_2}_\sigma)$ hom of deg 0 in (θ,σ)

so
$$\tilde{\chi} = \begin{cases} 1 & |\theta| \leq \frac{\delta}{2}|\sigma| \\ 0 & |\theta| \geq \delta|\sigma| \end{cases}$$

& then integrating by parts as before,

$$\int e^{i(\phi_1+\phi_2)} |(\theta,\sigma)|^{-n_y} \tilde{\chi} a_1 a_2 \, d\tilde{y} \, d\theta \, d\sigma \in C^\infty(X \times Z)$$

So far we have

$$K_{A_1 \circ A_2}(x,z) = \int e^{i\phi(x,z,\tilde{\theta})} |(\theta,\sigma)|^{-n_y} \tilde{a}_1 \tilde{a}_2(x,z,\tilde{\theta}) \, d\tilde{\theta}$$

with $\tilde{a}_1 \tilde{a}_2$ supported where $C_2|\theta| \leq |\sigma| \leq C_1|\theta|$
modulo a smooth function.

$$(\tilde{a}_1 = (1-\chi)a_1, \quad \tilde{a}_2 = (1-\tilde{\chi})a_2).$$

Conclusion: $b(x,z,\tilde{\theta}) = |(\theta,\sigma)|^{-n_y} \tilde{a}_1 \tilde{a}_2(x,z,\tilde{\theta})$
$$\in S^{M_1+M_2-n_y}(X \times Z \times \mathbb{R}^{N_1+N_2+n_y})$$

It remains to check that $\phi(x,z,\tilde{\theta})$ is a phase fc^{n} on supp b:

1) ϕ is $\mathbb{R}$-valued ✓
2) ϕ hom. of deg 1 in $\tilde{\theta}$ ✓
3) ϕ is smooth: $\phi \in C^\infty(X \times Z \times \mathbb{R}^N - 0)$

$$N = N_1 + N_2 + n_y$$

$$\phi(x,z,\tilde{\theta}) = \phi_1\left(x, \frac{\tilde{y}}{|\theta,\sigma|}, \theta\right) + \phi_2\left(\frac{\tilde{y}}{|\theta,\sigma|}, z, \sigma\right)$$

$$\phi_1 \in C^\infty(X \times Y \times \mathbb{R}^{N_1} \setminus \{0\})$$
$$\phi_2 \in C^\infty(Y \times Z \times \mathbb{R}^{N_2} \setminus \{0\})$$

What if $\theta \neq 0$, $\sigma = 0$? or $\theta = 0$, $\sigma \neq 0$?
But on supp b, these cannot occur.

4) $d\phi \neq 0$ on supp b
if $d_\theta \phi = 0$, then $d_\theta \phi_1 = 0$
$d_\sigma \phi = 0$, then $d_\sigma \phi_2 = 0$
$d_y \phi = 0 = d_y \phi_1 + d_y \phi_2$, then $d_y \phi_1 = -d_y \phi_2$.

$d\phi = 0 \Rightarrow d_\theta \phi_1 = 0$, $d_\sigma \phi_2 = 0$ & $d_y \phi_1 = -d_y \phi_2$.
Then $\quad d_x \phi_1 = 0 \quad d_z \phi_2 = 0$

$$(x,y;\xi,\eta) \in \Lambda'_{\phi_1} \ \& \ (y,z;\eta,\zeta) \in \Lambda'_{\phi_2}$$

with $\xi = 0$, $\zeta = 0$

since

$$\Lambda'_{\phi_1} = \{(x,y,\overset{=0}{d_x\phi_1}, \underline{-d_y\phi_1}); d_\theta\phi_1 = 0\}$$

$\Rightarrow \eta$ appears in both points

$$\Lambda'_{\phi_2} = \{(y,z,\underline{d_y\phi_2}, \underset{0}{\underset{\parallel}{-d_z\phi_2}}); d_\sigma\phi_2 = 0\}$$

But $\xi = 0$ & $\zeta = 0$ contradicts (c).

<u>Claim</u> $\Lambda'_\phi = \Lambda'_{\phi_1} \circ \Lambda'_{\phi_2}$

Proof// $K_{A_1 \circ A_2} = \int e^{i\phi(x,z,\tilde{\theta})} b(x,z,\tilde{\theta})\, d\tilde{\theta} \mod C^\infty(X,Z)$.

$$\Lambda'_\phi = \{(x,z; d_x\phi, -d_z\phi);\ d_{\tilde{\theta}}\phi = 0\}$$

$$\Lambda'_{\phi_1} = \{(x,y; d_x\phi_1, -d_y\phi_1);\ d_\theta\phi_1 = 0\}$$

$$\Lambda'_{\phi_2} = \{(y,z; d_y\phi_2, -d_z\phi_2);\ d_\sigma\phi_2 = 0\}.$$

$$\Lambda'_{\phi_1} \circ \Lambda'_{\phi_2} = \{(x,z,\xi,\zeta) \in X \times Z \times \mathbb{R}^{n_x + n_z} \setminus \{0\} \text{ st. } \exists (y,\eta) \text{ so that } (x,y;\xi,\eta) \in \Lambda'_{\phi_1},\ (y,z;\eta,\zeta) \in \Lambda'_{\phi_2}\}$$

But $d_x\phi = d_x\phi_1$,

we need $d_y\phi_1 = -d_y\phi_2$ $\quad d_{\theta_1}\phi_1 = 0$, $d_\sigma\phi_2 = 0$

have $d_z\phi = d_z\phi_2$

But $d_{\tilde{\theta}}\phi = 0 \Rightarrow d_{\tilde{y},\theta,\sigma}\phi = 0$

$\Rightarrow d_\theta\phi = 0$, $d_\sigma\phi = 0$, $d_{\tilde{y}}\phi = 0$

$d_{\tilde{y}}\phi = 0 \Rightarrow \frac{1}{|\theta,\sigma|} d_y\phi_1 + \frac{1}{|\theta,\sigma|} d_y\phi_2 = 0$

i.e. $d_y\phi_1 = -d_y\phi_2$ as needed.

So, $d_{\tilde{\theta}}\phi = 0 \Leftrightarrow d_\theta\phi_1 = 0$, $d_\sigma\phi_2 = 0$

$d_y\phi_1 = -d_y\phi_2$

Finally, we must check that $\phi(x,z,\tilde{\theta})$ is non-degenerate in some $\Gamma \subset X \times Z \times \mathbb{R}^N \setminus \{0\}$.
That is,

if $\quad C_\phi = \{(x,z,\tilde{\theta}) ; d_{\tilde{\theta}}\phi = 0\}$

then

$$\left\{ d\left(\frac{\partial \phi}{\partial \theta_1}\right), \dots, d\left(\frac{\partial \phi}{\partial \tilde{\theta}_N}\right) \right\}$$

are linearly independent on $C_\phi \cap \Gamma$.

(1/27)

$$\Lambda'_{\phi_1} \times \Lambda'_{\phi_2} \cap (T^*X \times \Delta_{T^*Y} \times T^*Z) \subset \{ d_y(\phi_1 + \phi_2) = 0, \ d_y\phi_1 = d_y\phi_2 = 0 \}$$

$$\subset \{ d_{\tilde{\theta}}\phi = 0 \} = C_\phi .$$

Recall (d) $\Rightarrow$ at points in the intersection,

$$\dim\left(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) + T(T^*X \times \Delta_{T^*Y} \times T^*Z)\right) = 2n_x + 4n_y + 2n_z$$

$$\dim(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2})) + \dim(T(T^*X \times \Delta_{T^*Y} \times T^*Z)) - \dim(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)) = 2n_x + 4n_y + 2n_z$$

i.e. $$\left((n_x + \cancel{n_y}) + (\cancel{n_y} + n_z)\right) + \cancel{2n_x} + \cancel{2n_y} + \cancel{2n_z} - \dim(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)) = \cancel{2n_x} + \cancel{4n_y} + \cancel{2n_z}$$

So (d) $\Rightarrow$ $$\dim T((\Lambda'_{\phi_1} \times \Lambda'_{\phi_2})) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)) = n_x + n_z .$$

Compute $\dim\left(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)\right)$

$$C_{\phi_1} \xrightarrow{F} \Lambda_{\phi_1}$$

$$(x, y, \theta) \longmapsto (x, y, d_x\phi_1, d_y\phi_1)$$

$$d_\theta \phi_1 = 0$$

Recall F is an immersion since ϕ_1 is non-deg.

$$T(\Lambda'_{\phi_1}) = \{ t = (\delta x, \delta y, d(d_x\phi_1)u, -d(d_y\phi_1)u);\ d(d_\theta\phi_1)u = 0 \quad u = (\delta x, \delta y, \delta\theta) \in C_{\phi_1} \}$$

$$T(\Lambda'_{\phi_2}) = \{ \tilde{t} = (\delta y, \delta z, d(d_y\phi_2)v, -d(d_z\phi_2)v);\ d(d_\sigma\phi_2)v = 0,\ v = (\delta y, \delta z, \delta\sigma) \}$$

$\therefore\ T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)$,

$$= \{ (t, \tilde{t});\ d(d_y\phi_2)v = -d(d_y\phi_1)u \quad d(d_\theta\phi_1)u = 0,\ d(d_\sigma\phi_2)v = 0 \}$$

<u>Claim</u> $\dim\left(T(\Lambda'_{\phi_1} \times \Lambda'_{\phi_2}) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)\right)$

$$= \dim\left(\ker d(d_{\tilde{\theta}}\phi)\right)$$

Assuming this, $\dim\left(\ker \underset{\uparrow}{d}(d_{\tilde{\theta}}\phi)\right) = n_x + n_z$.

$x, z, \tilde{\theta}$ $\qquad \tilde{\theta} = (\underset{\uparrow}{\tilde{y}}, \theta, \sigma)$

$n_y + N_1 + N_2$

$$\begin{array}{c c} & n_y + N_1 + N_2 \\ \begin{array}{c} n_x \\ n_z \\ n_y + N_1 + N_2 \end{array} & \left(\begin{array}{c} \\ \\ \\ \end{array} \right) \end{array} \leftarrow \text{matrix of } d(d_{\tilde{\theta}}\phi)$$

so $\dim(\operatorname{rank} d(d_{\tilde\theta}\phi)) = n_y + N_1 + N_2$

$\iff$ ϕ is non-degenerate on $C_\phi \cap \Gamma$ for some Γ

($C_\phi = \{x, d_{\tilde\theta}\phi = 0\}$ non-degenerate if $d\left(\frac{\partial\phi}{\partial\tilde\theta_j}\right)$ lin. ind. for $j = N_1 + N_2 + n_y$)

Proof of claim:

$d_{\tilde\theta}\phi = (d_y(\phi_1+\phi_2), d_\theta\phi_1, d_\sigma\phi_2)$

It is clear the spaces are in fact the same //

(end of proof of "Comp. of FIOP'S I")

(1/29)

<u>Example</u> $\varphi : X \to Y \quad X \subseteq \mathbb{R}^{n_x} \quad Y \subseteq \mathbb{R}^{n_y}$

X, Y open, φ smooth.

Consider $\varphi^* f(x) = f(\varphi(x))$, $f \in C_c^\infty(Y)$.

Claim: the map

$$f \xmapsto{\phi^*} \phi^* f$$

is a FIOp.

$$f(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\hat f(\xi)d\xi$$

$$\phi^* f(x) = \frac{1}{(2\pi)^n}\int e^{i\phi(x)\xi}\hat f(\xi)d\xi$$

$$= \frac{1}{(2\pi)^n}\int e^{i(\phi(x)-y)\cdot\xi} f(y)dy\,d\xi$$

$\phi(x,y,\xi) = (\phi(x) - y)\cdot\xi$

$d_y\varphi = -\xi \neq 0$ if $\xi \neq 0$

$C_\varphi = \{ y = \varphi(x) \}$

We need $\quad \frac{\partial \phi}{\partial \xi_i} = y_i - \frac{\partial \phi}{\partial x_i}$

$d_{x,y,\xi}\left(y_i - \frac{\partial \phi}{\partial x_i}\right)$ lin. ind. for $i = 1, \cdots, n$.

$d_x\left(y_i - \frac{\partial \phi}{\partial x_i}\right) = -d_x\left(\frac{\partial \phi}{\partial x_i}\right) = \left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right]$

$d_y\left(y_i - \frac{\partial \phi}{\partial x_i}\right) = \text{identity}$

$d_\xi\left(y_i - \frac{\partial \phi}{\partial x_i}\right) = 0.$ $\quad$ this assures rank n.

Remark (under cond^{n}s (a), ..., (d))

$$\Lambda'_\phi = \Lambda'_{\phi_1} \circ \Lambda'_{\phi_2}$$

We would like to know when $A_1 \circ A_2 \in \Psi^m(X)$.

Then we need $\Lambda'_\phi = \{ (x, y; \xi, \eta) : y = x, \eta = \xi \}$.

i.e. $\Lambda'_{\phi_1} \circ \Lambda'_{\phi_2} = \{ (x, x; \xi, \xi), \xi \neq 0 \}$.

One situation: $A_1 = A_2^t$ since ...

$K_{A_1}(x,y) = \int e^{i\phi_1(x,y,\xi)} a_1(x,y,\xi) d\xi.$

$K_{A^t}(y,x) = \int e^{i\phi_1(x,y,-\xi)} a(x,y,-\xi) d\xi$

put $\phi_1^t(y, x, \xi) = \phi_1(x, y, -\xi)$

$\Lambda_{\phi_1^t} \overset{\text{claim}}{=} \{ (y, x; \eta, \xi) \mid (x, y, \xi, \eta) \in \Lambda'_{\phi_1} \} = I$

Then

$$\Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t} = \{ (x, z; \xi, \zeta) \mid \exists (y, \eta) \text{ with } (x, y, \xi, \eta) \in \Lambda'_{\phi_1} \text{ \& } (y, z; \eta, \zeta) \in \Lambda'_{\phi_1 t} \}$$

i.e. $(z, y; \zeta, \eta) \in \Lambda'_{\phi_1}$

If $\Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t} = I$, then $(z, \zeta) = (x, \xi)$

<u>Claim</u> $\Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t} = I$

given $(x, y; \xi, \eta) \in \Lambda'_{\phi_1}$,

then $(y, x, \eta, \xi) \in \Lambda'_{\phi_1 t}$

so $(x, x; \xi, \xi) \in \Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t}$.

Conclusion, $\Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t} \supseteq I$

But if the comp^n makes sense, then

$$\Lambda'_{\phi_1} \circ \Lambda'_{\phi_1 t} = I$$

by dimension count.

But we need to satisfy (a) ... (d).

In particular,

$$\Lambda'_{\phi_1} \times (\Lambda'_{\phi_1 t}) \text{ intersects } T^*X \times \Delta_{T^*(Y)} \times T^*(X)$$

transversally.

When is this true?

Λ'_{ϕ_1} has dim. $2n$

$\Lambda'_{\phi_1 t}$ dim $2n$.

So we need dim^n of intersection to be 0.

This is true in the following case:

$$\Lambda'_{\phi_1} = \{ (x, \xi; \chi(x, \xi)) \}$$

where $\chi : T^*X \to T^*X$ is a local diffeo,

$(y, \eta) = \chi(x, \xi)$

In this case Λ'_ϕ, will be called a <u>local</u> <u>canonical</u> <u>graph</u>.

<u>Example</u> Generalized Radon Transform

$$Rf(s,\omega) = \int \delta(s - \phi(x,\omega))\, a(x,\omega) f(x)\, dx.$$

$$R : C_0^\infty(\mathbb{R}^n) \to C^\infty(\underbrace{\mathbb{R} \times S^{n-1}})$$

we need def^{n}s for manifolds.

Under cond^{n}s imposed on $\phi(x,\omega)$, R will be a FIOp & the transversality cond^n is satisfied for R & R^t i.e.

$$R^t R \in I^m(\mathbb{R}^n)$$

for some m.

We need to extend all concepts to manifolds, & define the principal symbol of a Fourier Integral distribution.

WFu, $u \in \mathcal{D}'(X)$, $X \subseteq \mathbb{R}^n$

$$\widehat{\phi u}(t\xi) = O(t^{-N}) \text{ if } (x_0, \xi_0) \notin WFu$$

$\phi(x_0) \neq 0$,

ξ close to ξ_0.

$$= (\phi u)(e^{-i\langle \cdot, t\xi \rangle})$$

$d_x \langle x, \xi \rangle = \xi$.

We'll consider
$(\varphi u)(e^{-it\psi(x,a)})$ as $t \to \infty$ for certain $\psi \in C^\infty(X \times A)$
$X \subseteq \mathbb{R}^n$, X open, $A \subseteq \mathbb{R}^p$, A open.

<u>Theorem</u> Let $(x_0, \xi_0) \in T^*X \setminus \{0\}$. Then
$(x_0, \xi_0) \notin WFu \iff \exists\ \mathcal{U}$ nbhd of x_0, (V open nbhd of ξ_0), A_0 nbhd of a_0 s.t.

$$d_x \psi(x_0, a_0) = \xi_0 \neq 0$$

and

$$\varphi u(e^{-it\psi(\cdot,a)}) = O(t^{-N}) \quad \forall N$$

uniformly for ~~$x \in \mathcal{U}$~~, $a \in A_0$.

(1/31) <u>Example</u> $\psi(x,\xi) = \langle x, \xi \rangle$.

<u>Proof</u>: Let $\alpha \in C_0^\infty(\mathcal{U})$. $\alpha = 1$ on supp φ
Then

$$(\varphi u)(e^{-it\psi(\cdot,a)}) = (\alpha\varphi u)(e^{-it\psi(\cdot,a)})$$

$$= (\varphi u)(\alpha e^{-it\psi(\cdot,a)})$$

$$= (\varphi u)^\wedge (\alpha e^{-it\psi(\cdot,a)})^\vee$$

$$= \int (\varphi u)^\wedge(\xi)\, (\alpha e^{-it\psi(\cdot,a)})^\vee(\xi)\, d\xi$$

$$= \int (\varphi u)^\wedge(\xi) \frac{1}{(2\pi)^n} \int e^{i(-t\psi(\cdot,a) + x\cdot\xi)} \alpha(x)\, dx\, d\xi$$

$$= \frac{1}{(2\pi)^n} \iint e^{i(\langle x,\xi\rangle - t\psi(x,a))} (\varphi u)^\wedge(\xi)\, \alpha(x)\, d\xi\, dx$$

$\xi \to t\xi \quad = \frac{t^{-n}}{(2\pi)^n}\iint e^{it(x\cdot\xi - \psi(x,a))}$
$$(\varphi u)^\wedge(t\xi)\,\alpha(x)\,dx\,d\xi.$$

Since $(x_0,\xi_0)\notin WFu$, $(\varphi u)^\wedge(t\xi)$ is rapidly dec. if $\varphi\in C_0^\infty(\mathcal{U})$ & $\xi\in V$ = nbhd of ξ_0.

$$= \frac{t^{-n}}{(2\pi)^n}\left(\iint_V + \iint_{V^c}\right)(\text{———})\,dx\,d\xi.$$

On V^c we'll get $O(t^{-N})$ $\forall N$ if (eg)
$$d_x(\langle x,\xi\rangle - \psi(x,a)) \neq 0$$
for $x\in\mathcal{U}$, $a\in A$ for some $\mathcal{U}, A$, $\xi\in V^c$.
i.e. $\quad \xi \neq d_x(\psi(x,a))$
But
$$d_x(\psi(x_0,a_0)) = \xi_0 \notin V^c$$
so we can choose $\tilde{\mathcal{U}}$ a nbhd of x_0 & A a nbhd of a_0 st.
$$|d_x\psi(x,a) - \xi| \geq \delta > 0 \quad \forall x\in\tilde{\mathcal{U}},\ a\in A,\ \xi\in V^c.$$

Spose A is a FIOp.
$$Af = \int e^{i\phi(x,y,\theta)}a(x,y,\theta)\,f(y)\,dy\,d\theta.$$

Compute $\sigma(A)$? We want this to make sense on manifolds.
We know
$$WF'A \subseteq \Lambda_\phi \text{, a Lagrangian manifold.}$$

We know

$$(\varphi K_A)(e^{-it\psi(\cdot,\cdot,a)}) = O(t^{-N}) \quad \forall N$$

if

$$d_{x,y}\psi(x,y,a) \notin \Lambda'_\phi$$

Can one compute

(*) $(\varphi K_A)(e^{-it\psi(\cdot,\cdot,a)})$

as a function of t, when

$$d_{x,y}\psi(x,y,a) \in \Lambda'_\phi\ ?$$

$$(*) = \int e^{i(\phi(x,y,\theta) - t\psi(x,y,a))} a(x,y,\theta)\varphi(x,y)\,dx\,dy\,d\theta$$

$$\theta \to t\theta$$

$$= t^{-N}\int e^{it(\phi(x,y,\theta) - \psi(x,y,a))} a(x,y,t\theta)\varphi(x,y)\,dx\,dy\,d\theta.$$

We need the critical points of

$$k(x,y,\theta,a) = \phi(x,y,\theta) - \psi(x,y,a)$$

to be non-degenerate w.r.t. x,y,θ (to apply st. phase?).

In terms of A itself,

$$(\varphi A)(e^{-it\psi(\cdot,a)})$$

$$= t^{-N}\int e^{it(\phi(x,y,\theta) - \psi(x,y,a))} a(x,y,\theta)\varphi(y)\,dy\,d\theta.$$

We need non-degenerate critical points in y & θ.

<u>Theorem</u> The following are equivalent:

① $\phi(x,\theta) - \psi(x,a)$ $\quad a \in \mathbb{R}^p, x \in \mathbb{R}^n$
has non-deg. crit. $\quad \psi \in C^\infty(X \times \mathbb{R}^p)$ $\mathbb{R}$-valued.
points as a fc^n of (x,θ), at (x_0, θ_0, a_0)
i.e. $$d_{x,\theta}(\phi(x,\theta) - \psi(x,a))\Big|_{(x_0,\theta_0,a_0)} = 0$$
& $d(d_{x,\theta}(\phi(x,\theta) - \psi(x,a)))$ is non-singular.

② (a) ϕ is a non-degenerate phase function in a conic nbhd of (x_0, θ_0).
(b) $(x, d_x\psi)$ intersects Λ_ϕ transversally at $(x_0, \xi_0) = (x_0, d_x\psi(x_0, a_0))$.
(recall $(x; d_x\psi)$ is a Lagrangian manifold).

Proof ① ⇒ ②: since non-singular,

if $$\begin{cases} d_x(d_x(\phi-\psi))\delta x + d_\theta(d_x\phi)\delta\theta = 0 \\ d_\theta(d_x\phi)\delta x + d_\theta(d_\theta\phi)\delta_\theta = 0 \end{cases}$$

then $\delta x = 0$ & $\delta\theta = 0$

Take $\delta x = 0$. Then
$$d_\theta(d_x\phi)\,\delta\theta = 0 \ \& \ d_\theta(d_\theta\phi)\delta_\theta = 0$$
$$\Longrightarrow$$
$$\delta\theta = 0.$$

$\left(C_\phi = \{(x,\theta), d_\theta \phi = 0\} \right)$

So we have $d_{x,\theta}\left(\frac{\partial \phi}{\partial \theta_i}\right)$ are lin. ind.

i.e. ① ⇒ ② (a).

Next, $C_\phi \xrightarrow{F} \Lambda_\phi$

$(x,\theta) \longmapsto (x, d_x\phi)$

Tangent space of Λ_ϕ at $(x_0, d_x\phi(x_0,\theta_0))$ with $d_\theta \phi(x_0,\theta_0) = 0$

$$\text{is } \{ (\delta x, d_x(d_x\phi)\delta x + d_\theta(d_x\phi)\delta_\theta); \; d_x(d_\theta\phi)\delta x + d_\theta(d_\theta\phi)\delta_\theta = 0 \}$$

(2/5)

(a) says $\left.\begin{array}{l} d_x(d_\theta\phi)\delta\theta = 0 \\ \& \; d_\theta(d_\theta\phi)\delta\theta = 0 \end{array}\right\} \Rightarrow \delta\theta = 0.$

(b) says:

Tangent space to Λ_ϕ at (x_0, ξ_0)

$$T_\phi = \{ (\delta x, d_x(d_x\phi)\delta x + d_\theta(d_x\phi)\delta_\theta); \; d_x(d_\theta\phi)\delta_x + d_\theta(d_\theta\phi)\delta_\theta = 0 \}$$

Tangent space to $\{(x, d_x\psi)\}$ at (x_0,ξ_0)

$$T_\psi := \{ (\delta x, d_x(d_x\psi)\delta x) \}$$

$T_\phi \cap T_\psi = \{0\}$. (each is of dim^{n} n).

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① ⇒ (b): $d_x(d_x\phi)\delta_x + d_\theta(d_x\phi)\delta\theta$

$= d_x(d_x\psi)\delta_x$

with $d_x(d_\theta\phi)\delta_x + d_\theta(d_\theta\phi)\delta_\theta = 0$ $\curvearrowleft$ similarly

<u>Moral</u> To find $\psi(x,a)$ we just need to check that $(x, d_x\psi)$ is transversal to Λ_ϕ at (x_0,ξ_0).

<u>Corollary</u> $\psi(x,\xi) = \langle x, \xi \rangle$.

$$u = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$$

ϕ non-deg. in a conic nbhd of (x_0, θ_0)

$d_x\phi(x_0,\theta_0) = \xi_0 \neq 0$.

Then transversal intersection in this case is

$$T_{(x_0,\xi_0)}\{(x,d_x\psi)\} \pitchfork \Lambda_\phi$$

transversal to

$\Leftrightarrow$ Λ_ϕ transversal to fiber.

<u>Theorem</u> Let A be a FIOp in some cone

$$Af(x) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta) f(y)\, dy\, d\theta$$

Let $\psi \in C^\infty(X \times \mathbb{R}^p_\sigma)$

Assume non-degenerate critical point at $(x_0, \theta_0, \sigma_0)$

i.e. $d_{x,\theta}(\phi - \psi)(x_0,\theta_0,\sigma_0) = 0$

& Hessian non-singular at (x_0,θ_0,σ_0).

Then $\exists$ $\mathcal{U}$ nbhd of x_0, A nbhd of σ_0 and Γ a conic nbhd of $(x_0,\theta_0) \in X \times \mathbb{R}^N\setminus\{0\}$ such that if $a \in S^{-\infty}$ outside Γ and $\varphi \in C_0^\infty(\mathcal{U})$, then

$$A(e^{it\Psi(\cdot,\sigma)}\varphi) \sim e^{it\Psi(x(\sigma),\sigma)} t^{\frac{1}{2}(N-n)} (2\pi)^{\frac{1}{2}(N+n)}$$
$$|\det Q(\sigma)|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q(\sigma)}$$
$$\sum_{k=0}^{\infty} \frac{1}{k!}(R(\sigma))^k g(y,\sigma,t)\Big|_{y=0} t^{-k}$$

uniformly in A

where $(x(\sigma), \theta(\sigma))$ is such that

$$d_{x,\theta}(\phi - \Psi)(x(\sigma),\theta(\sigma),\sigma) = 0$$
$$Q(\sigma) = d^2_{x\theta}(\phi - \Psi)(x(\sigma),\theta(\sigma),\sigma)$$

$$g(y(x,\theta,\sigma),\sigma,t)\,|\det d_{x,\theta}\, y(x,\theta,\sigma)| = a(x,t\theta)\varphi(x)$$

where $(x,y) \longrightarrow y(x,\theta,\sigma)$ is a diffeomorphism of a nbhd of (x_0,θ_0) to a nbhd of 0, with

$$y(x(\sigma),\theta(\sigma),\sigma) = 0$$

&

$$d_{x,\theta}\, y(x(\sigma),\theta(\sigma),\sigma) = I$$

$$R(\sigma) = \frac{1}{2} i \left\langle Q^{-1}(x)\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle.$$

n

<u>Corollary</u> $e^{-it\Psi(x(\sigma),\sigma)} A(e^{it\Psi(\Psi(\cdot,\sigma)}\varphi)$

(*) $$\underset{t\to\infty}{\sim} t^{\frac{1}{2}(N-n)}(2\pi)^{\frac{1}{2}(N+n)}|\det Q(\sigma)|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} Q(\sigma)}$$
$$* a(x(\sigma), t\theta(\sigma))\varphi(x)$$
$$+ O(t^{\frac{1}{2}(N-n)-1+m}) \qquad (a \in S^m)$$

RHS depends on a, φ, $Q(\tau) = d^2_{x,\theta}(\phi - \psi)\Big|_{(x(\tau), \theta(\tau), \tau)}$

<u>Remark</u> $\phi(x(\tau), \theta(\tau)) = 0$

$d_\theta \phi(x(\tau), \theta(\tau)) = 0$

& ϕ hom. of deg. 1

so by Euler's Formula,

$$\phi(x, \theta) = \sum_{i=1}^{N} \theta_i \frac{\partial \phi}{\partial \theta_i}$$

<u>Goal</u>: To make invariant sense of RHS of $(*)$

(2/7) Show that given $\Lambda \subseteq T^*X \setminus \{0\}$, Λ closed Lag. & $(x_0, \xi_0) \in \Lambda$. Then $\exists\, \phi(x, \theta)$ non-degenerate in a conic neighborhood of (x_0, θ_0), $d_x \phi(x_0, \theta_0) = \xi_0$ such that $\Lambda = \Lambda_\phi$ in a conic neighborhood of (x_0, ξ_0).

We first prove this in a special case.

$$\Lambda \xrightarrow{\pi} \mathbb{R}^n \setminus \{0\}$$

$$(x, \xi) \longmapsto \xi.$$

assume & π is a local diffeo. near (x_0, ξ_0)
(i.e. Λ is transversal to horizontal lagrangian).

<u>Proposition</u> $X \subseteq \mathbb{R}^n$ open, $\Lambda \subset T^*X \setminus \{0\}$ Lagrangian. Then Λ is conic $\iff$ $\alpha = \sum_{i=1}^{n} \xi_i dx_i = 0$ on Λ.

<u>Proof</u>: $V \subset T^*X \setminus \{0\}$ a submanifold
conic $\iff Y = \sum \xi_i \partial_{\xi_i}$ tangent to V

since $\gamma(t) = (x, t\xi) \in V \quad \forall t$

$$\frac{d\gamma}{dt}\Big|_{t=0} = \sum \xi_i \frac{\partial}{\partial \xi_i}$$

Let $t \in T_{(x,\xi)}(\Lambda)$

$$d\alpha(Y,t) = \sum d\xi_i \wedge dx_i(Y,t)$$

$$= \sum d\xi_i(Y)\, dx_i(t) - d\xi_i(t)\, \cancel{dx_i(Y)}^{\,0}$$

$$= \sum \xi_i\, dx_i(t) = \alpha(t)$$

Λ conic, Lagrangian $\Rightarrow d\alpha(Y,t) = 0 \Rightarrow \alpha(t) = 0$

Conversely, $\alpha(t) = 0 \quad \forall\, t \in T_{(x,\xi)}(\Lambda)$

$\Rightarrow d\alpha(Y,t) = 0 \quad \forall\, t \in T_{(x,\xi)}\Lambda$

$\Rightarrow Y \in T_{(x,\xi)}(\Lambda)$ since Λ is Lag. $\therefore$ maximal space where $d\alpha$ vanishes.

$\Rightarrow Y$ conic. $\square$

<u>Proposition</u> Let $\Lambda \subset T^*X \setminus \{0\}$ conic Lagrangian, Λ, $(x_0, \xi_0) \in \Lambda$.
Assume $\Lambda \xrightarrow{\pi} \mathbb{R}^n$, $(x,\xi) \longmapsto \xi$ is a local diffeo at (x_0,ξ_0)
Then $\exists\, \phi(x,\theta)$, a non-degenerate phase function in a conic nbhd of (x_0, θ_0), $d_\theta \phi(x_0,\theta_0) = 0$, $d_x \phi(x_0, \theta_0) = \xi_0$ such that

$$\Lambda = \Lambda_\phi \quad \text{near } (x_0, \xi_0)$$

<u>Proof</u> Locally, by inverse fc^n th^m,

$$\Lambda = \{(h_1(\xi), \dots, h_n(\xi); \xi)\} \quad \text{i.e. } x_i = h_i(\xi)$$

Try as a phase function

$$\phi(x,\theta) = \langle x, \theta \rangle - \sum h_i(\theta)\theta_i$$

(h_i hom. of deg 0 since Λ conic $\Rightarrow$ $h = \sum h_i(\theta)\theta_i$ hom. of deg 1 $\Rightarrow$ $h = \sum \theta_i \frac{\partial h}{\partial \theta_i}(\theta)$ $\Rightarrow$)

To check, since Λ is conic-Lag., $h_i(\theta) = \frac{\partial h}{\partial \theta_i}(\theta)$ with h homog. of deg. 1 in θ.

$$C_\phi = \{(x,\theta);\ x_i - h_i(\theta) - \sum \frac{\partial h_k(\theta)}{\partial \theta_i}\theta_k = 0\}$$

$$\Lambda_\phi = \{(x, d_x\phi(x,\theta));\ (x,\theta) \in C_\phi\} = \{(x,\theta) : \in C_\phi\}$$

To check that $\Lambda_\phi = \Lambda$, we need $\sum \frac{\partial h_k(\theta)}{\partial \theta_i}\theta_k = 0\ \forall i$. $(*)$

$$\Lambda \xrightarrow{\pi} \mathbb{R}^n_\theta \setminus \{0\}$$
$$(x,\xi) \longmapsto \xi = \theta$$

$\alpha = 0$ on Λ

know: $\alpha((\pi^{-1})_* \partial_{\theta_i}) = 0$

Calc $(\pi^{-1})_* \partial_{\theta_i} = \partial_{\xi_i} + \sum_{k=1}^{n} \frac{\partial h_k(\xi)}{\partial \xi_i}\frac{\partial}{\partial x_k}$

$$\alpha\left(\frac{\partial}{\partial \xi_i} + \sum \frac{\partial h_k(\xi)}{\partial \xi_i}\frac{\partial}{\partial x_k}\right) = \sum \xi_\ell\, dx_\ell\left(\partial_{\xi_i} + \sum \frac{\partial h_k}{\partial \xi_i}\partial_{x_k}\right)$$

$$= \sum_{k,\ell} \xi_\ell \frac{\partial h_k}{\partial \xi_i}(\xi)\,\delta_{k\ell} = \sum \xi_k \frac{\partial h_k(\xi)}{\partial \xi_i}$$

$$= 0.$$

which is $(*)$ as we needed. $\square$

Corollary: Under the cond^{n}s of the prop^{n},

$$\phi(x,\xi) = \langle x,\xi\rangle - h(\xi)$$

$$h(\xi) = \sum_{i=1}^{n} h_i(\xi)\xi_i$$

$\square$

Next time: Λ Lagrangian satisfying cond^{n}s of prop^{n}.

$(x_0,\xi_0)\in\Lambda$

$\Lambda_\phi = \Lambda_{\tilde\phi} = \Lambda$ near (x_0,ξ_0) $\quad \phi(x,\theta)$ $\tilde\phi(x,\tilde\theta)$

$I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta \quad a\in S^M(X\times\mathbb{R}^N)$

(more assumpt^{n}s on a)

then $\exists\ \tilde a \in S^{\tilde M}(X\times\mathbb{R}^{\tilde N})$ so that

$I_{a,\phi} = I_{\tilde a,\tilde\phi}$ mod. C^∞.

(2/12) Theorem Let $\phi(x,\theta)$, $x\in X\subseteq\mathbb{R}^n$ open, $\theta\in\mathbb{R}^N$ be a phase function, non-degenerate in a conic nbhd of $(x_0,\theta_0)\in X\times\mathbb{R}^N$. ~~Let $a\in S^{m+(\frac{n-2N}{4})}(X\times\mathbb{R}^N)$. Then $\exists\ \mathcal{U}$, a conic nbhd of (x_0,θ_0) such that if supp $a\subseteq\mathcal{U}$, $\exists$ a conic nbhd~~

Assume $\Lambda_\phi = \Lambda_{\tilde\phi}$ in a conic nbhd of (x_0,ξ_0);

$(x_0,\xi_0) = (x_0, d_x\phi(x_0,\theta_0))$, $d_\theta\phi(x_0,\theta_0) = 0$,

where $\tilde\phi \in C^\infty(X\times\mathbb{R}^{\tilde N}\setminus\{0\})$ is a non-degenerate phase function in a conic nbhd of $(x_0,\tilde\theta_0)$,

$(x_0,\xi_0) = (x_0, d_x\tilde\phi(x_0,\tilde\theta_0))$, $d_{\tilde\theta}\tilde\phi(x_0,\tilde\theta_0) = 0$.

Then $\exists\ \mathcal{U}$, a conic nbhd of (x_0,θ_0) s.t. if $a\in S^{M+(\frac{n-2N}{4})}(X\times\mathbb{R}^N)$, supp $a\subseteq\mathcal{U}$, $\exists$ conic nbhd $\tilde V$ of $(x_0,\tilde\theta_0)$ and $\tilde a\in S^{M+(\frac{n-2\tilde N}{4})}(X\times\mathbb{R}^{\tilde N})$, supp $\tilde a\subseteq\tilde V$ s.t.

$$I_{a,\phi} = I_{\tilde a,\tilde\phi} \mod C^\infty.$$

We'll prove this under the assumption that

$$\begin{aligned} \Lambda_\phi &\longrightarrow \mathbb{R}^n\setminus\{0\} \quad (*) \\ (x,\xi) &\longmapsto \xi \end{aligned}$$

is a local diffeo. near (x_0,ξ_0)

Proof: Write $\Lambda = \Lambda_\phi = \Lambda_{\tilde\phi}$. Recall that

$$\Lambda = \Lambda_\psi \quad \text{where} \quad \psi = \langle x,\xi\rangle - h(\xi).$$

Also, from (*),

$$\phi(x,\theta) - \langle x,\xi\rangle \quad (\text{\& cor. p70})$$

is a non-degenerate phase function (as a fc^n of (x,θ); i.e. non-degenerate critical points).

Consider

$$\widehat{I_{a,\phi}}(t\xi) = \int e^{i(\phi(x,\theta) - t\langle x,\xi\rangle)} a(x,\theta)\,d\theta\,dx$$

(WLOG, a compactly sptd in x).

$$= t^N \int e^{it(\phi(x,\theta) - \langle x,\xi\rangle)} a(x,t\theta)\,d\theta\,dx.$$

(apply stationary phase in (x,θ))

$$= t^{\frac{1}{2}(N-n)} (2\pi)^{\frac{1}{2}(n+N)} e^{it\overbrace{\langle x(\xi),\xi\rangle}^{=h(\xi)}} |\det \phi''(x(\xi),\theta(\xi))|^{-1/2}$$

$$e^{\frac{\pi i}{4}\operatorname{sgn}(\phi''(x(\xi),\theta(\xi)))} a(x(\xi), t\theta(\xi))$$

$+$ lower order terms (in t).

$$\Lambda_\phi = \{(x, d_x\phi(x,\theta)) : d_\theta\phi = 0\}$$

$$\parallel$$

$$\Lambda_\psi = \{(h'(\xi),\xi)\} \quad \text{but} \quad h(\xi) = \sum \xi_i h_i(\xi)$$

$$= \{(h_1(\xi),\dots,h_n(\xi),\xi)\}.$$

Write $\omega = \xi/|\xi|$, $t = |\xi|$.

$$\widehat{I_{a,\phi}}(\xi) = |\xi|^{1/2(N-n)} (2\pi)^{1/2(N+n)} e^{i\overbrace{h(\xi)}^{=|\xi|h\left(\frac{\xi}{|\xi|}\right)}} a\Big(\overbrace{\cancel{x\left(\tfrac{\xi}{|\xi|}\right)}}^{x(\xi)}, \overbrace{\cancel{|\xi|\theta\left(\tfrac{\xi}{|\xi|}\right)}}^{\theta(\xi)}\Big)$$

$$|\det\phi''\Big(\underbrace{x\left(\tfrac{\xi}{|\xi|}\right)}_{x(\xi)}, \theta\left(\tfrac{\xi}{|\xi|}\right)\Big)|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sgn}\left(\cancel{\phi''(x(\frac{\xi}{|\xi|})}^{\,x(\xi)}, \theta\left(\frac{\xi}{|\xi|}\right)\right)}$$

$+$ lower order

but $|\xi|\theta\left(\frac{\xi}{|\xi|}\right) = \theta(\xi)$, $\quad x\left(\frac{\xi}{|\xi|}\right) = x(\xi)$.

Order of $\widehat{I_{a,\phi}}(\xi)$ in ξ $= \frac{1}{2}(N-n) + \text{order } a + -\frac{1}{2}(n-N)$

[claim: $|\det \phi''(x,\theta)|$ hom of deg. $n-N$

since $d^2_\theta \phi$ hom. of deg. -1 (N of these)

$d^2_{x\theta}\phi$ 0

$d^2_x \phi$ ... ... 1 (n of these)

]

(*) Order of $e^{-ih(\xi)}\widehat{I_{a,\phi}}(\xi) = \frac{1}{2}(N-n) + m + \frac{n-2N}{4}$

$= m - \frac{n}{4}$.

So $e^{-ih(\xi)}\widehat{I_{a,\phi}}(\xi) = b(x,\xi)$ + lower order

/// $e^{-ih(\xi)}\widehat{I_{\tilde{a},\tilde{\phi}}}(\xi) = \tilde{b}(x,\xi)$ + lower order.

So we must choose $\tilde{a}$ so that

$b(x,\xi)$ + lower $= \tilde{b}(x,\xi)$ + lower.

(2/14)

(*) $|\xi|^{\frac{N-n}{2}}\left|\det \phi''\left(x(\xi), \theta\left(\frac{\xi}{|\xi|}\right)\right)\right|^{-1/2} = \left|\det \phi''(x(\xi), \theta(\xi))\right|^{-1/2}$

Claim $e^{-ih(\xi)}\widehat{I_{a,\phi}}(\xi) = b(\xi) \in S^{m-n/4}(X\times\mathbb{R}^n)$

&

$$b(\xi) = C\left|\det \phi''(x(\xi),\theta(\xi))\right|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn}\phi''\left(x(\xi),\theta\left(\frac{\xi}{|\xi|}\right)\right)}$$
$$a(x(\xi),\theta(\xi)) + S^{m-\frac{n}{4}-1}(X\times\mathbb{R}^n).$$

We have $\tilde{\phi}(x,\tilde{\theta})$ with $\Lambda_\phi = \Lambda_{\tilde{\phi}}$. We want to find $\tilde{a} \in S^{\tilde{m}}(X \times \mathbb{R}^{\tilde{N}})$ so that

$$I_{a,\phi} = I_{\tilde{a},\tilde{\phi}} \quad \text{mod} \quad C^\infty$$

We know
$$e^{-ih(\xi)}\widehat{I_{\tilde{a},\tilde{\phi}}}(\xi) = \tilde{C}\,\tilde{a}(x(\xi),\tilde{\theta}(\xi))\,|\det \tilde{\phi}''(x(\xi)\tilde{\theta}(\xi))|^{-1/2}$$
$$e^{\frac{\pi i}{4}\operatorname{sgn}\tilde{\phi}''(x(\xi),\tilde{\theta}(\xi/|\xi|))} + S^{\tilde{m}-\frac{n-1}{4}}(X\times\mathbb{R}^n)$$

Write $\tilde{a} \sim \sum_{j\le 0}' \tilde{a}_j$, $\tilde{a}_j \in S^{\tilde{m}-j}(X\times\mathbb{R}^n)$.

Put $\tilde{a}_0(x(\xi),\tilde{\theta}(\xi)) = \dfrac{C}{\tilde{C}}\, b(\xi)\, e^{-\frac{\pi i}{4}\operatorname{sgn}(\tilde{\phi}''(x(\xi),\tilde{\theta}(\xi/|\xi|))}\,|\det(--)|^{1/2}$

$b(\xi) \in S^{m-n/4}$ $\qquad$ $|\det(--)|^{1/2}$: hom deg $\dfrac{n-\tilde{N}}{2}$

$$\Rightarrow \tilde{a}_0(x(\xi),\tilde{\theta}(\xi)) \in S^{m+\frac{n}{4}-\frac{\tilde{N}}{2}}$$

So far only defined @ critical pts. $\in S^{m+\frac{n-2\tilde{N}}{4}}$

i.e. $\tilde{m} = m + \left(\frac{n-2\tilde{N}}{4}\right)$

Extend $\tilde{a}_0(x,\tilde{\theta}) \in S^{\tilde{m}}$ any way at all.
Continue inductively....
We get
$$\cancel{e^{-ih(\xi)}}\widehat{I_{\tilde{a},\tilde{\phi}}}(\xi) = \cancel{e^{-ih(\xi)}}\widehat{I_{a,\tilde{\phi}}}(\xi) \quad \text{mod } S^{-\infty}$$

i.e. $\quad I_{\tilde{a},\tilde{\phi}} = I_{a,\phi}$ mod C^∞. ∎

So far we have been assuming
$$\Lambda_\phi = \Lambda_{\tilde{\phi}} \longrightarrow X \text{ projects as}$$
a local diffeo near (x_0,ξ_0).
Now we must show we can find coords (y,η) so that in these coords, Λ is transversal

to $\eta = \eta_0$.

Symplectic Geometry.

Def^n Let V be a finite dim^{n}al vector space. A symplectic form on V is a bilinear, anti-symmetric non-degenerate form on V

$$\sigma : V \times V \to \mathbb{R}$$

(V, σ) is called a symplectic vector space.

Def^n Let X be a (Paracompact) C^∞ manifold. A symplectic form on X is a C^∞ non-degenerate closed 2-form ω so that

$$\omega\big|_{T_x(X)} \text{ is a symplectic form.}$$

(X, ω) = Symplectic Manifold

Eg $\mathbb{R}^{2n}$, $\omega = \sum dy_i \wedge dx_i$

Eg X a smooth manifold, T^*X is a symplectic manifold
(when $X = \mathbb{R}^n$, $\omega = d\alpha$, $\alpha = \sum_{i=1}^n y_i dx_i$)

We define the canonical 1-form α on T^*X.
If $t \in T_{(x,\xi)}(T^*X)$,

$$\alpha(t) = \xi(\pi_* t) \qquad \pi : T^*X \to X \text{ proj}^n.$$

Then put $\omega = d\alpha$; ω is a symplectic form

In local coords (x,ξ),

$$t = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial \xi_i}$$

$$\pi_*(t) = \sum a_i \frac{\partial}{\partial x_i}$$

Then $\xi(\pi_* t) = \sum a_i \xi_i \quad (\xi = \xi_i dx_i)$

$$= \sum \xi_i dx_i(t) = \alpha(t)$$

<u>Def^n</u> Let X be C^∞ Manifold, $(x,\xi) \in T^*X$
Then the <u>vertical space</u> at (x,ξ) is

$$V_{(x,\xi)} = \{ t \in T_{(x,\xi)}(T^*X) : \pi_* t = 0 \}$$

In local coords, (x,ξ)

$V_{(x,\xi)}$ is spanned by $\frac{\partial}{\partial \xi_i}$

The <u>horizontal space</u> (defined only in local coords) is given by

$$H_{(x,\xi)} = \operatorname{span}\left(\frac{\partial}{\partial x_i}\right).$$

<u>Def^n</u> (1) Let $(\cancel{X},\omega)$ be a symplectic manifold of dim $2n$.
Λ be a submanifold of X.
Then Λ is Lagrangian if
(a) $\dim \Lambda = n$
(b) ω vanishes on Λ

(2) $Y \subset X$, Y a submanifold. Y is called <u>isotropic</u> if ω vanishes on Y.

(2/19)

Proposition. Let X be a C^∞-manifold, θ a one form on X.

$$\tilde{\theta} : X \to T^*X$$
$$x \mapsto (x, \sum \theta_i dx^i) = (x, \theta(x))$$
$$N = \tilde{\theta}(X).$$

Then: (1) N is always transversal to $V_{(x,\xi)}$ $\forall (x,\xi) \in N$

(2) N is Lagrangian $\Leftrightarrow$ $\theta = d\varphi$ for some φ Locally.

Proof. (1) Let (x_i) be local coords in x & induced coords in T^*X (x,ξ).

Compute $\tilde{\theta}_*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + \sum \frac{\partial \theta_j}{\partial x_i} \frac{\partial}{\partial \xi_j}$

So tangent space to N is spaned by
$$\{ \frac{\partial}{\partial x_i} + \sum \frac{\partial \theta_j}{\partial x_i} \frac{\partial}{\partial \xi_j} \}$$

which is clearly transversal to $V_{x,\xi} = \langle \frac{\partial}{\partial \xi_j} \rangle$.

(2) $\omega = d\alpha$, $\alpha_{(x,\xi)}(t) = \xi(\pi_* t)$, $t \in T_{(x,\xi)}(T^*X)$.

$$d\alpha(\theta_* \tfrac{\partial}{\partial x_i}, \theta_* \tfrac{\partial}{\partial x_j}) = d\alpha(\partial_{x_i} + \sum \tfrac{\partial \theta_k}{\partial x_i} \partial_{\xi_k}, \partial_{x_j} + \sum \tfrac{\partial \theta_k}{\partial x_j} \partial_{\xi_k})$$
$(d\alpha = d\xi_i \wedge dx^i)$
$$= \pm \left(\frac{\partial \theta_j}{\partial x_i} - \frac{\partial \theta_i}{\partial x_j} \right)$$

So N Lagrangian $\Leftrightarrow \frac{\partial \theta_j}{\partial x_i} = \frac{\partial \theta_i}{\partial x_j} \Leftrightarrow \theta$ locally exact. $\square$

Aside: $\Delta u - qu = 0$, $\mathcal{C}_q = \{ (u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega}) \} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Define ω on $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$

$$\omega((f,g),(\tilde{f},\tilde{g})) = \int_{\partial\Omega} (f\tilde{g} - \tilde{f}g)\, ds$$

C_q is Lagrangian in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$

$$\int_{\partial\Omega} \left(f \frac{\partial u}{\partial \nu} - g \frac{\partial}{\partial \nu}\right) dS = 0$$

Then (!) C_q is maximal isotropic.

<u>Gunther's Conjecture</u>:

$R \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$

Then $R = C_q \ \exists q \iff R$ is Lagrangian.

<u>Prop^n</u>. Let L be a Lagrangian subspace of $T_{(x_0,\xi_0)}(T^*X)$, transversal to $V_{(x_0,\xi_0)}$. Then $\exists$ a Lagrangian manifold N through (x_0,ξ_0) so that $T_{(x,\xi)} N = L$.

Proof, We want $T_{(x,\xi)} N$ to be ~~of the~~ spanned by

$$\frac{\partial}{\partial x_i} + \sum a_{ki} \frac{\partial}{\partial \xi_k}$$

N is Lagrangian $\iff a_{ki} = a_{ik}$.

Let $g \in C^\infty(X)$ be s.t. $dg(x) = \xi$ and

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = a_{ij} \text{ at } (x_0).$$

(L is spanned by $\left.\frac{\partial}{\partial x_i}\right|_{(x_0,\xi_0)} + \sum a_{ij}(x_0) \left.\frac{\partial}{\partial \xi_j}\right|_{(x_0,\xi_0)}$

with $a_{ij}(x_0) = a_{ji}(x_0)$)

Then $N = \{(x, dg(x))\}$.

$\square$

<u>Exercise</u>: Let V be a symplectic vector space and L_0, L_1 Lagrangian subspaces of V. Then $\exists$ L Lagrangian, L transversal to L_0 & L_1

<u>Theorem</u>. Let Λ be a Lagrangian manifold of $T^*X \setminus \{0\}$. $(x_0, \xi_0) \in \Lambda$. We can find coords (x, ξ) near (x_0, ξ_0) so that

$$\begin{aligned} \Lambda &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ (x, \xi) &\longrightarrow \xi \end{aligned}$$

is a local diffeo.

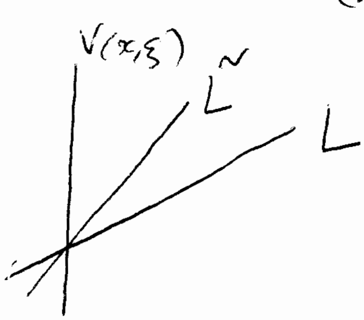
<u>Corollary</u> (a) Λ_ϕ, ϕ non-deg. phase fc^{n}. Then $\exists$ coords $w(x)$ so that locally $\Lambda_{\tilde\phi} = \Lambda_\psi$ where
$\psi(x, \theta) = \langle w(x), \theta \rangle - h(\theta)$
h homogeneous of degree 1.

(b) End of proof of theorem that says that if $\Lambda_\phi = \Lambda_{\tilde\phi}$, then

$$I_{a,\phi} = I_{\tilde a, \tilde\phi} \ \text{mod. } C^\infty.$$

Proof,, (of Th^{m})

Aim: Choose coords so that Λ is transversal to $H_{(x,\xi)}$:
$L = T_{(x_0,\xi_0)}(\Lambda)$ is Lagrangian.
Let $\tilde L$ be a Lagrangian subspace transversal to both $V_{(x_0, \xi_0)}$ & L



So we find coords so that $\tilde L = H(x,\xi)$ as follows, then
By previous prop^{n} $\exists$ Lagrangian manifold N so that
$$T_{(x,\xi)} N = \tilde L$$

and $N = \{(x, dg(x))\}$.

So, put coords:

$$\omega_1(x) = g(x)$$

& complete to a coord. system.

$\tilde{L}$ is spanned by $\frac{\partial}{\partial x_i} + \sum \underbrace{\frac{\partial^2 g}{\partial x_i \partial x_j}}_{\text{in } \omega \text{ coord's } = 0} \frac{\partial}{\partial \xi_j}$

So $\tilde{L} = H_{(x,\xi)}$ in these coords.

$\square$

(3/21)

<u>Toward Darboux's Th^m</u> $((M,\omega)$ symplectic $\Rightarrow$ locally $M = T^*X)$

Let (X,ω) be a symplectic manifold, $f : X \to \mathbb{R}$ C^∞

<u>Def^n</u>

$$\omega_x(H_f, t) = -df(t), \quad t \in T_x(X)$$

i.e.

$$H_f \lrcorner \omega = -df. \quad \left(\begin{array}{ll} X = T^*\mathbb{R}^n & \omega = \sum d\xi_i \wedge dx_i \\ H_f = \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i} \end{array} \right)$$

<u>Prop^n</u> $\mathcal{L}_{H_f}\omega = 0$

Proof// $\mathcal{L}_{H_f}\omega = \frac{d}{dt}\Big|_{t=0} \left(\exp(tH_f)^* \omega \right)$, $\mathcal{L}_v \omega = [v, w]$.

$\cancel{\text{Let } \phi_t = \exp(tH_f),\ \phi_t : X \to X}$

$\cancel{\frac{d}{dt}\phi_t^*\omega = \frac{d}{dt}\phi_t^*(\phi_s^*)\omega\big|_{s=0}}$

$\cancel{= \frac{d}{dt}\phi_{t+s}^*\omega\big|_{s=0}}$

$\cancel{= \frac{d}{ds}\phi_{t+s}^*\omega\big|_{s=0}}$

Recall $\mathcal{L}_{H_f}\omega = H_f \lrcorner \underset{=0}{d\omega} + d(\underset{=d(-df)}{H_f \lrcorner \omega})$

so

$= 0.$ $\square$

<u>Prop</u>

Prop$^{\underline{n}}$ Let $\phi_t = \exp tH_f$, $\phi_t : X \to X$

$\phi_t^* \omega = \omega \quad \forall t$

Proof//

$$\frac{d}{dt}\phi_t^*\omega = \frac{d}{dt}\phi_t^*\phi_s^*\omega\Big/_{s=0} = \frac{d}{dt}\phi_{t+s}^*\omega\Big/_{s=0}$$

$$= \phi_t^*\left(\frac{d}{ds}\phi_s^*\omega\right)\Big/_{s=0} = \phi_t^*\mathcal{L}_{H_f}\omega = 0$$

$$\Rightarrow \phi_t^*\omega = \phi_t^*\omega\Big/_{t=0} = \omega.$$

$\square$

Prop$^{\underline{n}}$ X, C^∞-manifold, $\Sigma \subset T^*X \setminus \{0\}$, isotropic wrt ω (standard symplectic form), i.e. $\omega = 0$. $\dim \Sigma = n-1$

Let $(x_0, \xi_0) = \rho_0 \in \Sigma$

Let $p : T^*X \setminus \{0\} \to \mathbb{R}$, $p \in C^\infty$

Let $H = \{(x,\xi) \in T^*X \setminus \{0\} : p = 0\}$.

Assume $\Sigma \subset H$, and that

$H_p(\rho_0) \notin T_{\rho_0}\Sigma$

Then in a nbhd of ρ_0 $\exists$ Lagrangian manifold Λ such that $\Sigma \subset \Lambda \subset H$

Proof// ω vanishes on Σ: $\omega(t,\tilde{t}) = 0$, $t, \tilde{t} \in T(\Sigma)$

Put $\Lambda = \exp(tH_p)(\Sigma)$, $|t|$ small, local to ρ_0.

Then Λ is an n-dim$^{\underline{n}}$al manifold. We check it is Lagrangian. Let $\rho \in \Sigma$:

$T_\rho(\Lambda) = T_\rho(\Sigma) \oplus \mathbb{R} \cdot H_p(\rho)$

so $t \in T_\rho(\Lambda)$ can be written

$$t = t_\Sigma + \alpha H_p(\rho)$$

/// $$\tilde{t} = \tilde{t}_\Sigma + \tilde{\alpha} H_p(\rho).$$

We know $\omega(t_\Sigma, \tilde{t}_\Sigma) = 0$

& $\omega(H_p, H_p) = 0$

Check: $$\underbrace{\omega(t_\Sigma, H_p)} = 0$$

$$= -dp(t_\Sigma) = 0 \quad \text{since } t_\Sigma \in T(H)$$

where $p \equiv 0$.

Now let $\rho_t = \exp(t H_p)(\rho)$, $\rho \in \Sigma$.

$$t, \tilde{t} \in T_{\rho_t}(\Lambda).$$

Then $t = (\exp t H_p)_*(t_\Sigma)$, $\tilde{t} = (\exp t H_p)_*(\tilde{t}_\Sigma)$.

& $t_\Sigma, \tilde{t}_\Sigma \in T_\rho(\Lambda)$, $\rho \in \Sigma$

$$\omega_{\rho(t)}(t, \tilde{t}) = \omega_\rho(t_\Sigma, \tilde{t}_\Sigma) = 0.$$

$\square$

<u>Remark</u> Suppose $\Sigma \subset H = \{p_1 = p_2 = \cdots = p_k = 0\}$, Σ isotropic.

To generate Λ as in the prop$^{\underline{n}}$, we need

$$H_{p_i} p_j = 0 \quad \text{on } H$$

& dp_i lin. ind. on Σ

<u>Def</u>

Def^n $\omega(H_f, H_g) \triangleq \{f,g\} = H_f g$ "Poisson Bracket."

Prop^n $H_{\{f,g\}} = [H_f, H_g]$

Proof, Check $[H_f, H_g] \lrcorner\, \omega = -d\{f,g\} = \omega(H_{\{f,g\}}, \cdot)$

$$\mathcal{L}_{H_f}(H_g \lrcorner\, \omega) = \mathcal{L}_{H_f} H_g \lrcorner\, \omega + H_g \lrcorner\, \underbrace{\mathcal{L}_{H_f}\omega}_{=0}.$$

$$= [H_f, H_g] \lrcorner\, \omega$$

On the other hand

$$\mathcal{L}_{H_f}(H_g \lrcorner\, \omega) = \mathcal{L}_{H_f}(-dg)$$

$$= -\big(H_f \lrcorner\, \underbrace{d(dg)}_{=0} + d(H_f \lrcorner\, dg)\big)$$

$$= -d\{f,g\}$$

So we have

$$\omega([H_f, H_g], \cdot) = \omega(H_{\{f,g\}}, \cdot)$$

$\square$

Def^n (X,ω) symplectic of dim^n $2n$. Suppose X is given locally by $(p_1, \cdots, p_n, q_1, \cdots, q_n)$.
Then (p_i, q_i) are called <u>symplectic</u> <u>coordinates</u> if

$$\omega = \sum dq_i \wedge dp_i$$

Prop^n (Jacobi Identity)

$$\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$$

Proof. $-\{f,\{g,h\}\} = \{\{g,h\},f\} = H_{\{g,h\}}f = [H_g, H_h]f$

$$= H_g(H_h f) - H_h(H_g f) = \{g\{h,f\}\} - \{h,\{g,f\}\}$$

$$= \{g,\{h,f\}\} + \{h,\{f,g\}\}.$$ $\square$

Prop$^{\underline{n}}$ $(q_1,\dots,q_n,p_1,\dots,p_n)$ local coords on (X,ω) are symplectic

$\iff$ $\{p_i,q_j\} = \delta_{ij}$, $\{p_i,p_j\}=0$, $\{q_i,q_j\}=0$

Proof (by def$^{\underline{n}}$). $\square$

Theorem (Darboux) Let (X,ω) be symplectic, dim$^{\underline{n}}$ $2n$. Then near every pt. $x_0 \in X$ we can find symplectic coords $(q_1,\dots,q_n,p_1,\dots,p_n)$.

Proof Let q_1 be any coordinate with $dq_1(x) \neq 0$ near x_0, $q_1(x_0)=0$. We want p_1 with dp_1, dq_1 lin. ind. near x_0 so that $\{p_1,q_1\}=1$ i.e.

$H_{p_1}q_1 = 1$, or $-H_{q_1}p_1 = 1$

Define p_1 solving $H_{q_1}p_1 = -1$: Let $\tilde{p}_1$ be such that $H_{\tilde{p}_1}$ is transversal to $\{q_1=0\}$,

solve $H_{\tilde{p}_1}\tilde{q}_1 = +1$, $\tilde{q}_1 = 0$ on $q_1=0$

Take coordinates $(\tilde{q}_1,\tilde{p}_1)$

~~Remark: $\Sigma_1 = \{\tilde{p}_1 = \tilde{q}_1 = 0\}$ is a symplectic manifold with $\omega_1 = d\tilde{p}_1 \wedge d\tilde{q}_1$.~~

Continuing inductively, we need $\{q_2,\tilde{q}_1\} = \{q_2,\tilde{p}_1\} = \{p_2,\tilde{q}_1\} = 0$

& $\{q_2,p_2\} = -1$.

(2/26).

Theorem (Darboux) (X,ω) symplectic, dim^n $2n$.
Near every $x_0 \in X$ we can find symplectic coords
$(q_1,\dots,q_n,p_1,\dots,p_n)$.

Proof//

Choose $q_1(x_0)=0$, $dq_1(x_0)\neq 0$, $q_1\in C^\infty(X)$.
Choose Σ transversal to H_{q_1} at (& hence near) x_0.
Solve $H_{q_1}p_1=-1$, $p_1|_\Sigma$ arbitrary, $p_1(x_0)=0$.
so $\{p_1,q_1\}=1$.

Remark Let $F_1=\{p_1=q_1=0\}$, of dim^n $2n-2$.
Then $\omega|_{F_1}$ is symplectic//
Now H_{p_1}, H_{q_1} commute ($[H_{p_1},H_{q_1}]=H_{\{p_1,q_1\}}=H_1=0$)
and H_{p_1}, H_{q_1} are transversal to F_1, so solve

$$H_{p_1}q_2=0$$
$$H_{q_1}q_2=0$$
$$q_2|_{F_1} \text{ arbitrary w/ } dq_2(x_0)\neq 0,\ q_2(x_0)=0$$

Check: dq_1, dq_2, dp_1 are lin. ind. at x_0
Next choose p_2:

$$H_{q_2}p_2=-1 \qquad p_2|_{\Sigma_2} \text{ arbitrary w/}$$
$$H_{q_1}p_2=0 \qquad dp_2(x_0)\neq 0,\ p_2(x_0)=0$$
$$H_{p_1}p_2=0$$

All three of $H_{q_1}, H_{q_2}, H_{p_1}$ commute, so by Frob.
choose Σ_2 transversal to all three at x_0
etc.

$\square$

Remark $\{p_i\}_{i\in I}$, $\{q_j\}_{j\in J}$, $|I|+|J|\le 2n$
$\{p_i,p_j\}=0$ $i,j\in I$ $\{q_i,q_j\}=0$ $i,j\in J$ $|I|\le n$, $|J|\le n$.

$\{p_i, q_j\} = \delta_{ij} \quad i \in I,\ j \in J.$

dp_i, dq_j lin. ind. at $x_0 \in X$.

Then we can always find symplectic coords $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ that complete the basis //

<u>Remark</u> T^*X, X C^∞ manifold. $\rho_0 = (x_0, \xi_0) \in T^*X \setminus \{0\}$

Given $\{p_i\}_{i \in I}$, $\{q_j\}_{j \in J}$, $|I| \le n$, $|J| \le n$. cond^{n}s as above,

p_i homogeneous of degree 1

q_j homogeneous of degree 0

Then we can find symplectic coords completing $\{p_i, q_j\}$ maintaining the homogeneity cond^{n}s if

$$H_{p_i},\ H_{q_j},\ \sum \xi_i \frac{\partial}{\partial \xi_i}$$

are linearly ind. at x_0, $i, j \in I \cap J$.

Same proof as before, solving ODE's with initial data on <u>conic</u> submanifolds //

<u>Def^n</u> (X, ω_X), (Y, ω_Y) symplectic, dim^n $2n$

$$\phi : (X, \omega_X) \longrightarrow (Y, \omega_Y)$$

is called a <u>canonical</u> transformation (a symplectomorphism) if

$$\phi^* \omega_Y = \omega_X$$

<u>Example</u> (a) $\phi_t = \exp(t H_f) \quad f : X \to \mathbb{R}$

$$\phi_t^* \omega = \omega \quad \text{on } (X, \omega),\ \forall t.$$

(b) $\phi : X \to X$ diffeo

$\Phi : T^*X \to T^*X \quad (x, \xi) \longmapsto (\phi(x), ({}^tD\phi(x))^{-1} \xi)$.

Propⁿ ϕ a canonical transⁿ $\Rightarrow$ ϕ local diffeo.

Proof// (Y,ω), symplectic coords $(q_1,\dots,q_n,p_1,\dots,p_n)$.

$\phi: X \to Y$, then

$$(\phi^* q_1,\dots,\phi^* q_n,\phi^* p_1,\dots,\phi^* p_n)$$

are sym. coords on X.

$$\omega_X = \sum_{i=1}^n d\phi^* p_i \wedge d\phi^* q_i$$

$$\omega_Y = \sum dp_i \wedge dq_i$$

Now

$$(\phi^* \omega_Y)^n = \omega_X^n$$

i.e. $(\pm)\, n!\,(\text{Jac.}\,\phi)\, dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2 \wedge \cdots \wedge dp_n \wedge dq_n$

$= (\pm)\, n!\, dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n$.

$\Longrightarrow$ $\text{Jac.}\,\phi = 1$

$\square$

Propⁿ. $\phi: Y \to X$ canonical. (Y,ω_Y) (X,ω_X)

Graph of $\phi = G_\phi = \{(\phi(y), y) \subseteq X \times Y\}$

Then G_ϕ is a Lagrangian submanifold of $X \times Y$ with symplectic form given by

$$\omega_X - \omega_Y;$$

more precisely, $X \times Y \xrightarrow{\pi_X} X$

$X \times Y \xrightarrow{\pi_Y} Y$

$$(\pi_X)^* \omega_X - (\pi_Y)^* \omega_Y.$$

Proof// obvious.

$\square$

Eg: $\phi: T^*Y \to T^*X$ canonical transⁿ.

On T^*Y, $\omega_Y = \sum d\eta_i \wedge dy_i$

T^*X, $\omega_X = \sum d\xi_i \wedge dx_i$.

$(\phi(y,\eta),(y,\eta))$ is Lagrangian w/ symplectic form $\sum d\eta_i \wedge dy_i - \sum d\xi_i \wedge dx_i$

Def^{n} (X,ω_X) (Y,ω_Y) symplectic, dim^{n} $2n$.
Then a Lag. manifold in

$$(X\times Y,\ \omega_Y-\omega_X)$$

is called a <u>canonical</u> <u>relation</u>

<u>Recall</u> Λ Lagrangian in T^*X.

if $\pi: T^*X \to \mathbb{R}^n$

$(x,\xi)\mapsto \xi$ diffeomorphism

$\Rightarrow$ $\Lambda=\Lambda_\phi$ with $\phi=\langle x,\xi\rangle-h(\xi)$

$$\Lambda_\phi = \{(x=h'(\xi),\ \xi)\} = \{(h'(\xi),\xi)\}$$

and

$$h(\xi)=\sum_{i=1}^{n} h_i(\xi)\xi_i \qquad =\{(h_1(\xi),\dots,h_n(\xi),\xi)\}.$$

In <u>general</u> we can make a change of var's $w(x)$ so that

$$\Lambda=\Lambda_\phi,\quad \phi(x,\xi)=\langle w(x),\xi\rangle-h(\xi)$$

&

$$\Lambda_\phi=\{(w(x)=h'(\xi),\ w'(x)\xi)\}$$

(2/28)

Let $\phi: T^*(\mathbb{R}^n)\longrightarrow T^*(\mathbb{R}^n)$ a canonical trans^{n}.

$$(y,\eta)\longmapsto(x,\xi)=\phi(y,\eta)$$

$G_\phi=\{(\phi(y,\eta),(y,\eta))\ ;\ \eta\neq0\}$ is Lagrangian w.r.t.

$$\sum d\xi_i\wedge dx_i-\sum d\eta_i\wedge dy_i$$

$$\phi(y_0,\eta_0)=(x_0,\xi_0)$$

Let $\tilde\Lambda_{x_0}=(x_0,\xi)$ - Lagrangian manifold wrt. $\sum d\xi_i \wedge dx_i$. Then $\phi^{-1}(\tilde{\Lambda}_{x_0}) = \Lambda$ is also Lagrangian

$$\Lambda = \{(y,\eta) \in T^*(\mathbb{R}^n)\setminus\{0\} : \phi(y,\eta) = (x_0,\xi)\ \exists \xi\}.$$

We can change variables in y so that

$$\Lambda \longrightarrow \mathbb{R}^n\setminus\{0\}$$
$$(y,\eta) \longmapsto \eta \qquad \text{(see p77)}.$$

is a local difeo near η_0.

Then $$G_\phi \longrightarrow \mathbb{R}^n_x \times \mathbb{R}^n_\eta \setminus\{0\}$$
$$(x,\xi;y,\eta) \longmapsto (x,\eta)$$

is a local diffeo under the induced c-of-vars in (y,η). So by imp. fc^n theorem.

$$G_\phi = \{(x, \xi(x,\eta), y(x,\eta), \eta)\}$$

Lagrangian wrt $\sum d\xi_i \wedge dx_i - \sum d\eta_i \wedge dy_i$

Using same method as in a previous th^m we can find $S(x,\eta)$ smooth in $\mathbb{R}^n \times \mathbb{R}^n \setminus\{0\}$, hom. of degree 1 in η so that

$$\xi(x,\eta) = \frac{\partial S}{\partial x}(x,\eta)$$

$$y(x,\eta) = \frac{\partial S}{\partial \eta}(x,\eta)$$

i.e. $$\phi : \left(x, \frac{\partial S}{\partial x}(x,\eta)\right) \longmapsto \left(\frac{\partial S}{\partial \eta}(x,\eta), \eta\right)$$

We have $\det \frac{\partial^2 S}{\partial x \partial \eta} \neq 0$; in other words

$$G_\phi = \Lambda'_\psi \quad \text{with}$$
$$\psi(x,y,\eta) = S(x,\eta) - y\cdot\eta$$

check: $d_\eta \psi = 0 \Leftrightarrow y = \frac{\partial S}{\partial \eta}$

$$d_x\psi = d_x S, \quad d_y \psi = -\eta$$

Defⁿ Let $\phi : T^*\mathbb{R}^n \setminus \{0\} \to T^*\mathbb{R}^n \setminus \{0\}$ be a canonical transⁿ. Then $S(x,\eta)$ is called the generating function of ϕ if under a change of variables in y & the induced c. of vars in (y,η)

$$G_\phi = \{(x, \frac{\partial S}{\partial x}(x,\eta), \frac{\partial S}{\partial \eta}(x,\eta), \eta)\}$$

$$= \Lambda'_\psi, \quad \psi = S(x,\eta) - y\cdot\eta.$$

Example Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeo.
Then $\tilde{f} : T^*\mathbb{R}^n \setminus \{0\} \to T^*\mathbb{R}^n \setminus \{0\}$

$$(y,\eta) \longmapsto (f(y), ({}^tDf_x)^{-1}\eta)$$

is a canonical transⁿ.

Claim $G_{\tilde{f}} = \Lambda'_\psi$

$$\psi(x,y,\eta) = f^{-1}(x)\cdot\eta - y\cdot\eta$$

Pf.

$$d_\eta\psi = f^{-1}(x) - y = 0 \iff y = f^{-1}(x)$$

$$d_x\psi = d_x f^{-1}\eta, \quad d_y\psi = -\eta.$$

Egorov's Theorem

say $a \in S^0$

$$Af(x) = \int e^{i(S(x,\eta) - y\cdot\eta)} a(x,y,\eta) f(y)\,dy\,d\eta$$

$$WF'A \subseteq \left\{\left(x, \frac{\partial S}{\partial \eta}, \frac{\partial S}{\partial x}, \eta\right)\right\}$$

Suppose $a(x_0, y_0, \eta_0) \neq 0$.
Check: A is microlocally invertible; i.e.
$\exists\ B$ s.t. $(x_0, \xi_0) \notin WF'(BA - I)$
$(x_0, \xi_0) \notin WF'(AB - I)$
Let $P \in \Psi^m(\mathbb{R}^n)$, we'll show
$APA^{-1} \in \Psi^m(\mathbb{R}^n)$
and $\sigma_m(APA^{-1}) = \sigma_m(P) \circ \phi$
ϕ the canonical trans$^{\underline{n}}$

$$\left(x, \frac{\partial S}{\partial \eta}\right) \longrightarrow \left(\frac{\partial S}{\partial x}, \eta\right)$$

Let $\phi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be the canonical trans$^{\underline{n}}$ whose generating function is $S(x,\eta)$
$\left(\det \frac{\partial^2 S}{\partial x \partial \eta} \neq 0\right)$

$$\phi(y_0, \eta_0) = (x_0, \xi_0)$$

$$Af(x) = \int e^{i(S(x,\eta) - y\cdot\eta)} a(x,y,\eta) f(y)\,dy\,d\eta$$

assume $a \in S^0(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^{2n}_\eta)$
a supported in a small conic neighborhood of (x_0, y_0, η_0).
Assume $a(x_0, y_0, \eta_0) \neq 0$.

(mimicing transpose).

Let

$$Bg(y) = \int e^{i(\langle y,\eta\rangle - S(x,\eta))} b(y,x,\eta)\, g(x)\, dx\, d\eta$$

$$b(y_0,x_0,\eta_0) \neq 0, \quad b_0 \in S^0(\mathbb{R}^n_y \times \mathbb{R}^n_x \times \mathbb{R}^{2n})$$

then

$$AB\,g(x) = \int e^{i(S(x,\eta) - \langle y,\eta\rangle)} a(x,y,\eta) \int e^{i(\langle y,\theta\rangle - S(z,\theta))} b(y,z,\theta)\, g(z)\, dz\, d\theta\, dy\, d\eta$$

$$= \int e^{i(\underbrace{S(x,\eta) - S(z,\theta) + \langle y,\theta\rangle - \langle y,\eta\rangle}_{\tau})} a\cdot b\cdot g(z)\, dy\, d\eta\, dz\, d\theta.$$

(3/3) Stationary phase in y,θ

Where are the crit. pts of $\tau(x,y,z,\eta,\theta)$ in z,θ ?

$$d_\theta(\tau) = 0 \iff y = \frac{\partial S}{\partial \theta}$$

$$d_y(\tau) = 0 \iff \eta = \theta$$

Check Hess. non-deg.

$$\begin{array}{c|cc} d_y^2 & 0 & I \\ d_\theta^2 & I & -d_\theta^2 S \end{array}$$

then $AB\,g(x) = \displaystyle\int e^{i(S(x,\eta) - S(z,\eta))} \, a\,b\, g(z)\, dz\, d\eta$

?

$$S(x,\theta) - S(z,\theta) = (x-z)\cdot \rho(x,z,\theta)$$

$$\rho(x,x,\theta) = d_x S(x,\theta)$$

$$\Rightarrow D_\theta \rho(x,x,\theta) = \left(\frac{\partial^2 S}{\partial x\, \partial \theta}(x,\theta)\right)$$

so $ABf(x) = \int e^{i(x-z)\cdot \rho(x,z,\theta)} c\, g(z)dzd\theta$

Put $\xi = \rho(x,z,\theta)$. $\quad c \in S^0(\mathbb{R}^n_x \times \mathbb{R}^n_z \times \mathbb{R}^n_\theta)$

$$c(x,z,\theta) = a(x,z,\theta)b(y,z,\theta), \quad y = \frac{\partial S}{\partial \theta}(z,\theta)$$

modulo lower order terms.

Then

$$ABf(x) = \int e^{i(x-z)\cdot\xi} \tilde{c}(x,z,\xi) g(z)dzd\xi$$

$\tilde{c} \in S^0$

$$\tilde{c} = ab\Big|_{y = \frac{\partial S}{\partial \theta}(z,\xi)} \cdot \det\left(\frac{\partial^2 S}{\partial x \partial \theta}\right)^{(?)}$$

i.e.

$AB \in \Psi^0(\mathbb{R}^n)$.

$\sigma_0(AB) = ab\Big|_{y = \frac{\partial S}{\partial \theta}(z,\xi)} \cdot \alpha \quad (\alpha \neq 0)$

mod. lower order.

$\sigma_0(AB)(x_0, \xi_0) \neq 0$

<u>Proposition</u> Let A be as above, then $\exists\, R$ so that

$(x_0,\xi_0) \notin WF'(AR - I)$

$(y_0,\eta_0) \notin WF'(RA - I)$

Moreover, R is of the form of B above.

<u>Proof</u> We must find $R = B$.

$ab\Big|_{y = \frac{\partial S}{\partial \theta}} \cdot \alpha = 1 \quad$ near (x_0, x_0, ξ_0)

so this gives the principal symbol of b.

Continue recursively.

∎

Def$^{\underline{n}}$ Such an R is called a <u>microlocal parametrix</u> for A at (x_0, ξ_0).

— " —

$P \in \Psi^m(\mathbb{R}^n)$

A as before

R as in prop$^{\underline{n}}$. (we'll write $R = A^{-1}$).

What is $A P A^{-1}$?

We'll show $\tilde{P} = A P A^{-1} \in \Psi^m(\mathbb{R}^n)$ & compute $\sigma_M(\tilde{P})(x,\xi)$

$$Pf(x) = \int e^{i(x-y)\cdot\xi} \underset{(\in S^m)}{p(x,y,\xi)} f(y)\, dy\, d\xi.$$

$$\sigma_m(P)(x,\xi) = p(x,x,\xi) + \text{lower order.}$$

$$PA^{-1}f(x) = \int e^{i(x-y)\cdot\xi} p(x,y,\xi) A^{-1}f(y)\, dy\, d\xi$$

$$= \int e^{i[\underbrace{(x-y)\cdot\xi + y\cdot\eta - S(z,\eta)}_{\uparrow\, \tau}]} p(x,y,\xi)\, b(y,z,\eta) f(z)\, dy\, d\xi\, dz\, d\eta.$$

do St. phase in y & η — Crit. pts of phase are $y = \frac{\partial S}{\partial \eta}(z,\eta)$

$$d_\eta \tau = 0 \iff y = \frac{\partial S}{\partial \eta}(z,\eta)$$

$$d_y \tau = 0 \iff \eta = \xi.$$

$$= \int e^{i(x\cdot\xi - S(z,\xi))} c(x,z,\xi) f(z)\, dz\, d\xi$$

$$c(x,z,\xi) \in S^m$$

&

$$c(x,z,\xi) = p\cdot b \Big|_{y = \frac{\partial S}{\partial \eta},\ \xi = \eta}.$$

but PA^{-1} is of the form of B as before, & we know

$$ABf(x) = \int e^{i(x-z)\cdot\xi} d(x,z,\xi)\, dz\, d\xi$$

for some d.

(3/5)

Now $APA^{-1}f(x)$

$$= \int e^{i(S(x,\eta) - y\cdot\eta + y\cdot\xi - S(z,\xi))} c(y,z,\xi)\, a(x,y,\eta) f(z)\, dz\, d\eta\, d\xi\, dy$$

Do stationary phase in y & η, critical pts of phase are $\eta = \xi$ & $y = \frac{\partial S}{\partial \eta}(x,\eta)$

(not d above)

$$= \int e^{i(S(x,\xi) - S(z,\xi))} d(x,z,\xi) f(z)\, dz\, d\xi$$

$$\& \quad d(x,z,\xi) = a.c \Big/_{\eta=\xi,\ y = \frac{\partial S}{\partial \eta}(x,\eta)}$$

Use Taylor series:

$$S(x,\xi) - S(z,\xi) = (x-z)\cdot\rho(x,z,\xi)$$

$$\& \quad \rho(x,x,\xi) = \nabla_x S(x,\xi)$$

$$\nabla_\xi \rho = \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi)$$

Then $APA^{-1}f(x)$ $\quad (\tilde{\eta} = \rho)$

$$= \int e^{i(x-z)\cdot\tilde{\eta}} \underbrace{l(x,z,\tilde{\eta})}_{(\in S^m)} f(z)\, dz\, d\tilde{\eta}$$

$$l(x,z,\tilde{\eta}) = d(x,z,\underbrace{\xi}_{\rho^{-1}(\tilde{\eta})}) \left|\frac{\partial \tilde{\eta}}{\partial \xi}\right|$$

Conclusion

$$APA^{-1} \in \Psi^m(\mathbb{R}^n)$$

and

$$\sigma_m(APA^{-1})(x,\xi) = \ell(x, x, \rho^{-1}(\xi)) \quad \text{mod } S^{m-1}$$

$$= d(x, x, \rho^{-1}(\xi)) / \left|\frac{\partial \tilde{\eta}}{\partial \xi}\right|(x, x, \rho^{-1}(\xi))$$

$$= a(x, y, \rho^{-1}(\xi))\, c(x, x, \rho^{-1}(\xi)) / \left|\frac{\partial \tilde{\eta}}{\partial \xi}\right|(x, x, \rho^{-1}(\xi))$$

$$= a(x, y, \rho^{-1}(\xi))\, p(x, y, \rho^{-1}(\xi))\, b(y, x, \rho^{-1}(\xi)) / \left|\frac{\partial \tilde{\eta}}{\partial \xi}\right|(x, x, \rho^{-1}(\xi)) \Big|_{y = \frac{\partial S}{\partial \eta}(x,\xi)}$$

$$(\rho^{-1}(\xi) \equiv \rho^{-1}(x, x, \xi))$$

From our choice of A^{-1},

$$a(x, y, \rho^{-1}(\xi))\, b(y, x, \rho^{-1}(\xi)) / \left|\frac{\partial \tilde{\eta}}{\partial \xi}(x, x, \rho^{-1}(\xi))\right| = 1$$

so $\sigma_m(APA^{-1})(x,\xi) = p(x, y, \rho^{-1}(\xi)) \quad y = \frac{\partial S}{\partial \eta}(x,\xi)$

$$= p\left(x, \frac{\partial S}{\partial \eta}(x,\xi), \rho^{-1}(\xi)\right)$$

$$= \sigma_m(P) \circ \phi^{-1}(x,\xi) \quad \text{modulo } S^{m-1}$$

since $\left(y, \frac{\partial S}{\partial y}(y,\eta)\right) \xrightarrow{\phi} \left(\frac{\partial S}{\partial \eta}(y,\eta), \eta\right)$

$$\& \quad \rho(x, x, \xi) = D_x S(x,\xi).$$

<u>Theorem</u> (Egorov).

Let ϕ be a canonical transformation,

$$\phi : T^*\mathbb{R}^n \setminus \{0\} \to T^*\mathbb{R}^n \setminus \{0\}$$

hom. of deg 1.

$$(y, \eta) \longmapsto (x, \xi)$$

Let $S(x,\eta)$ be the generating function of ϕ

i.e. $\left(x, \frac{\partial S}{\partial \eta}(x,\eta)\right) \xmapsto{\phi^{-1}} \left(\frac{\partial S}{\partial \eta}(x,\eta), \eta\right)$

Let $Af(x) = \int e^{i(S(x,\eta) - y\cdot\eta)} a(x,y,\eta) f(y)\,dy\,d\eta$

$$a(x_0, y_0, \eta_0) \neq 0$$

$$\& \quad \phi^{-1}(x_0, \xi_0) = (y_0, \eta_0).$$

Then $\exists$ FIO B s.t.

$$WF'B \subseteq \text{graph } \phi^{-1}$$

so that $\quad AB = BA = I$

microlocally near $(x_0, \xi_0, y_0, \eta_0)$.

Let $P \in \Psi^m(\mathbb{R}^n)$. Then

$$APB \in \Psi^m(\mathbb{R}^n)$$

&

$$\sigma_m(APB) = \sigma_m(P) \circ \phi^{-1}$$

(3/7) <u>Remark</u>

$$Au(x) = \int e^{i(S(x,\eta) - y\cdot\eta)} a(x,y,\eta)\, u(y)\,dy\,d\eta$$

$$a(x_0, y_0, \eta_0) \neq 0, \quad a \in S^0$$

$$Bv(y) = \int e^{i(y\cdot\eta - S(x,\eta))} b(y,x,\eta)\, v(x)\,dx\,d\eta$$

$$\phi : T^*\mathbb{R}^n \setminus \{0\} \to T^*\mathbb{R}^n \setminus \{0\}$$

$$(y, \eta) \longmapsto (x, \xi)$$

B chosen so that $AB \equiv I$ near (x_0, ξ_0)
$BA \equiv I$ near (y_0, η_0).

Let $P \in \Psi^m(\mathbb{R}^n)$, $BPA \in \Psi^m(\mathbb{R}^n)$.
with

$$\sigma(BPA) = \sigma_m(P) \circ \phi$$

$$\phi(\partial_\eta S, \eta) = (x, \partial_x S).$$

$$(x_0, \xi_0, y_0, \eta_0) \notin WF'(PA - A(BPA))$$

— n —

Example $f: \mathbb{R}^n \to \mathbb{R}^n$ diffeo,

$$Au(x) = f^* u(x) = \int e^{i(f(x)-y)\cdot\eta} \frac{u(y)}{(2\pi)^n} \, dy \, d\eta$$

$$S(x, \eta) = f(x) \cdot \eta$$

then

$$\phi(f(x), \eta) = (x, {}^t f'(x) \cdot \eta)$$

Thus:

$$(f^*)^{-1} P f^* \in \Psi^m(\mathbb{R}^n) \quad \text{if } P \in \Psi^m(\mathbb{R}^n)$$

&

$$\sigma_m((f^*)^{-1} P f^*)(f(x), \eta) = \sigma_m(P)(x, {}^t f'(x)\eta)$$

or

$$\sigma_m((f^*)^{-1} P f^*)(x, \xi) = \sigma_m(P)(f^{-1}(x), ({}^t f'(x))\xi)$$

(i.e. $\sigma_m(P)$ is invariantly defined on $T^*\mathbb{R}^n$)
(Seeley '62).

<u>Application</u> : (Reduce operators to simpler ones.)

$$V = \sum a_i \frac{\partial}{\partial x_i}, \qquad V(x_0) \neq 0.$$

Change var's so that $V = \frac{\partial}{\partial x_1}$

Generalize this to a canonical transformation in the phase space.

Given $P \in \Psi^1(\mathbb{R}^n)$,

$$H_p(x_0, \xi_0) \neq 0$$

Ch. of vars in (x, ξ) so that $H_p = \frac{\partial}{\partial x_1}$ (not nec. symplectic)

Suppose $H_p(x_0, \xi_0)$, $\sum \xi_i \frac{\partial}{\partial \xi_i}(x_0, \xi_0)$ are lin. ind.

Then we know we can find (by Daboux) an homogeneous canonical trans$^{\underline{n}}$

$$\phi : T^*\mathbb{R}^n \setminus \{0\} \longrightarrow T^*\mathbb{R}^n \setminus \{0\}$$

$$\phi(0, (0, \cdots, 1)) = (x_0, \xi_0)$$

$$(y, \eta)$$

$$p \circ \phi = \xi_1 \qquad (p = \sigma_1(P)).$$

Take A, B as before (assoc. to ϕ).

$$\sigma_1(BPA) = \sigma_1(P) \circ \phi = \xi_1 = \sigma_1(D_{x_1})$$

$$\Rightarrow \quad BPA = D_{x_1} + R(x, D_x), \quad R \in \Psi^0(\mathbb{R}^n)$$

<u>Proposition</u>: $\exists$ $C \in \Psi^0(\mathbb{R}^n)$ elliptic so that

$$(D_{x_1} + R(x, D_x))C = CD_{x_1} \mod \Psi^{-\infty}$$

Thus

$$C^{-1}BPAC = D_{x_1} \text{ mod smoothing.}$$

i.e.

$$F^{-1}PF = D_{x_1} \mod \Psi^{-\infty} \text{ (microlocally).}$$

We want to solve $Pu = f$.

$D_{x_1} u = f$, $\quad u = \int_{-\infty}^{x_1} f(s, x')\,ds$ "forward fund. solⁿ"

or $\quad u = -\int_{x_1}^{\infty} f(s, x')\,ds$ "backward fund solⁿ"

so $u = E_{\pm} f$ (a for distⁿs etc).

$$D_{x_1} E_{\pm} = I.$$

So, $\quad F^{-1} P F E_{\pm} = I \Rightarrow P F E_{\pm} F^{-1} = I$

$\Rightarrow P_{\pm}^{-1} := F E_{\pm} F^{-1}$ (a Right Parametrix).

So we can solve, microlocally, $Pu = f$ modulo smoothing.

<u>Corollary</u> (Propagⁿ of singularities).
Let $P \in \Psi^m(\mathbb{R}^n)$, $dp \neq 0$ on $p = 0$,
$p = \sigma_m(P)$, real valued.
(operators of "Real principal type).
Assume H_p & $\sum \xi_i \frac{\partial}{\partial \xi_i}$ are linear independent

$Pu = f$, $\quad (x_0, \xi_0) \in WFu - WFf$.

$\Rightarrow p(x_0, \xi_0) = 0$

Then the whole connected null-bicharacteristic through (x_0, ξ_0) that does not intersect WFf is in WFu

<u>Remark</u>: We proved this earlier under the slightly weaker assumption that $\partial_\xi p \neq 0$ on $p = 0$

<u>Outline of proof</u>: Using $F^{-1}PF \equiv D_{x_1}$.

$$\underset{(y,\eta)}{\phi : ((0,(0,\cdots,0,1))} \to (x_0,\xi_0)$$

Choose A, B associated to ϕ, ϕ^{-1}.
Then

$$BPA = D_{x_1} + R$$

$\exists C$: $\quad C^{-1}BPAC \equiv D_{x_1}$, $\quad F = AC$

$$Ff(x) = \int e^{i(s(x,\eta) - y\cdot\eta)} \ell(x,y,\eta) f(y)\, dy\, d\eta$$

$$F^{-1}PF \equiv D_{x_1}, \quad P = FD_{x_1}F^{-1} \text{ (microlocally)}$$

Suppose $Pu = f$; then

$$FD_{x_1}F^{-1}u = f \quad ; \quad D_{x_1}F^{-1}u = F^{-1}f$$

$$(x_0,\xi_0) \in WFu - WFf$$

Check: $(y_0,\eta_0) = (0,(0,\cdots,0,1)) \in WF(F^{-1}u) - WF(F^{-1}f)$
Then prove prop$^{\underline{n}}$ of sing. for D_{x_1}.
$WF(F^{-1}u) - WF(F^{-1}f)$ is invariant under H_{ξ_1}

$\Rightarrow$ $WFu - WFf$ is invariant under H_p
(ϕ^{-1} maps integral curves of H_{ξ_1} to those of H_p ($p \circ \phi = \xi_1$)).

———— " ————

<u>Example</u> $\quad P = P_1P_2 + R, \; R, P_i \in \Psi'(\mathbb{R}^n) \quad i = 1,2$

Assume $\{p_1, p_2\} = 0$ near (x_0,ξ_0), with $p_i = \sigma(P_i)$ $\mathbb{R}$-valued. $\quad p_1(x_0,\xi_0) = p_2(x_0,\xi_0) = 0$.

$H_{p_1}(x_0,\xi_0)$, $H_{p_2}(x_0,\xi_0)$, $\sum \xi_i \frac{\partial}{\partial \xi_i}$ all lin. ind (at (x_0,ξ_0))

A simpler example is $\tilde{P} = D_{x_1} D_{x_2}$ (all the cond$^{\underline{n}}$s above are satisfied by $\tilde{P}$)

Egorov $\Rightarrow$ $\exists$ canonical trans$^{\underline{n}}$ s.t.

$$p \circ \phi = \xi_1 \xi_2 \qquad (\rho_1 \to \xi_1,\ \rho_2 \to \xi_2)$$

(3/10)

<u>Remark</u> $E_+ f(x) = \int_{-\infty}^{x_1} f(s, x') \, ds$

$$E_- f(x) = -\int_{x_1}^{\infty} f(s, x') \, ds$$

$$E_+ f(x) = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{x_1} \int e^{i x' \cdot \xi'} \hat{f}(s, \xi) \, d\xi' \, ds$$

$$= \frac{1}{(2\pi)^{n-1}} \iint H(x_1 - s) e^{i x' \cdot \xi'} \hat{f}(s, \xi') \, d\xi' \, ds$$

$$\left(H = \begin{matrix} 1 & \text{for} \geq 0 \\ 0 & \text{for} < 0 \end{matrix} \right)$$

Note that $\iint e^{i x' \cdot \xi'} a(s, x, \xi') \hat{f}(s, \xi') \, d\xi' \, ds = Af(x)$

$s \in S^m(\mathbb{R}_s \times \mathbb{R}^n \times \mathbb{R}^{n-1}_{\xi'})$

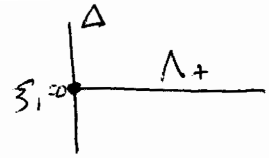
is a FIO since

$$Af(x) = \iint e^{i(x' - y') \cdot \xi'} a(s, x, \xi') f(s y') \, dy' \, ds \, d\xi'$$

$$WF'A \subseteq \{(x_1, x', y_1, x'; 0, \xi', 0, \xi')\}$$

= "flow out under H_{ξ_1} starting at

$$\{(x_1, x'; x_1, x'; 0, \xi'; 0, \xi')\} = \Delta \cap \{\xi_1 = 0\}"$$



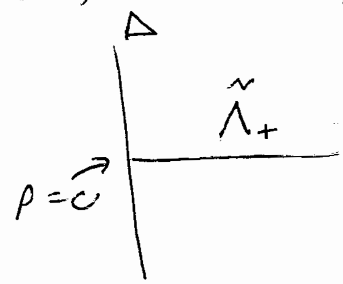
$$WF'E_+ = \Delta \cup \{(x_1, x', x_1+s, x'; 0, \xi', 0, \xi'), s \geq 0\}$$
$$WF'E_- = \Delta \cup \{(x_1, x', x_1+s, x'; 0, \xi', 0, \xi'), s \leq 0\}$$

$E_\pm$ is not an FIO because $WF'E_\pm$ consists of intersecting Lag. manifolds.

$P_\pm^{-1} \triangleq F^{-1} E_\pm F$, P of real princ. type.

$WF' P_\pm^{-1} \subset \Delta \cup \tilde{\Lambda}_\pm$

$\tilde{\Lambda}_\pm$ = forward (back) flow out by H_p from $\Delta \cap \{p = 0\}$.



Calculus of F.I.O.'s

Recall $I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$.

ϕ non-deg. phase fc^n w/ non-deg. crit. points.

We have

$$I_{a,\phi}(t) \sim C t^{-N/2} e^{\frac{\pi i}{4} \operatorname{sgn} Q(-)} |\det Q(-)|^{-1/2} a(-) + \text{lower}.$$

"$(-)$" = at crit. point.

Take $I_{a,\varphi} = \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$

φ non-deg. phase function

Question: What is the princ. symbol of $I_{a,\phi}$.

$$WF\, I_{a,\varphi} \subseteq \Lambda_\varphi = \{(x, d_x\varphi) ; d_\theta \varphi = 0\}.$$

<u>Recall</u> We can choose coodinates x so that

$$\Lambda = \Lambda_\varphi = \Lambda_\psi \;,\quad \psi(x,\xi) = \langle x, \xi \rangle - h(\xi)$$

$$\Lambda = \{(h'(\xi), \xi) ; \xi \in \mathbb{R}^n \setminus \{0\}\}.$$

$$\begin{aligned} \Lambda_\varphi &\to \mathbb{R}^n \setminus \{0\} \quad \text{local diffeo.} \\ (x,\xi) &\to \xi \end{aligned}$$

Give an interpretation to $|\det Q|^{-1/2}$

<u>Half Densities</u>.

Let V be a (fin. dim^{n}al) vecter space $\mathbb{C}$

$\beta V = \{$ set of bases on $V\}$.

An α-density on V is

$$\gamma : \beta V \to \mathbb{C}$$

such that if $A : V \to V$ non-singular, then

$$\gamma(Ae_1, \dots, Ae_n) = |\det A|^\alpha \gamma(e_1, \dots, e_n)$$

eg. $\alpha = 1$ standard density.

Now let X be a C^∞ manifold.

Take $V_x = T_x X$.

$$\Omega^\alpha(T_x X) = \alpha\text{-densities on } T_x X$$

$$\Omega^\alpha(X) = \cup_{x \in X} \Omega^\alpha(T_x X)$$

Note that $\Omega^\alpha(V)$ is one dimensional.

<u>Def^n</u> An α-density on X a C^∞-manifold is a smooth section of

$$X \to \Omega^\alpha(X).$$

$C_\varphi = \{ {}^{(x,\theta):} d_\theta \varphi = 0 \}$. Define a density $d\varphi$ on C_φ.

$$d\varphi \wedge d\left(\frac{\partial \varphi}{\partial \theta_1}\right) \wedge \cdots \wedge d\left(\frac{\partial \varphi}{\partial \theta_N}\right) = dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_N$$

(in fact $d\varphi = \delta_{\left(\frac{\partial \varphi}{\partial \theta_j} = 0\right)}$).

(3/12) $d\varphi$ depends on the coordinates x, but otherwise invariantly defined on C_φ.

Let $\lambda_1, \dots, \lambda_n$ be local coords on C_φ. Then

$$d\varphi = f \, d\lambda_1 \wedge \cdots \wedge d\lambda_n$$

& $$f = \left| \det \begin{pmatrix} \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial \theta} \\ \frac{\partial^2 \varphi}{\partial x \partial \theta} & \frac{\partial^2 \varphi}{\partial \theta^2} \end{pmatrix} \right|^{-1}$$

Take $\lambda_j = \frac{\partial \varphi}{\partial x_j}$ (by identifying C_φ & Λ_φ)

so $$f = \left| \det \begin{pmatrix} \frac{\partial^2 \varphi}{\partial x^2} & \frac{\partial^2 \varphi}{\partial x \partial \theta} \\ \frac{\partial^2 \varphi}{\partial x \partial \theta} & \frac{\partial^2 \varphi}{\partial \theta^2} \end{pmatrix} \right|^{-1}$$

The density on Λ_φ is $|d\varphi| = |f| \, |d\xi_1 \wedge \cdots \wedge d\xi_n|$

Now we consider

$$I_{a,\varphi} \in \mathcal{D}'(X, \Omega_{1/2}) = (C_0^\infty(X, \Omega_{1/2}))^*$$

(if $u, f \in C^\infty(X, \Omega_{1/2})$ then

$u(f) = \int \underbrace{u(x) f(x)}_{\text{this is a density}} dx$ makes sense since this is a density)

$u = I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\,d\theta$

Let $\tilde{x}$ be a change of vars in x

& $\tilde{u} = \int e^{i\phi(\tilde{x},\theta)} a(\tilde{x},\theta)\,d\theta$

Then $\tilde{u}(\tilde{x}) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{1/2} u(x)$

Let $\tilde{\phi}(x,\theta) = \phi(\tilde{x},\theta)$.

Recall

$$d\varphi \wedge d\left(\frac{\partial \varphi}{\partial \theta_1}\right) \wedge \cdots \wedge d\left(\frac{\partial \varphi}{\partial \theta_N}\right) = dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_N$$

so $d\varphi = \left|\frac{\partial x}{\partial \tilde{x}}\right| d\tilde{\varphi}$

Conclusion $\tilde{a}(x,\theta) = a(\tilde{x},\theta)$

$$\tilde{a}|d\tilde{\varphi}|^{1/2} = a|d\varphi|^{1/2}$$

i.e. $a|d\varphi|^{1/2}$ is invariantly defined under changes of variables.

& $a|d\varphi|^{1/2} = a|\det Q|^{-1/2}$

where $Q = \left(\frac{\partial^2 \varphi}{\partial x \partial \theta}\right)$.

To be precise, $I_{a,\varphi} = (2\pi)^{-\frac{n+2N}{4}} \int e^{i\varphi} a(x,\theta)\,d\theta$.

i.e. this factor is nec.

So far $I_{a,\varphi} \in \mathcal{D}'(X, \Omega_{1/2})$
& princ symbol $\in S^{\tilde{m}}(\Lambda, \Omega_{1/2})$
This doesn't take into account the factor
$$e^{\frac{\pi i}{4} \operatorname{sgn} Q(\cdots)}$$

Suppose $\Lambda_\varphi = \Lambda_\alpha$; φ, α non-deg. phase fc^{n}s.
$$I_{a,\varphi} \sim e^{\pi i/4 \operatorname{sgn} Q_\varphi} (\cdots) + \text{lower order.}$$
$$I_{a,\alpha} \sim e^{\pi i/4 \operatorname{sgn} Q_\alpha} (\cdots)$$
actually $I_{a,\alpha}(e^{-it\psi(\cdot,\xi)})$
& $I_{a,\varphi}(e^{-it\psi(\cdot,\xi)})$.

There is a shift of phase when the phase fc^n is changed from φ to α.
So what is $e^{\frac{\pi i}{4}(\operatorname{sgn}(Q_\varphi - Q_\alpha))}$

- how can we interpret this?

<u>Keller-Maslov Line Bundle over Λ</u> $\mathcal{L}$.

<u>Description of $\mathcal{L}$</u>:

Two phase functions φ, α, non-degenerate.
$\mathcal{L}$ is equipped with the transition functions
$$e^{\frac{i\pi}{4}(\operatorname{sgn} \varphi''|_\Lambda - \operatorname{sgn} \alpha''|_\Lambda)}$$
on $\Lambda_\varphi \cap \Lambda_\alpha$ & with transitions ~~functions~~
$$e^{\frac{\pi i}{4}(\operatorname{sgn} \varphi''(x,\theta) - \operatorname{sgn} \varphi''(y,\theta))}$$
when we change variables from x to y on Λ_φ.

<u>Theorem</u> There exists such a line bundle $\mathcal{L}$.
(Duistermaat, Hörmander Vol III, IV)

<u>Theorem</u> Let $u = I_{a,\varphi} \in \mathcal{D}'(X, \Omega_{1/2})$. Then we have a well defined principal symbol

$$\sigma(u) \in S^{\tilde{m}}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})$$

(3/14)

$$u = I_{a,\varphi} = (2\pi)^{-\frac{n+2N}{4}} \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$$

$$a \in S^{m - \frac{N}{2} + \frac{n}{4}}(X \times \mathbb{R}^N)$$

$$u \in \mathcal{D}'(X, \Omega_{1/2})$$

$$\sigma_{m+\frac{n}{4}}(u) \in S^{m+\frac{n}{4}}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})$$

$$\Lambda = \Lambda_\varphi$$

$\mathcal{L}$ = Keller-Maslov bundle.

In local coordinates

$$\sigma_{m+\frac{n}{4}}(u) = a(h'(\xi), \xi)\, e^{\frac{\pi i}{4} \operatorname{sgn} Q(h'(\xi);\xi)}$$

$$|\det Q(h'(\xi), \xi)|^{-1/2} \mod S^{m+\frac{n}{4}-1}$$

<u>Defn</u> Let X be a C^∞ manifold, Λ an immersed conic Lagrangian manifold, $\Lambda \subset T^*X \setminus \{0\}$.

$$u \in \mathcal{D}'(X, \Omega_{1/2})$$

Then u is a Lagrangian distribution if

$$WF\, u \subseteq \Lambda$$

$$u = \sum_j u_j \quad \text{locally finite}$$

with

$$u_j = (2\pi)^{-\left(\frac{n+2N_j}{4}\right)} \int e^{i\varphi_j(x,\theta)} a_j(x,\theta)\,d\theta$$

with φ_j a non-degenerate phase function on an open cone Γ_j in $X \times \mathbb{R}^{N_j}$,

$a_j \in S^{m - \frac{N_j}{2} + n/4}(X \times \mathbb{R}^{N_j})$, $\Lambda = \Lambda_{\varphi_j}$ locally

supp $a_j \subseteq \Gamma_j$

— " —

<u>Def^n</u> $\sigma_{m+\frac{n}{4}}(u) = \sum_j \sigma_{m+\frac{n}{4}}(u_j)$

$I^m(X,\Lambda) = \{u \cdot$ Lag. dist^{ns} of order $m\}$

<u>Calculus of Lagrangian Distributions</u>

<u>Theorem</u> a) Λ is an embedded Lagrangian conic submanifold of $T^*X \setminus \{0\}$.

$$\frac{I^m(X,\Lambda)}{I^{m-1}(X,\Lambda)} \underset{\sigma_{m+\frac{n}{4}}}{\cong} \frac{S^{m+\frac{n}{4}}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})}{S^{m+\frac{n}{4}-1}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})}$$

$$\sigma_{m+\frac{n}{4}}[u] = \sigma_{m+n/4}(u)$$

where $[\;]$ denotes the equivalence class in $I^m(X,\Lambda)$

b) Λ as in (a);

$P \in \Psi^m(X)$

$u \in I^{\tilde m}(X,\Lambda)$

$\Rightarrow Pu \in I^{m+\tilde m}(X,\Lambda)$

and if $\sigma_m(P) = 0$ on Λ,

$$\sigma_{m+\tilde m}(Pu) = \sigma_m(P)\,\sigma_{\tilde m}(u) = 0 \quad \text{on } \Lambda$$

so $Pu \in I^{m+\tilde m - 1}(X,\Lambda)$

$$\sigma_{m+n-1}(Pu) = \frac{1}{i}\mathcal{L}_{H_p}\sigma_m(u) + \sigma_{sub}(P)\,\sigma_m(u)$$

where $\sigma_{sub}(P) = p_{m-1} - \frac{1}{2i}\sum_{i=1}^{n}\frac{\partial^2 p_m}{\partial x_i \partial \xi_i}$

and the full symbol of P is given in local coordinates by

$$p_m + p_{m-1} + p_{m-2} + \cdots$$

p_j hom. of deg j.

$\sigma_{sub}(P)$ is well defined (invariant) on $\frac{1}{2}$ densities.

Next quarter...

- Composition of FIO's
- Applications:

X: C^∞-manifold with Riem. metric g.

Solve

$$(*)\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta_g\right)u = 0 & \text{on } X\times\mathbb{R} \\ u\big|_{t=0} = u_0 \in \mathcal{D}'(X) \\ \frac{\partial u}{\partial t}\big|_{t=0} = u_1 \in \mathcal{D}'(X) \end{cases}$$

Locally $u = u_+ + u_-$ (Lax parametrix)

$$u_\pm = \int e^{i\varphi_\pm(t,x,\xi)}a_\pm(t,x,\xi)\,d\xi.$$

$\varphi_\pm$ solve

$$\left(\frac{\partial\varphi_\pm}{\partial t}\right)^2 = \sum g^{ij}(x)\frac{\partial\varphi_\pm}{\partial x_i}\frac{\partial\varphi_\pm}{\partial x_j}, \quad \varphi_\pm(0,x,\xi) = \langle x,\xi\rangle.$$

$a_\pm \in S^0(\mathbb{R}\times X\times \mathbb{R}^n_\xi)$.

$a_\pm$ were obtained by solving ODEs.

Problem: Find global sol$^{\underline{n}}$s.

Construct u solving $(*)$

$$u = u_+ + u_-$$

$$u_\pm \in I^m(X, \Lambda_\pm)$$

We need to find global Lagrangian manifolds $\Lambda_\pm$.

$\Lambda_\pm$ will be the flow out by $H_{p_\pm}$ from $t=0$

$$p_\pm = \tau \pm \sqrt{g^{ij}(x)\xi_i\xi_j}$$

We will need conditions so that $\Lambda_\pm$ is a global Lagrangian manifold.

Also, what will $\sigma_m(u_\pm)$ be?

$P = \frac{\partial^2}{\partial t^2} - \Delta_g$, $\quad \sigma(P) = 0$ on $\Lambda_\pm$.

$$P_+ = \frac{\partial}{\partial t} + \sqrt{\Delta_g}$$

$$P_- = \frac{\partial}{\partial t} - \sqrt{\Delta_g}$$

$$\sigma_m(P_\pm u_\pm) = \frac{1}{i}\mathcal{L}_{H_{p_\pm}}\sigma(u_\pm) + \sigma_{sub}(P_\pm)\sigma(u_\pm) \overset{\text{set}}{=} 0$$

an invariantly defined ODE, globally.

$$\sigma(u_\pm)\big|_{t=0} = \text{given}.$$

(4/2). Recall from last quarter...

$I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$

ϕ phase fc^{n}, non-degenerate in $\Gamma \subset X \times (\mathbb{R}^N - 0)$
Γ open cone, $X \subset \mathbb{R}^n$ open.

$a \in S^{m + n/4 - N/2}(X \times \mathbb{R}^N)$

$\Lambda_\phi = F_\phi(C_\phi)$

$C_\phi = \{(x,\theta) \in X \times \mathbb{R}^N \setminus \{0\} ; \, d_\theta \phi = 0\}$

$$F_\phi : X \times \mathbb{R}^N \setminus \{0\} \longrightarrow T^*X \setminus \{0\}$$
$$(x,\theta) \longmapsto (x, d_x\phi(x,\theta))$$

Λ_ϕ an immersed Lagrangian manifold.

Recall: $\Lambda \subset T^*X \setminus \{0\}$ conic Lagrangian, then given any $(x_0, \xi_0) \in \Lambda$ $\exists$ Γ conic nbhd of (x_0,ξ_0) s.t. $\Lambda = \Lambda_\phi$ in Γ, with ϕ non-degenerate in some open conic set.

Given Λ, in local coord's, $\Lambda = \Lambda_\psi$ with $\psi(x,\xi) = \langle x, \xi\rangle - h(\xi)$ locally.

$C_\psi = \{(h'(\xi), \xi)\}$

$\Lambda = \Lambda_\psi = \{(h'(\xi), \xi)\}$

$$\int e^{i(\phi(x,\theta) - \psi(\xi))} a(x,\theta)\, d\theta$$
$$= c\, e^{\frac{\pi i}{4} \operatorname{sgn} Q(x(\xi), \theta(\xi))} |\det Q(x(\xi), \theta(\xi))|^{-1/2} a(x(\xi), \theta(\xi))$$
$$+ \text{ lower order terms}$$

Conclusion:

$$I_{a,\phi} \longrightarrow \sigma(I_{a,\phi}) \in S^{m+n/4}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})$$

<u>Def^n</u> Let X be a smooth manifold, Λ a conic immersed Lagrangian manifold in $T^*X \setminus \{0\}$. A <u>Fourier Integral Dist^n</u>, $u \in \mathcal{D}'(X, \Omega_{1/2})$

$$u = \sum_{j \in J} u_j,$$

$u_j \in \mathcal{D}'(X, \Omega_{1/2})$, with locally finite support

$$u_j = \int e^{i\phi_j(x,\theta)} a_j(x,\theta)\, d\theta$$

ϕ_j non-deg. phase fc^ns in some non-deg. cone Γ_j

$a_j \in S^{m - (\frac{N_j}{2}) + n/4}(X \times \mathbb{R}^{N_j})$

$\operatorname{supp} a_j \subseteq \Gamma_j$

$F_{\phi_j} : C_{\phi_j} \to \Lambda_{\phi_j}$ is a diffeo onto an open cone Λ_{ϕ_j} in Λ.

We say $u \in I^m(X, \Lambda)$

<u>Theorem.</u> (Symbol Calculus of FI Dist^ns)

$$\frac{I^m(X,\Lambda)}{I^{m-1}(X,\Lambda)} \simeq \frac{S^{m+n/4}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})}{S^{m+n/4-1}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})}$$

a linear isomorphism σ_m called the symbol map, and $\sigma_m(u)$ the principal symbol of $u \in I^m(X, \Lambda)$. (Λ must be closed)

<u>Check</u>: In the case u = Schwartz kernel of $P \in \Psi^m(X)$, this is the standard symbol.

<u>Proof</u> given $u \in I^m(X,\Lambda)$, $\sigma_m(u) = \sum \sigma_m(u_j)$.

$$T^*X \xrightarrow{\pi} S^*X \xrightarrow{\pi^s} X$$

cosphere bundle. $(x,\xi) \sim (x, \lambda\xi)$ $\lambda > 0$.

Note: π^s is proper.

Let V_j, $j \in J$ be a locally finite covering of $\pi(\Lambda)$ such that

$$\Lambda_j = \pi^{-1}(V_j) = \Lambda_{\phi_j}$$

for some non-deg. phase fc^n ϕ_j on some open cone $\Gamma_j \subset T^*X \setminus \{0\}$.

Let $\chi_j \in C_0^\infty(\pi(\Lambda))$ be a part^n of unity on $\pi(\Lambda)$ w/ $\operatorname{supp} \chi_j \subseteq V_j$.

Let $a \in S^{m+n/4}(\Lambda, \Omega_{1/2} \otimes \mathcal{L}) / S^{m+n/4-1}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})$

$\exists$ $u_j \in I^m(X,\Lambda)$ s.t.

$$\sigma_m(u_j) = (\chi_j \circ \pi) a$$

Since π^s is proper,

$$\{\pi^s(V_j)\}$$

form a locally finite system in X

Since $\operatorname{supp} u_j \subseteq \pi^s(V_j)$, $\operatorname{supp} u_j$ form a locally finite system.

Then $u = \sum u_j$, $\sigma_m(u) = \sum \sigma_m(u_j) = a$.

$\blacksquare$

For applications,

$$\left(\frac{\partial}{\partial t^2} - \Delta_g\right) u = 0$$

Δ_g Laplace-Baltrami operator of a Riem. metric.

$$u|_{t=0} = 0$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \delta_0$$

$$u = u_+ + u_-, \quad u_+ = \int e^{i\varphi_+(t,x,\xi)} a_+(t,x,\xi)\,d\xi.$$

$$p_\pm(t, x, d_{t,x}\varphi_\pm) = 0$$

$$p_\pm(t,x,\tau,\xi) = \tau \pm \sqrt{\textstyle\sum g^{ij}(x)\xi_i\xi_j}$$

$$WF(I_{a_\pm}, \varphi_\pm) \subseteq \Lambda_{\varphi_\pm}$$

$p_\pm$ vanishes on $\Lambda_{\varphi_\pm}$.

Next $u \in I^m(X, \Lambda)$ (from now on, Λ closed conic Lag. subman. of $T^*X \setminus \{0\}$.

$P \in \Psi^{\tilde m}_{cl}(X)$.

$p_{\tilde m} = \sigma_{\tilde m}(P)$ vanishes on Λ.

Question: What is Pu?

We'll find $Pu \in I^{m+\tilde m-1}(X, \Lambda)$

and

$$\sigma_{m+\tilde m-1}(Pu) = \mathcal{L}_{H_{p_{\tilde m}}} \sigma_m(u) + c\,\sigma_m(u)$$

c called the subprincipal symbol of p. $\in S^{\tilde m - 1}(T^*X)$.

Let $u = I_{a,\phi} = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$

$a \in S^{m - N/2 + n/4}(X \times \mathbb{R}^N)$, $X \subset \mathbb{R}^n$ open.

Let $P \in \Psi_{cl}^{\tilde{m}}(X)$, properly supported.

$$Pu = \int e^{i\phi(x,\theta)} \underbrace{e^{-i\phi(x,\theta)} P(e^{i\phi(x,\theta)} a(x,\theta))}_{b(x,\theta) \text{ say.}} d\theta$$

$b(x,\theta) \in S^{\tilde{\tilde{m}}}(X \times \mathbb{R}^N)$?

$$Pv(x) = \int e^{i(x-y)\cdot\xi} p(x,\xi) v(y)\, dy\, d\xi$$

$\uparrow$ full symbol of P.

$p(x,\xi) = p_{\tilde{m}}(x,\xi) + r(x,\xi)$

$r \in S^{\tilde{m}-1}(X \times \mathbb{R}^n)$

$p_{\tilde{m}}$ hom. of degree $\tilde{m}$

$$e^{-i\phi(x,\theta)} P(e^{i\phi(x,\theta)} a) = \int e^{i((x-y)\cdot\xi + \phi(y,\theta) - \phi(x,\theta))} p(x,\xi)\, a(y,\theta)\, dy\, d\xi$$

$= b(x,\theta)$

St. phase in y, ξ:

$d_\xi(-) = x - y = 0 \iff x = y$.

$d_y(-) = -\xi + d_y\phi = 0 \iff \xi = d_y\phi$

$$\text{Hess} = \begin{pmatrix} d^2_\phi y & -I \\ -I & 0 \end{pmatrix}$$

so

$b(x,\theta) \sim \cancel{p(x, d_x\phi)\, a(x,\theta) +}$

$$\sum \frac{1}{\alpha!} \left(\frac{1}{i}\frac{\partial}{\partial\xi}\right)^\alpha (p(x, d_x\phi)) e^{-i\phi(x,\theta)} \left(\frac{\partial}{\partial x}\right)^\alpha \left(e^{i\phi(x,\theta)} a(x,\theta)\right)$$

highest order term is

$$p_{\tilde m}(x, d_x\phi)\, a(x,\theta)$$

We have proven so far that

$$Pu \in I^{m+\tilde m}(X,\Lambda)$$

& $\sigma_{m+\tilde m}(Pu) = p_{\tilde m}(x, d_x\phi)\, a(x,\theta)$.

(4/4)

$$b(x,\theta) = p_{\tilde m}(x, d_x\phi)\, a + p_{\tilde m -1}(x, d_x\phi) a$$
$$+ \frac{1}{i}\sum_j \frac{\partial p_{\tilde m}}{\partial \xi_j}(x, d_x\phi)\frac{\partial a}{\partial x_j} - \frac{i}{2}\left(\sum_{j,k} \frac{\partial^2 p_{\tilde m}}{\partial \xi_k \partial \xi_j}(x, d_x\phi)\frac{\partial^2 \phi}{\partial x_j \partial x_k}\right) a$$
$$\text{mod. } S^{m+\tilde m + \frac{n}{4} - \frac{N}{2} - 2}(X\times\mathbb{R}^N).$$

In particular if $p_{\tilde m}(x, d_x\phi) = 0$, then we want to interpret this invariantly (in terms of a). So we use the fact that

$$\Lambda_\phi = \Lambda_\psi \quad w/ \quad \psi = \langle x\, \xi\rangle - h(\xi).$$
$$\Lambda = \{(h'(\xi), \xi)\} = \Lambda_\psi = C_\psi.$$

Write $u = \int e^{i\psi(x,\xi)} a(x,\xi)\, d\xi, \quad a \in S^{m - n/4}(T^*X)$.

Next goal, replace $a(x,\xi)$ by $a_0(\xi)$.
We develope a in a Taylor series around $x = h'(\xi)$.
Then

$$a(x,\xi) = a(h'(\xi),\xi) + (x - h'(\xi))\cdot a_r(x,\xi).$$

$$x - h'(\xi) = \partial_\xi \psi$$

so

$$u = \int \frac{1}{i}\partial_\xi e^{i\psi} a_r(x,\xi) + \int e^{i\psi} a(h'(\xi),\xi)\, d\xi$$

$$= \int e^{i\psi} a(h'(\xi),\xi)\,d\xi + \frac{1}{i}\int e^{i\psi} \underbrace{\partial_\xi a(x,\xi)}_{\text{lower order.}}\,d\xi$$

could continue all the way to $S^{-\infty}$.

$$a_0(\xi) = a(h'(\xi),\xi) + \sum \partial_{\xi_i} a_r(h'(\xi),\xi)$$

$$\mod S^{m-\frac{n}{4}-2}(T^*X).$$

So,
$$u = \int e^{i\psi(x,\xi)} a_0(\xi)\,d\xi \quad \mod C^\infty$$

$$= \int e^{i(\langle x,\xi\rangle - h(\xi))} a_0(\xi)\,d\xi.$$

$$Pu = \int e^{i(\langle x,\xi\rangle - h(\xi))} e^{-i\langle x,\xi\rangle} P(e^{i\langle x,\xi\rangle}) a_0(\xi)\,d\xi$$

$$= \int e^{i(\langle x,\xi\rangle - h(\xi))} p(x,\xi) a_0(\xi)\,d\xi.$$

$$= \int e^{i(\langle x,\xi\rangle - h(\xi))} \left(p_{\tilde m}(x,\xi) + p_{\tilde m-1}(x,\xi)\right) a_0(\xi)\,d\xi$$

$$\mod I^{m+\tilde m-2}(X,\Lambda)$$

Since $p_{\tilde m}$ vanishes on $\Lambda = \{(h'(\xi),\xi)\}$

$$(*) \quad p_{\tilde m}(x,\xi) = \sum_{j=1}^{n} \left(x_j - \frac{\partial h}{\partial \xi_j}(\xi)\right) \tilde p_j(x,\xi)$$

so
$$u = \frac{1}{i}\sum \int \partial_{\xi_j}(e^{i\psi(x,\xi)}) \cancel{\left(x_j - \frac{\partial h}{\partial \xi_j}(\xi)\right)} \tilde p_j(x,\xi)\, a_0(\xi)\,d\xi$$

$$+ \int e^{i\psi(x,\xi)} p_{\tilde m-1}(x,\xi)\, a_0(\xi)\,d\xi.$$

$$= \int e^{i\psi} p_{\tilde m -1}(x,\xi) a_0(\xi)\, d\xi \qquad \text{mod } I^{m+\tilde m-2}(X,\Lambda)$$
$$-\frac{1}{i}\int \sum e^{i\psi(x,\xi)} \frac{\partial \tilde p_j}{\partial \xi_j}(x,\xi)\, a_0(\xi)\, d\xi$$
$$-\frac{1}{i}\int \sum e^{i\psi(x,\xi)} \tilde p_j(x,\xi) \frac{\partial a_0}{\partial \xi_j}(\xi)\, d\xi .$$

We can replace $a_0(\xi)$ by $a(h'(\xi),\xi)$ since a_r is lower order.

Consider

$$\sum \tilde p_j(x,\xi)\frac{\partial a_0}{\partial \xi_j}(\xi) = \sum \tilde p_j(x,\xi)\frac{\partial a}{\partial \xi_j}(h'(\xi),\xi)$$
$$\text{mod. } S^{m+\tilde m-\frac{n}{4}-2}.$$

$$\frac{\partial a}{\partial \xi_j} = \sum_k \frac{\partial a}{\partial x_k}\frac{\partial^2 h}{\partial \xi_j \partial \xi_k} + \frac{\partial a}{\partial \xi_j} \quad \text{at } x = h'(\xi)$$

$$\Rightarrow \sum \tilde p_j(x,\xi)\frac{\partial a_0}{\partial \xi_j}(\xi) = \sum_{jk} \tilde p_j \frac{\partial a}{\partial x_k}\frac{\partial^2 h}{\partial \xi_j \partial \xi_k} + \sum_j \tilde p_j \frac{\partial a}{\partial \xi_j}.$$

But $\dfrac{\partial \tilde p_m}{\partial x_j} = \tilde p_j$ on $x = h'(\xi)$ (see (*)).

& $\dfrac{\partial \tilde p_m}{\partial \xi_k} = -\sum \dfrac{\partial^2 h}{\partial \xi_j \partial \xi_k}\tilde p_j(x,\xi)$ on $x = h'(\xi)$ ('').

Conclusion:

$$\sum \tilde p_j(x,\xi)\frac{\partial a}{\partial \xi_j}(h'(\xi),\xi)$$
$$= \sum \frac{\partial p_{\tilde m}}{\partial \xi_j}\frac{\partial a}{\partial x_j} - \frac{\partial p_{\tilde m}}{\partial x_j}\frac{\partial a}{\partial \xi_j} \quad \text{on } x = h'(\xi).$$
$$= H_{p_{\tilde m}}\, a \quad \text{on } x = h'(\xi).$$

Conclusion: $\sum \tilde{\rho}_j(x,\xi)\frac{\partial}{\partial \xi_j} = H_{p_{\tilde{m}}}$ on $C_\psi = \Lambda_\psi$.

Recall that $\sigma_M(u) \in S^{M-\frac{n}{4}}(\Lambda, \Omega_{1/2} \otimes \mathcal{L})$

$\mathcal{L}_{H_{p_{\tilde{m}}}}$ = Lie der. is well defined on $\Omega_{1/2}(\Lambda)$.

Excercise: V: vector field on C^∞ manifold X.
a, a density of order α on X.
If u is a non-vanishing density & $a = fu$
then

$$\mathcal{L}_V a := \frac{d}{dt}(\varphi_t^* a)\Big|_{t=0} \quad \leftarrow \text{1 par. group def. by } V$$

$$= \{V(f) + (\alpha \operatorname{div} V)\}u.$$

So

$$(4.19) \quad \sigma_{M+\tilde{m}-1}(Pu) = \left(\frac{1}{i}\mathcal{L}_{H_{p_{\tilde{m}}}} + \gamma\right)\sigma_M(u)$$

w/ $\displaystyle \gamma = p_{\tilde{m}-1} + \frac{1}{2i}\sum_{j=1}^{n}\frac{\partial p_{\tilde{m}}}{\partial \xi_j}(h'(\xi),\xi) - i\sum \frac{\partial \tilde{\rho}_j}{\partial \xi_j}(h'(\xi),\xi)$

$$= p_{\tilde{m}-1} + \frac{1}{2i}\left(\sum_j \frac{\partial p_{\tilde{m}}}{\partial \xi_j} + \sum \frac{\partial p_{\tilde{m}}}{\partial x_k}\frac{\partial^2 h}{\partial \xi_j \partial \xi_k}\right) - i\sum \frac{\partial \tilde{\rho}_j}{\partial \xi_j}(h'(\xi),\xi)$$

$$= p_{\tilde{m}-1} + \frac{1}{2i}\left(\sum_j \frac{\partial}{\partial \xi_j}\left(-\tilde{\rho}_j - \cancel{\sum_k \frac{\partial \tilde{p}_m}{\partial x_k}\left(x_k - \frac{\partial h}{\partial \xi_k}\right)}\right)\right)$$

$$- i\sum \frac{\partial \tilde{\rho}_j}{\partial \xi_j} + \cancel{\sum x_j(x - h')}$$

$$= p_{\tilde{m}-1} - \frac{1}{2i}\sum_j \frac{\partial \rho_j}{\partial \xi_j} + \frac{1}{i}\sum_j$$

$$= p_{\tilde{m}-1} - \frac{1}{2i}\sum \frac{\partial^2 p_{\tilde{m}}}{\partial x_j \partial \xi_j}$$

<u>Theorem</u>
Let X be a C^∞ manifold, $P \in \Psi^m_{cl}(X, \Omega_{1/2})$
$u \in I^{\tilde{m}}(X, \Lambda)$, $\Lambda \subseteq T^*X \setminus \{0\}$.

Λ embedded closed Lagrangian manifold.
Then
(a) $Pu \in I^{m+\tilde{m}}(X, \Lambda)$,
$$\sigma_{m+\tilde{m}}(Pu) = \sigma_m(P)\,\sigma_{\tilde{m}}(u)$$
(b) If $\sigma_m(p) = 0$ on Λ then
$Pu \in I^{m+\tilde{m}-1}(X, \Lambda)$ &
$$\sigma_{m+\tilde{m}-1}(Pu) = \left(\frac{1}{i}\mathcal{L}_{H_{p_m}} + c\right)\sigma_{\tilde{m}}(u)$$

w/ $$c = p_{m-1} - \frac{1}{2i}\sum_{j=1}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}$$

in local coord's; $p = p_m + p_{m-1} + \cdots$ etc.

<u>Def^n</u> c in Th^m above is called the <u>subprincipal symbol</u> of P.

<u>Prop^n</u> If $P \in \Psi^m(X, \Omega_{1/2})$ then c is invariantly defined.
<u>Proof</u>. Let $V = \left(\frac{\partial p_m}{\partial \xi_1}(x, d_x\varphi), \ldots, \frac{\partial p_m}{\partial \xi_n}(x, d_x\varphi)\right)$
$$e^{-i\tau\varphi(x)} p(e^{i\tau\varphi(x)}\omega) \qquad \omega \in C_0^\infty(x, \Omega_{1/2})$$
$$= \left(\tau^m p_m(x, d_x\varphi) - \frac{i\tau^{m-1}}{2}\sum \frac{\partial^2 p}{\partial \xi_j \partial \xi_k}(x, d_x\varphi)\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right)\omega$$
$$+ \frac{\tau^{m-1}}{i}\sum \frac{\partial p_m}{\partial \xi_j}(x, d_x\varphi)\frac{\partial \omega}{\partial x_j} + \tau^{m-1} p_{m-1}$$
$$(\text{mod. } \tau^{m-2})$$

$$e^{-i\tau\phi} P(e^{i\tau\phi}\omega) = \tau^m (P_m(x, d_x\varphi))\omega + \tau^{m-1} P_{m-1}(x, d_x\varphi)\omega$$
$$- \frac{1}{2i} \tau^{m-1} \sum \frac{\partial^2 P_m}{\partial \xi_j \partial x_j} \omega - i \tau^{m-1} \mathcal{L}_v \omega \mod \tau^{m-2}.$$

$\blacksquare$

Composition

Theorem X, Y, Z C^∞ manifolds, $A_i \in I^{m_i}(X, Y, C_i)$

C_i : canonical relat^{ns}.

C_i homog. Lagrangians w.r.t.

$\omega_{T^*(X)} - \omega_{T^*(Y)}$.

$k_{A_i} \in I^{m_i}(X \times Y, C_i)$

Assume

(a) proj^n from $\operatorname{supp}(k_{A_1} \times k_{A_2}) \cap X \times (\operatorname{diag} Y) \times Z$ $\to X \times Z$ is proper.

(b) $\eta \neq 0$ if $((x, \xi), (y, \eta)) \in C_1$ or $((y, \eta), (z, \zeta)) \in C_2$

(c) $C_1 \circ C_2 \subseteq (T^*X \times T^*Y) \setminus \{0\}$.

(d) $C_1 \times C_2$ intersects $T^*X \times (\operatorname{diag} T^*Y) \times T^*Z$ transversally.

Then $A_1 \circ A_2 \in I^{m_1+m_2}(X, Z, C_1 \circ C_2)$

$$(\Omega_{1/2} \otimes \mathcal{L}_{C_1}) \times (\Omega_{1/2} \otimes \mathcal{L}_{C_2}) \to (\Omega_{1/2} \otimes \mathcal{L}_{C_1 \circ C_2}).$$

Theorem (L^2-estmates).

$A \in I^0(X, Y, C)$, C: local canonical graph.

Then

$$A : L^2_{comp}(Y, \Omega_{1/2}) \to L^2_{loc}(X, \Omega_{1/2})$$

Proof $\langle Au, v\rangle = \langle u, A^* v\rangle$, $u, v \in C^\infty(X, \Omega^c_{1/2})$

$$Au(x) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(y)\, dy\, dy$$

$$\in S^{0 + n/2 - N/2}$$

Check: $A^* u(y) = \int e^{-i\phi(x,y,\theta)} \overline{a(x,y,\theta)}\, v(x)\, dx\, d\theta$

$$A^* A \in I^0(X \times X, C^* \circ C) \qquad C^* = C^{-1}$$

$$\in I^0(X \times X, \Delta)$$

Then, $A^* A \in I^0(X)$

$$A^* A : L^2_{comp}(X, \Omega_{1/2}) \to L^2_{loc}(X, \Omega_{1/2})$$

$$\langle A^* A u, u\rangle = \|Au\|^2_{L^2}$$

The Cauchy Problem for Strictly Hyperbolic Eq^n's.

Model Problem.

(X, g) : Riemannian manifold

Δ_g Laplace-Baltrami operator.

In local coords, $g = g_{ij}$

$$\Delta_g u = \sum \sqrt{\det g}\, \frac{\partial}{\partial x_i}\left(g^{ij} \frac{1}{\sqrt{\det g}} \frac{\partial u}{\partial x_j}\right)$$

$$= g^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + (\text{first order})u$$

Try to find E so that

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_g\right) E = 0$$

$$E\big|_{t=0} = 0$$

$$\frac{\partial E}{\partial t}\Big|_{t=0} = I$$

$$E : C_0^\infty(X) \longrightarrow C^\infty(\mathbb{R} \times X).$$

First, we try to solve modulo smoothing :

$$(\partial_t^2 - \Delta_g) E = \text{smoothing}$$

$$E|_{t=0} = \text{"}$$

$$\frac{\partial E}{\partial t}\Big|_{t=0} = I \quad \text{mod. smoothing}.$$

Such an E will be called a <u>parametrix</u> for the Cauchy problem for $P = (\partial_t^2 - \Delta_g)$.

Write $(\partial_t^2 - \Delta_g) = (\frac{1}{i}\partial_t - \sqrt{-\Delta_g})(\frac{1}{i}\partial_t + \sqrt{-\Delta_g})$

We consider $L = (\frac{1}{i}\partial_t - A(t,x,D_x))$ where
$A(t,x,D_x) \in \Psi^1_{cl}(X)$ $\forall t$
& depends smoothly on t.

So find $LE = 0$ mod. smoothing.
$E|_{t=0} = I$

Then to solve $(\partial_t^2 - \Delta_g)u = 0$
$u|_{t=0} = u_0$
$\partial_t u|_{t=0} = u_1$,

$(\frac{1}{i}\partial_t - \sqrt{-\Delta_g})(\frac{1}{i}\partial_t + \sqrt{-\Delta_g})u = 0$.

Solve $(\frac{1}{i}\partial_t - \sqrt{-\Delta_g})v = 0$
$v|_{t=0} = \tilde{u}_0$

Then solve $(\frac{1}{i}\partial_t + \sqrt{-\Delta_g})w = v$ (Duhamel's Princ)
$w|_{t=0} = \tilde{u}_1$

Then $Pw = 0$ &
$w|_{t=0} = \tilde{u}_1 \triangleq u_0$
$\partial_t w|_{t=0} = -\frac{1}{i}\sqrt{-\Delta_g}\, w|_{t=0} + \tilde{u}_0$

$= i\sqrt{-\Delta_g}\, u_0 + \tilde{u}_0$
$= u_1$

$L = D_t - A(t,x,D_x)$

If $X = \mathbb{R}^n$;

Ansatz: $Ef = \int e^{i\varphi(t,x,\xi)} e(t,x,\xi)\hat{f}(\xi)\,d\xi$

Then $LEf = \int e^{i\varphi(t,x,\xi)} e^{-i\varphi(t,x,\xi)} L(e^{i\varphi}e)\hat{f}(\xi)d\xi$.

But $e^{-i\varphi} L(e^{i\varphi}e)$

$$= \left(\frac{\partial\varphi}{\partial t} e + \frac{\partial e}{\partial t}\right) - e^{-i\varphi} A(e^{i\varphi}e)$$

Now

$$e^{-i\varphi}A(e^{i\varphi}e) = a_1(t,x,d_x\varphi)e + a_0\,(t,x,d_x\varphi)e$$
$$+ \frac{1}{i}\sum_j \frac{\partial a_1}{\partial\xi_j}(t,x,d_x\varphi)\frac{\partial e}{\partial x_j} - \frac{i}{2}\sum \frac{\partial^2 a_1}{\partial\xi_j\partial\xi_k}(t,x,d_x\varphi)\frac{\partial^2\varphi}{\partial x_j\partial x_k}e$$

mod. S^{m-2}

see p 95/96

Clearly $e \in S^0_{cl}$ since $E|_{t=0} = I$.

Then $\left(\frac{\partial\varphi}{\partial t} - a_1(t,x,d_x\varphi)\right)e_0 = 0$

$e_0|_{t=0} = 1$

note: we know a_1 must determine e & φ.

$$\Rightarrow \frac{\partial\varphi}{\partial t} - a_1(t,x,d_x\varphi) = 0$$

$l(t,x,\tau,\xi) = \tau - a_1(t,x,\xi)$ mod S^0

$l_1(t,x,d_t\varphi,d_x\varphi) = 0$ the Eikonal eq$^{\underline{n}}$!

Next,

$$\frac{\partial e_0}{\partial t} - a_0(t,x,d_x\varphi)e_0 - \frac{1}{i}\sum\frac{\partial a_1}{\partial\xi_j}(t,x,d_x\varphi)\frac{\partial e_0}{\partial x_j} + \tilde{C}e = 0$$

& $e_0|_{t=0} = 1$

i.e. $Ve_0 + \tilde{\tilde{C}}e_0 = 0$ $\quad V$ a vector field.

$e_0|_{t=0} = 1$

etc.

So in $\mathbb{R}^n$, $LE = 0$ mod. smoothing

$E|_{t=0} = I$ " "

(locally near $t=0$, $x = x_0$)

with $E \in I^{\frac{1}{4}}(\mathbb{R}^{n+1}, \mathbb{R}^n, \Lambda_{\tilde{\ }})$

$= 0 - \frac{n}{2} + \frac{2n+1}{4}$ $\quad \varphi(t,x,\xi) - \langle y, \xi \rangle$

———— " ————

Now X: C^∞ manifold, $L = D_t - A(t,x,D_x)$.

Try:

$E \in I^m(\mathbb{R}\times X, X, C)$ C a canonical rel^{n}.

To determine: C, m, $\sigma_m(E)$

What is C?

$\ell_1(t,x,d_t\varphi, d_x\varphi) = 0$ in $\mathbb{R}^n$ (or locally)

$C \subset T^*(\mathbb{R}\times X \times X) \setminus \{0\}$.

Define

$\tilde{\ell}_1(t,x,\tau,\xi,y,\eta) = \ell_1(t,x,\tau,\xi)$

$\tilde{\ell}_1(t,x,d_t\varphi, d_x\varphi, y, \eta) = 0$

so $\tilde{\ell}_1$ must vanish on C

$\Rightarrow$ $H_{\tilde{\ell}_1}$ tangent to C.

We also want $E|_{t=0} = \mathrm{Id}$.

i.e. $RE = \mathrm{Id}$. R = restrict^{n} to $t=0$.

$Ru(x) = u(0,x) \in I^{m'}(X, \mathbb{R}\times X, \mathcal{R})$

Thus $\mathcal{R}\circ C = \Delta_{T^*X\times T^*X}$

<u>Answer</u> $C = \{(t,x,\tau,\xi,y,\eta)$:

$(t,x,\tau,\xi) \in$ null bich. curve of H_{ℓ_1}

thru' $(0,y,a_1(0,y,\eta),\eta)\}$

<u>Check</u> $\quad R : C_0^\infty(X \times \mathbb{R}) \to C_0^\infty(X)$

$$Ru(x) = u(0,x)$$

$X = \mathbb{R}^n$ say,

$$u(t,x) = \int \frac{1}{(2\pi)^n} e^{i(t\tau + x\cdot\xi)} \hat{u}(\tau,\xi)\, d\xi\, d\tau.$$

$$Ru(x) = u(0,x) = \frac{1}{(2\pi)^n}\int e^{i(\langle x-y,\xi\rangle - s\tau)} u(s,y)\, ds\, dy\, d\xi\, d\tau$$

Canonical rel^n of this is:

$$d_\xi = 0 \iff x = y$$
$$d_\tau = 0 \iff s = 0.$$

$$\Rightarrow \mathcal{R} = \{((x,\xi),(s,y,\tau,\xi)) ;\ x = y,\ s = 0\}$$

$$k_R(x,s,y) = \int e^{i(\langle x-y,\xi\rangle - s\tau)} d\xi\, d\tau.$$

$$\Rightarrow \quad \mathcal{R} = \{((x,\xi),(0,x,\tau,\xi))\}$$

Now $\mathcal{R}\circ C = \{(x,\xi,y,\eta) : \exists\ (x,\xi,(s,z),(\tau,\zeta)) \in \mathcal{R}$
w/ $((s,z),(\tau,\zeta),(y,\eta)) \in C\}$

$$\Rightarrow z = x,\ s = 0,\ \zeta = \xi$$

$$= \{((x,\xi),(y,\eta)) : (0,x,\tau,\xi),(y,\eta) \in C\}$$

need $(0,x,\tau,\xi)$ in same null bich. as
$(0,y,a,(0,y,\eta),\eta)$.

$\Rightarrow y = x$ & $\tau = a$, & $\eta = \xi$ since both @ $t=0$.

$\Rightarrow$ get diagonal

(4/16) Last time, locally we solved

$$Ef(t,x) = \int e^{i\varphi(t,x,\xi)} e(t,x,\xi)\hat{f}(\xi)\,d\xi$$

$$e \in S^0_{cl}(\mathbb{R} \times X \times \mathbb{R}^n)$$

$$\ell(t,x,d_t\varphi, d_x\varphi) = 0$$

<u>Guess</u>: $E \in I^{(?)}(X \times \mathbb{R}, X, C)$ for some canonical relation C.

We want $\mathcal{R} \circ C = \Delta$

where $\mathcal{R}$ = canonical rel^n associated to $Ru(x) = u(0,x)$.

C is invariant under $H_{\tilde{\ell}}$

where $\tilde{\ell}(t,x,y,\tau,\xi,\eta) = \ell(t,x,\tau,\xi)$

$$C = \{ ((t,x,\tau,\xi),(y,\eta)) \in (T^*(\mathbb{R} \times X) \times T^*(X)) \setminus \{0\} : (t,x,\tau,\xi) \text{ in null bichar. of } H_\ell \text{ passing through } (0,y,a(0,y,\eta),\eta) \}$$

<u>To check</u>: C is an embedded Lag. submanifold

(a) of $T^*(\mathbb{R} \times X) \times T^*(X) \setminus \{0\}$; imposed cond^{n}s on ℓ

(b) Since $LE = 0$, $\sigma_{(?)}(LE) = 0$

$$0 = \mathcal{L}_{H_{\tilde{\ell}}} \sigma(E) + c\,\sigma(E) \qquad c = \text{sub princ. symbol of } \tilde{L}$$

& $\sigma(E)\big|_{t=0} = \sigma(Id)$; i.e. $\sigma(RE) = \sigma(Id) = 1$

Is this globally solvable?

—————— u ——————

Let X be a C^∞-manifold, & L a v.f. on X. Under what cond^{n}s is L globally solvable?

<u>Theorem</u> (D-H, FIO II) (Acta Math. '71).
(Semiglobal solvability).
Let $K \subset\subset X$ compact. TFA Equiv.

(a) $L\, C^\infty(K) = C^\infty(K)$

(b) $(L+a)\, C^\infty(K) = C^\infty(K) \quad \forall\, a \in C^\infty(K)$

(c) (Pseudo convexity) $\exists\, \varphi \in C^\infty(X)$ s.t.
$L^2\varphi > 0$ in K

(d) No complete integral curve of L is contained in K.

<u>Proof</u> (a) $\Rightarrow$ (b) Let f be a sol^n of $Lf = a$; Let g be given.

$$(L+a)(e^{-f}\omega) = e^{-f}(-(Lf)\omega + L\omega + a\omega)$$

$$= e^{-f} L\omega \quad \text{since } Lf = a.$$

Thus
$$(L+a)e^{-f} = e^{-f}L.$$

So first solve $L\omega = e^{f} g$
$\Rightarrow (L+a)e^{-f}\omega = g.$

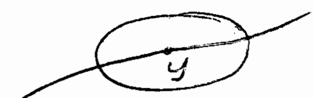
(b) $\Rightarrow$ (c). Solve $Lf = 1$ & $L\varphi = f \Rightarrow L^2\varphi = 1$.

(c) $\Rightarrow$ (d) Suppose $\exists$ complete int. curve γ remaining in K $\forall$ time.

Consider $\max_{\gamma} \varphi = \varphi(x_0)$; but then

$L^2\varphi(x_0) < 0$ $\times$.

(d) $\Rightarrow$ (a). <u>Claim</u>: (d) $\Leftrightarrow$ (d_1). No integral curve of L is contained in K for all pos. or neg. values of the parameter.
Therefore, every $y \in K$ lies on an interval of integral curve whose end pts are not in K.



Let $f \in C^\infty(X)$ supported near y. Then we can solve $Lu = f$ in a nbhd of K. Now use a partition of unity.

(d) $\Rightarrow$ (d_1) Let $y_0 \in K$ be a limit point of a complete half integral curve of L
Then $L(y_0) = 0$ since otherwise γ continues in both dir^{n}s.
Then int. curve thru' y_0 remains in K i.e. is $\{y_0\}$. $\square$

Th^m (<u>Global Solvability</u>) X: C^∞-manifold, L vector field,
Then the following are equiv:

(a) $L C^\infty(X) = C^\infty(X)$

(b) $(L+a) C^\infty(X) = C^\infty(X)$ $\forall$ $a \in C^\infty(X)$

(c) $\exists$ $\varphi \in C^\infty(X)$: $L^2\varphi > 0$ & $\forall c$, & $\{y \in X : \varphi(y) \le c\}$ compact

(d)

(d_1) No complete integral curve of L is contained in a compact set of X

(d_2) $\forall\ K \subset\subset X$ K compact, $\exists\ K' \subset\subset X$, K' compact s.t. every compact interval on an interval curve with end points in K is contained in K'.

(e) there are no periodic integral curves of L and
$$R = \{(x_1, x_2) \in X \times X : x_1, x_2 \in \text{same integral curve of } L\}$$
is a closed C^∞ submanifold of $X \times X$

(f) $\exists$ a manifold X_0, an open nbhd X_1 of $X_0 \times \{0\}$ in $X_0 \times \mathbb{R}$ & a diffeo
$$X \to X_1$$
w/
$$L \longmapsto \frac{\partial}{\partial t}$$
The points in $X_0 \times \mathbb{R}$ are denoted by (x_0, t).

(4/17) Proof

(a) $\Rightarrow$ (b) as before.

(a) $\Rightarrow$ (d_1) by last th$\underline{\text{m}}$.

(a) $\Rightarrow$ (d_2) Suppose not (d_2). Then $\exists$ intervals $[t_j', t_j'']$ w/ $\gamma(t_j'), \gamma(t_j'') \in K$ & $y_j = \gamma(t_j)$, $t_j \in [t_j', t_j'']$ & $\nexists\ K' \subset\subset X$ containing all the y_j. Taking a subseq.,
$$y_j' = \gamma(t_j') \to y'$$
$$y_j'' = \gamma(t_j'') \to y''$$
& that every compact set contains only a finite number of the y_j.

Let $f \in C^\infty(X)$, $f \geq 0$ so that $f(y_j) \to \infty$ as $j \to \infty$.

Let $u \in C^\infty(X)$ solve $Lu = f$. Then

$$u(y_j'') - u(y_j') = \int_{t_j'}^{t_j''} f(\gamma(t))\,dt \to \infty$$

$$\downarrow$$

$$u(y') - u(y''), \qquad \otimes.$$

(c) $\Rightarrow$ (d_1) by last th$^{\underline{m}}$.

(c) $\Rightarrow$ (d_2) Same situation: $y_j' = \gamma(t_j')$, $y_j'' = \gamma(t_j'')$

$y_j', y_j'' \in K$. Then

$$\varphi(y_j') \,\&\, \varphi(y_j'') \leq C$$

Claim:

$$\varphi(\gamma(t)) \leq C \quad \text{for} \quad t \in [t_j', t_j''].$$

Since φ is convex along γ ($L^2\varphi > 0$), the integral curve.

Now use $\{x \in X : \varphi(x) \leq C\}$ is compact, & (c).

(d) $\Rightarrow$ (e) clearly no periodic int-curves.

Let $\varphi(x, t)$ denote the flow of L.

i.e. $\dfrac{d\varphi}{dt} = L\varphi$, $\quad \varphi(0, x) = x$.

Let $D\varphi$ be the domain of φ. Then

$$R = \{(\varphi(x,t), x) : (x,t) \in D\varphi\}$$

Consider $\quad (x, t) \xmapsto{F} (\varphi(x,t), x)$

1st, F is injective: $(\varphi(x_1, t_1), x_1) = (\varphi(x_2, t_2), x_2)$

$\Rightarrow x_1 = x_2$ so $\varphi(x_1, t_1) = \varphi(x_1, t_2)$

$\Rightarrow t_1 = t_2$ since $\nexists$ periodic int-curves.

2nd, F is proper. $\{(x_i, t_i)\} \subset D\varphi$, $x_i \to x$,

$\varphi(x_i, t_i) \to y$ To check, $t_i \to t$.

By taking a subsequence $t_i \to T \in [-\infty, \infty]$

We must eliminate $T = \pm\infty$. But this contradicts

(d_2) ; $\exists$ comp. set K' s.t. $\varphi(x_i, t)$, $\forall t \in [0, t_i]$.
if $t_i \to \infty$, then $\varphi(x, s) \in K'$ $\forall s > 0$; then this contradicts (d_1).

3rd dF injective:

$$dF = \begin{pmatrix} D_x\varphi & \frac{\partial\varphi}{\partial t} \\ I & 0 \end{pmatrix} = \begin{pmatrix} D_x\varphi & L\varphi \\ I & 0 \end{pmatrix} \qquad \frac{\partial\varphi}{\partial t} = 0$$

(4/24)

(e) $\Rightarrow$ (f) Define $X/\sim$ by $x \sim y$ if in same int. curves of L.
$X/\sim$ is a manifold (!) $= X_0$ (since R is closed)

Claim: $\exists$ global C^∞ section

$$\begin{aligned} X_0 &\longrightarrow X \\ [x] &\longmapsto s(x) \end{aligned}$$

idea: first construct a continuous section (Steenroad, Topology of Fiber Bundles), then modify the section to obtain a smooth section.

Then $X_1 = \{\varphi(s(x)), t)$ & $L \to \partial/\partial t$.

Rest of implications left as exercise. $\square$

Recall, we want

$$(D_t - A(t, x, D_x))E = 0$$

$$E|_{t=0} = \mathrm{Id}$$

$$E \in I^{-1/4}(\mathbb{R} \times X, X, C)$$

$C = \{(t, x, \tau, \xi), (y, \eta) \mid (t, x, \tau, \xi)$ in same null bichar. curve as $(0, y, a(0, y, \eta), \eta)\}$.

- Conditions for C to be a closed C^∞ submanifold of $T^*(\mathbb{R}\times X)\times T^*(X)\setminus 0$:

(a) - no proj$^{\underline{n}}$ of null-bicharacteristic starting on X $(=\{0\}\times X)$ stays in a compact subset of $\mathbb{R}\times X$.

(b) - for every compact set K_0 in X $(=\{0\}\times X)$ there is a compact set K' in $\mathbb{R}\times X$ such that if γ is an interval of a proj$^{\underline{n}}$ of a null-bichar. curve with one end point in K_0 & another in $K \subset\subset \mathbb{R}\times X$, then the whole proj$^{\underline{n}}$ of the null-bichar. is contained in K'

(c) - In order to guarantee that $C \to T^*(\mathbb{R}\times X) \xrightarrow{\pi} \mathbb{R}\times X$ proper, for every compact $K\subset \mathbb{R}\times X$ $\exists$ a compact subset $K_0 \subset X$ s.t. every proj$^{\underline{n}}$ of bich. curves starting in K only hits X in K_0.

$(K_0 \Subset \mathbb{R}\times X$ compact, $\pi^{-1}(K_0)\cap C$ compact
$= \{(t,x,\tau,\xi),(y,\eta) \mid (t,x,\tau,\xi) \in$ same null-bichar. as $(0,y,a,(0,y,\eta),\eta)$ & $(y,\eta)\in K_0\})$

(d) Every null-bich. intersects $\{0\}\times X$ exactly once.

Claim: C is a Lagrangian submanifold of $T^*(\mathbb{R}\times X)\times T^*(X)\setminus 0$

$\mathcal{R}$ = canonical rel$^{\underline{n}}$ associated to rest$^{\underline{n}}$
$\subseteq T^*X\times T^*(\mathbb{R}\times X)\setminus\{0\}$.

$\mathcal{R} = \{(x,\xi),(0,x,\tau,\xi)\}$, a canonical rel$^{\underline{n}}$.

C = flow out along integral curves of $H_{\tilde{\ell}}$ from $\Lambda_0 = \{\ell(t,\tau,x,\xi)=0\}\times T^*(X) \cap \mathcal{R}^{-1}$

R^{-1} is Lagrangian, so Λ_0 is isotropic
$\therefore$ the flow-out is a Lagrangian manifold.

Now we find $\sigma(E) \in S(C; \Omega_{1/2} \otimes \mathcal{L}_C)$.

$$LE = 0$$
$$E|_{t=0} = I \qquad (\sigma_1(L) = \ell = 0 \text{ on } C)$$

$$\sigma(E) = \frac{1}{i}\mathcal{L}_{H_{\tilde{\ell}}}\sigma(E) + \tilde{C}\sigma(E)$$

$$\tilde{C}(t, x, \tau, \xi, y, \eta) = c(t, x, \tau, \xi)$$
= subprincipal symbol of L.

To solve: $\frac{1}{i}\mathcal{L}_{H_{\tilde{\ell}}}\sigma(E) + \tilde{C}\sigma(E) = 0$, $\sigma(E)|_{t=0} = 1$

$\mathcal{L}_C$ is a trivial bundle over C
The Global solvability th$^{\underline{m}}$ guarantee global solvability of this.

$E = E_0$,
$F_0 = LE_0 \in I^{-1/4-1}(\mathbb{R}\times X, X, C)$
& $RE_0 = Id + K_{-1}$, $K_{-1} \in \Psi^{-1}(X)$.

Next, $\sigma(L(E_0 + E_{-1})) = 0$
$$\Rightarrow \sigma(F_0) + \sigma(LE_{-1}) = 0$$
so solve
$$\frac{1}{i}\mathcal{L}_{H_{\tilde{\ell}}}\sigma(E_{-1}) + \tilde{C}\sigma(E_{-1}) = -\sigma(F_0)$$
$$\sigma(E_{-1})|_{t=0} = -\sigma(K_{-1}).$$

etc...

Proposition.

Under the cond^{n}s (a), (b), (c) above, $\exists\ E \in I^{-1/4}(\mathbb{R}\times X, X, C)$ such that $LE = 0$ modulo smoothing & $E|_{t=0} = \mathrm{Id}$. " "

Now we try to do the same for more general eq^{n}s.

Let X be a C^∞-manifold,

$P \in \Psi^m_{cl}(\mathbb{R}\times X)$

$p_m(t, x, \tau, \xi)$ the principal symbol.

Def^n P is called <u>strictly hyperbolic</u> if given $(t, x, \xi) \in \mathbb{R}\times(T^*X\setminus\{0\})$, the equation

$$p_m(t, x, \tau, \xi) = 0$$

has m different roots $\tau = \lambda_1(t, x, \xi), \dots, \tau = \lambda_m(t, x, \xi)$, all <u>real</u>.

Remark: $\lambda_i \in C^\infty(\mathbb{R}\times T^*X\setminus 0)$.

& homogeneous of degree 1 in ξ.

<u>Goal</u> Construct $E_k \in I(\mathbb{R}\times X, X, C)$ so that

$$PE_k = 0$$

$$\left(\frac{\partial}{\partial t}\right)^{m-j} E_k = \delta_{jk}(\mathrm{Id})$$

Let $N = \{(t, \tau, x, \xi) \overset{\xi\neq 0}{:} p_m = 0\}$

$$= \bigcup_{i=1}^m \{\tau = \lambda_i(t, x, \xi)\}_{\xi\neq 0} = \bigcup_{i=1}^m N_i$$

and $N_i \cap N_j = \emptyset$

Guess: $C = \bigcup_{i=1}^{m} \{ (t,x,\tau,\xi),(y,\eta) \mid (t,x,\tau,\xi)$ in same null-bichar. of H_{ℓ_i} as $(0,y,\lambda_i(0,y,\eta),\eta)$ where $\ell_i = \tau - \lambda_i(t,y,\eta) \}$

Write $E_k = \sum_{j=1}^{m} E_{kj}$, $E_{kj} \in I(\mathbb{R}\times X, X, C_j)$

(4/25)

Solve $PE_k = 0$:

$$(\mathcal{L}_{H_{\tilde{p}_m}} + \tilde{c})\,\sigma(E_k) = 0$$

$$\tilde{p}_m(t,\tau,x,\xi,y,\eta) = p_m(t,\tau,x,\xi)$$
$$\tilde{c}(t,\tau,x,\xi,y,\eta) = c(t,\tau,x,\xi)$$

c = sub princ. of P

It suffices to solve

$$(\mathcal{L}_{H_{\tilde{\ell}_i}} + \tilde{\tilde{c}})\,\sigma(E_{kj}) = 0 \qquad \sigma(E_{k_j}) \in S(C_j, \Omega_{1/2}\otimes\mathcal{L}_{C_j})$$
$$\ell_i = \tau - \lambda_i(t,x,\xi)$$

$$H_{p_m} = H_{(\tau-\lambda_1)\cdots(\tau-\lambda_m)} = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_m)\,H_{\tau-\lambda_1}$$

on C_1

so $$\tilde{\tilde{c}} = \tilde{c} \times \frac{1}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_m)}.$$

$\Gamma^{\frac{1}{4}} \quad \Gamma^{m-s} \qquad \Gamma^{0} \quad \Rightarrow E_{ks} \in I^{-m+s-\frac{1}{4}}$

$$R\left(\frac{\partial}{\partial t}\right)^{m-s} E_k = \delta_{ks} Id$$

$$E_{kj} f(t,x) = \int e^{i\varphi_j(t,x,\xi)} e_j(t,x,\xi) \hat{f}(\xi)\, d\xi \quad \text{microlocally}$$

$$\frac{\partial \varphi_j}{\partial t} = \lambda_j(t,x,d_x\varphi_j)$$

$$\varphi_j(0,x,\xi) = \langle x, \xi \rangle$$

$$e_j(t,x,\xi) \sim \sum_{\ell \le 0} e_j^{(\ell)}(t,x,\xi)$$

$\uparrow$ hom. of deg. ℓ

Goal

Compute $\sigma\left(R\left(\frac{\partial}{\partial t}\right)^{m-s} E_{kj}\right)$

$$\left(\frac{\partial}{\partial t}\right)^{m-s} E_{kj} f$$

$$= \int \left(i \frac{\partial \varphi_j}{\partial t}\right)^{m-s} e^{i\varphi_j} e_j \hat{f}(\xi)\, d\xi \quad + \text{ lower order}$$

$$R\left(\frac{\partial}{\partial t}\right)^{m-s} E_{kj} f = \int \left(\left(i\frac{\partial \varphi_j}{\partial t}\right)^{m-s} e_j\right)\Big|_{t=0} e^{i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi$$

$+$ lower.

But $\left(\left(i\frac{\partial \varphi_j}{\partial t}\right)^{m-s} e_j\right)\Big|_{t=0} = (i\lambda_j(0,x,\xi))^{m-s} e_j(0,x,\xi)$

We claim
$$\sigma\left(R\left(\frac{\partial}{\partial t}\right)^{m-s} E_{kj}\right) = (i\lambda_j(0,x,\xi))^{m-s} e_j^{(0)}(0,x,\xi)$$

We want
$$\sum_{j=1}^{m} \sigma\left(R\left(\frac{\partial}{\partial t}\right)^{m-s} E_{kj}\right) = \delta_{sk}.$$

$$\sum_{j=1}^{m} (i\lambda_j(0,x,\xi))^{m-s} e_j^{(0)}(0,x,\xi) = \delta_{sk}$$

for $s = 1, \dots, m$. (remember $e_j^{(0)} \approx e_{kj}^{(0)}$).

i.e.

$$\begin{bmatrix} (i\lambda_1)^{m-1} & \cdots\cdots & (i\lambda_m)^{m-1} \\ (i\lambda_1)^{m-2} & \cdots & (i\lambda_m)^{m-2} \\ \vdots & & \\ 1 & \cdots\cdots & 1 \end{bmatrix} \begin{bmatrix} e_1^{(0)}(0,x,\xi) \\ \vdots \\ e_m^{(0)}(0,x,\xi) \end{bmatrix} = \text{“}\delta\text{”}$$

(at $(0,x,\xi)$)

det. of this $= \Pi$ of $(i\lambda_s - i\lambda_t)$ $s \neq t$

$\neq 0$.

So here we are using the strict hyperbolicity

$R \in I^{1/4}(X, \mathbb{R}\times X, \mathbb{R})$

$R\left(\frac{\partial}{\partial t}\right)^{m-s} E_k \in \mathcal{I}^0(X)$.

$$E_k = \sum_j E_{kj} \quad , \quad E_k \in I^{-(m-k)-\frac{1}{4}}(\mathbb{R}\times X, X, C).$$

(inductively continue).

So, we have

$$PE_k = 0 \qquad \text{mod. smoothing.}$$

$$R\left(\frac{\partial}{\partial t}\right)^{m-j} E_k = \delta_{jk}\, Id.$$

$$PE_k = R_k \in \Psi^{-\infty}$$

$$R\left(\frac{\partial}{\partial t}\right)^{m-j} E_k = \delta_{jk}\, Id + \tilde{R}_{kj}, \quad \tilde{R}_{kj} \in \Psi^{-\infty}.$$

Suppose $\begin{pmatrix} u_m \\ \vdots \\ u_1 \end{pmatrix} \in (\mathcal{E}'(X))^m$

We want to solve

$$(*) \quad \begin{cases} Pu = 0 \\ u|_{t=0} = u_m \\ \quad\vdots \\ \partial_t^{m-1} u|_{t=0} = u_1 . \end{cases}$$

Take $\displaystyle\sum_{k=1}^{m} E_k(u_k)$.

Then $P(\sum E_k(u_k)) = f \in C^\infty$

$$\sum E_k(u_k)\big|_{t=0} = \sum (RE_k)(u_k) = u_m + V_m$$
$$V_m \in C^\infty$$

etc.

Now using regularity theorem for strictly hyperbolic equations, $\exists!\, \omega \in C^\infty$ such that

$$P\omega = -f$$
$$\& \quad \left(\frac{\partial}{\partial t}\right)^{m-j} \omega = V$$

<u>Therefore</u> Unique sol^n to problem $(*)$

$$u = \sum_{k=1}^{m} E_k(u_k) \quad \text{modulo } C^\infty$$

In particular,

$$WF u \subset \bigcup_{k=1}^{m} C_k \circ WF(u_k).$$

We now take P strictly hyperbolic diff. operator. (We assume P is strictly hyperbolic to $\{s\} \times X$ $\forall s$.)

To solve

$$Pu = f$$
$$\left(\frac{\partial}{\partial t}\right)^{m-j} u = u_j \in \mathcal{D}'(X)$$

Use Duhamel's principle.

Recall:

$$Pu = f$$
$$u|_{t=0} = 0 \qquad P \text{ second order.}$$
$$\partial_t u|_{t=0} = 0$$

Solve

$$Pv(t,s,x) = 0$$
$$v|_{t=s} = 0$$
$$\frac{\partial v}{\partial t}\Big|_{t=s} = f(s,x)$$

& put

$$u(t,x) = \int_0^t v(t,s,x)\,ds$$

$$\left(\frac{\partial}{\partial t} \int_0^t f(s,t)\,ds = f(t,t) + \int_0^t \frac{\partial}{\partial t} f(s,t)\,ds.\right)$$

As before we construct $E_k(s)$ so that

$P E_k = 0$ modulo C^∞

$$R_s\left(\left(\tfrac{\partial}{\partial t}\right)^{m-j} E_k(s)\right) = \delta_{jk}\, Id + R_{jk}^{(s)} \in C^\infty$$

$$\left(R_s\, u(x) = u(s,x)\right)$$

We must show $E_k^{(s)}$ depends smoothly on s.(!)

Then consider

$$PEf(t,x) = \int_0^t E_1(s) f(s,x)\, ds \qquad f \in C^\infty(\mathbb{R} \times X)$$

$$PEf = \int_0^t PE_1(s) f(s,x)\, ds + f(t,x)$$

$$\& \quad R_0\, \partial_t^j E = 0 \qquad j = 0, \cdots, m-1.$$

(1/5) Define $\tilde{E}_k^{(s)} = E_k^{(s)} - \sum_{j=1}^{m} \frac{(t-s)^{m-j}}{(m-j)!} R_{jk}(s)$.

$P\tilde{E}_k = 0$ modulo C^∞ still,

& now $R_s\left(\left(\tfrac{\partial}{\partial t}\right)^{m-j} \tilde{E}_k(s)\right) = \delta_{jk}\, Id$

E_k & $\tilde{E}_k \in I^{-m+k-1/4}(\mathbb{R} \times X, X, C)$.

Now apply Duhamel's Principle ($\tilde{E}_k \to E_k$).

$$Ef(t,x) = \int_0^t \left(E_1(\tau) f(\tau, \cdot)\right)(t,x)\, d\tau \qquad f \in C^\infty \text{ say}$$

What then is PE?

$$PEf = f + Rf$$

$$R_s E f = \partial_t R_s E f = \cdots = \partial_t^{m-1} R_s E f = 0.$$

(for $E_s f = \int_s^t (E_1^{(\tau)} f(\tau, \cdot))(t,x)\,d\tau$.

Remember we want to solve

$$(*) \begin{cases} Pu = g \\ u|_{t=0} = u_m \\ \vdots \\ \partial_t^{m-1} u|_{t=0} = u_1. \end{cases}$$

If $u = \sum_{k=1}^{m} E_k(u_k) + E(g)$

then we get $Pu = g + Rg$, R smoothing & all initial cond^{n}s.

<u>Theorem</u>

Suppose $(u_m, \cdots, u_1) \in \mathcal{E}'(X)$, $g \in C^\infty(\mathbb{R}_t; \mathcal{E}'(X))$. Then the solution of $(*)$ can be written in the form

$$u = \sum_{k=1}^{m} E_k(u_k) + E(g) + v$$

where $v \in C^\infty(\mathbb{R}_t \times X)$

To remove the assumption of compact support, see (Chazarain - Piriou)

Operators of Real Principal Type
(§6 of Duistermaat - Hörmander FIO II)

<u>Def^n</u> Let X be a smooth manifold, $P \in \Psi^m_{cl}(X)$; then P is said to be of <u>real principal type</u> if no complete null-bichar. of H_{p_m} stays in a compact set; p_m $\mathbb{R}$-valued.

eg $\Box$, $\frac{\partial^2}{\partial t^2} - \Delta_g$, are RPT
But not $\Box^2$: $\quad p = \tau^2 - |\xi|^2$
$H_{p^2} = 0$ on $p = 0$.
nor $\left(\frac{\partial}{\partial t} - \Delta\right)$; $p = -|\xi|^2$ $\quad p = p(t, x, \tau, \xi)$
so if $\tau \neq 0$ & $\xi = 0$, $H_p \equiv 0$.

<u>Theorem</u> (Prop^n of sing's). Let P be of RPT, $u, f \in \mathcal{D}'(X)$ s.t.
$$Pu = f.$$
then $\quad WF\,u \setminus WF\,f$ is invariant under H_{p_m}

We proved this in the case that $d_\xi p \neq 0$ on $p = 0$.

Sketch of proof:
- First, prove theorem for a simple example $P = D_{x_n}$ in $\mathbb{R}^n$
- Second, reduce general case to $\frac{\partial}{\partial x_n}$ using Egorov's theorem ($BPA = D_{x_n}$ microlocally).

<u>Step 1</u> Prove for $P = D_{x_n}$.

$D_{x_n} u = f$

$D_{x_n} E_n^{\pm} = I$; f smooth, compact supp

$$E_n^+ f = i\int_{-\infty}^{x_n} f(s, x')\,ds$$

$$E_n^- f = -i\int_{x_n}^{\infty} f(s, x')\,ds$$

$$K_{E_n^+} = i\,\delta(x'-y')\,H(x_n - y_n)$$

$$K_{E_n^-} = -i\,\delta(x'-y')\,H(y_n - x_n)$$

$$K_{E_n^+ - E_n^-} = i\,\delta(x'-y')$$

$$(E_n^+ - E_n^-)f = \frac{i}{(2\pi)^{n-1}}\int e^{i\langle x'-y', \theta'\rangle}\,d\theta'$$

$$\phi(x, y, \theta') = \langle x'-y', \theta'\rangle$$

$$d_\theta \phi = 0 \iff x' = y'$$

$$\Lambda_\phi' = \{(x', x_n, \underset{\uparrow \atop d_{x'}\phi}{\theta'}, \underset{\uparrow \atop d_{x_n}\phi}{0}); (x', \underset{\uparrow \atop d_\theta \phi = 0}{y_n}, \theta', 0)\} \quad (\theta' \neq 0)$$

$$= C_n.$$

$$E_n^+ - E_n^- \in I^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n, C_n)$$

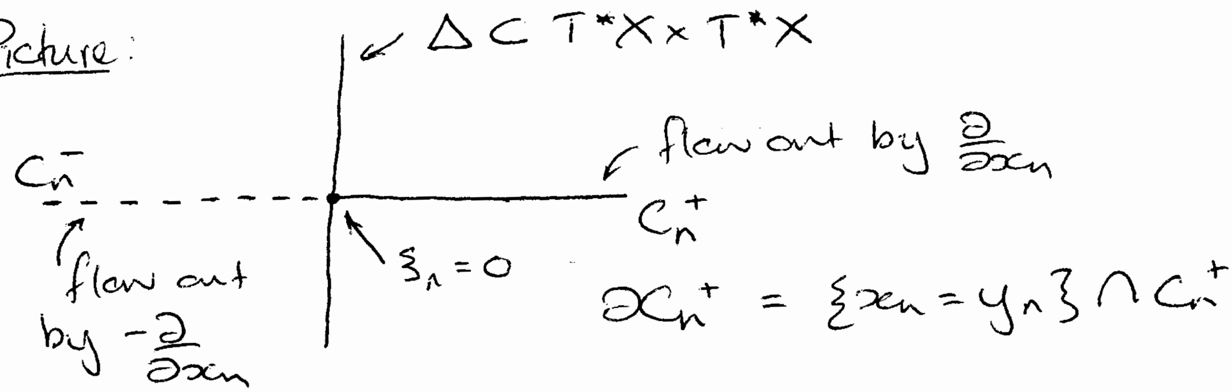
$$\text{since } \sigma(E_n^+ - E_n^-) = i\sqrt{2\pi}\,\underbrace{\sqrt{dx'\,dx_n\,dy_n\,d\theta'}}_{\in S^{\frac{n-1}{2}}}$$

Prop^n $WF'(E_n^\pm) = \Delta \cup C_n^\pm$

where

$$C_n^\pm = \{ (x', x_n, \xi', 0), (x', y_n, \xi', 0) : \begin{array}{l} y_n \leq x_n \ (+) \\ y_n \geq x_n \ (-1) \end{array} \}$$

Picture:



$\partial C_n^+ = \{x_n = y_n\} \cap C_n^+$

Remark (a) $E_n^\pm$ are non FIO's (WF certainly not a Lagrangian manifold)

(b) $C_n^\pm$ are Lag. manifolds with bdry.

Proof (of containment)

$$WF'(K_{E^+}) \subset \big(WF'(\delta(x'-y')) \cup WF(H(x_n - y_n)) \cup (WF'(\delta(x'-y')) + WF'(H(x_n-y_n))) \big) \cap \{x_n \geq y_n\} \cap \{x' = y'\}$$

$$WF'\,\delta(x'-y') = \{ (x', x_n, \xi', 0)(x', y_n, \xi', 0) \ \xi' \neq 0 \}$$

$$WF'\,H(x_n - y_n) = \{ (x', x_n, 0, \xi_n)(y', x_n, 0, \xi_n) \ \xi_n \neq 0 \}$$

$$WF'\,\delta(x'-y') + WF'\,H(x_n - y_n)$$
$$= \{ (x', x_n, \xi', \xi_n), (x', x_n, \xi', \xi_n) ; \xi' \neq 0, \xi_n \neq 0 \}$$
$$\in \Delta$$

supp $K_{E_n^+} \subset \{x_n \geq y_n\}$, supp $\delta = \{x' = y'\}$

Now $D_{x_n} E_n^+ = I$

i.e.

$$i\partial_{x_n} \delta(x'-y') H(x_n - y_n) = \delta(x-y)$$

Then

$$WF(D_{x_n} \quad) = WF'(\delta(x-y)) = \Delta$$

&

$$WF(D_{x_n} \quad) \subseteq WF' E_I$$

$$\Rightarrow \Delta \subseteq WF' E_I .$$

Thus the claim.
(exercise: equality). $\square$

We prove Prop$^{\text{n}}$ of Sing's for $P = D_{x_n}$.

Idea: $D_{x_n} u = f$, $\quad (x_0, \xi_0) \in WF u \setminus WF f$

$$E_\pm D_{x_n} u = E_\pm f \Rightarrow u = E_\pm f$$

$$\Rightarrow WF u = WF(E_\pm f)$$

?

Claim $D_{x_n} v = g$, $v \in \mathcal{E}'(\mathbb{R}^n)$

$\Rightarrow$ $WF\, v \subseteq (WF g \cup C_n^+ \circ WF g)$

$(v = E_\pm g)$ $\quad\quad \cap (WF g \cup C_n^- WF g)$

~~Proof. $D_{x_n} E_\pm g = g \Rightarrow D_{x_n}(v - E_\pm g) = 0$.~~

$D_{x_n} u = f \qquad (x_0, \xi_0) \in WF u \setminus WF f$.

Let $\varphi \in C_0^\infty$, $\varphi = 1$ near x_0, supported in a suff-small nbhd of x_0.

Then $(x_0, \xi_0) \notin WF(\varphi f)$. Let

$$v = \varphi u$$

$$D_{x_n} v = (D_{x_n}\varphi) u + \varphi (D_{x_n} u) = g \text{ say.}$$

$$= (D_{x_n}\varphi) u + \varphi f. \qquad \in \mathcal{E}'(\mathbb{R}^n)$$

~~$\Rightarrow (x_0, \xi_0) \notin WF(D_{x_n} v)$~~

~~but $(x_0, \xi_0) \in WF v$.~~

~~Assuming we know thm for compactly supp. v,~~

~~$(x_0, \xi_0) + t((0, \dots, 1), 0) \in WF u \quad |t| \leq \delta$.~~

(5/7) Claim $(x_0, \xi_0) \in WF u \setminus WF g$, clear.

$(D_{x_n}\varphi = 0 \text{ near } x_0)$.

$$D_{x_n} E_n^\pm = I, \quad \text{or} \quad (E_n^\pm)^* D_{x_n} = I$$

$$\left((E_n^\pm)^* = E_n^\mp\right)$$

Thus

$$\varphi u = E_n^\mp g$$

"$WF\Delta \circ WF g$"

$$WF(E_n^{\mp} g) \subset (WF g \cup \mathring{C}_n^{+} \circ WF g) \cap (WF g \cup \mathring{C}_n^{-} \circ WF g)$$

$$\mathring{C}_n^{\pm} = C_n^{\pm} \setminus \partial C_n^{\pm}$$

$$WF(\varphi u) = WF(E_n^{\mp} g)$$

Thus $(x_0, \xi_0) \in (WF g \cup \mathring{C}_n^{+} \circ WF g) \cap (WF g \cup \mathring{C}_n^{-} \circ WF g)$

We know $(x_0, \xi_0) \notin WF g$ so

$$(x_0, \xi_0) \in (\mathring{C}_n^{+} \circ WF g) \cap (\mathring{C}_n^{-} \circ WF g)$$

Also,

$$(x_0, \xi_0) \notin WF(\varphi (D_{x_n} u))$$

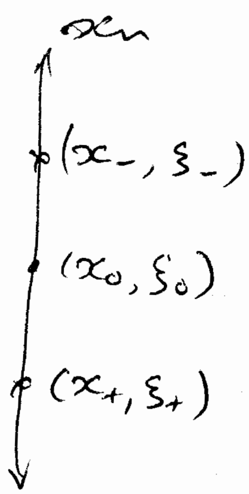
$\Rightarrow$

$$(x_0, \xi_0) \in (\mathring{C}_n^{+} \circ WF((D_{x_n}\varphi) u)) \cap (\mathring{C}_n^{-} \circ WF((D_{x_n}\varphi) u))$$

$$WF((D_{x_n}\varphi) u) \subset WF u$$

$$\Rightarrow \quad (x_0, \xi_0) \in (\mathring{C}_n^{+} \circ WF u) \cap (\mathring{C}_n^{-} \circ WF u)$$

i.e. $\exists \ (x_\pm, \xi_\pm) : (x_0, \xi_0) \in$ forward/backward flow out from $(x_\pm, \xi_\pm)$



$x_\pm \in \operatorname{supp} D_{x_n} \varphi$

Since $WF u$ is closed, we get complete integral curve of $D_{x_n} \subseteq WF u$.
(as long as it does not intersect $WF f$).

Now for the general case.

$Pu = f$, $\quad P \in \Psi^m_{cl}(X)$, p_m $\mathbb{R}$-valued

$(x_0, \xi_0) \in WFu \setminus WFf$

$\Rightarrow$ int. curve thru $(x_0,\xi_0) \in WFu$ so long as it does not intersect WFf.

i.e. $WFu \setminus WFf$ is invariant under H_{p_m}.

We can assume WLOG that $m=1$ & P is properly supported.

$H_p(x_0,\xi_0)$ & the cone-axis are lin. ind.

$\quad\quad\quad\quad\quad\quad\quad\quad \hookrightarrow \sum \xi_i \frac{\partial}{\partial \xi_i}$

By using Egorov's theorem / Darboux's th$^{\underline{m}}$, $\exists$ a canonical transformation, homogeneous

$$\chi : T^*X \setminus 0 \longrightarrow T^*\mathbb{R}^n \setminus 0$$

s.t.

$$p \circ \chi = \xi_n, \quad \chi(x_0,\xi_0) = (0;(0,\cdots,1))$$
$$= (y_0, \eta_0) \text{ say.}$$

and $\exists$ FIO $A \in I^k(\mathbb{R}^n \times X, (\text{graph}\, \chi)')$ so that

$WF'(PA - AD_{x_n}) \not\ni (x_0, \xi_0, y_0, \eta_0)$

& $\exists$ $B \in I^{-k}(X \times \mathbb{R}^n, (\text{graph}\, \chi^{-1})')$

s.t.

$$(x_0,\xi_0) \notin WF(AB - I)$$
$$(y_0,\eta_0) \notin WF(BA - I)$$

<u>Claim</u> $\quad (y_0,\eta_0,x_0,\xi_0) \notin WF(BP - D_{x_n}B)$

Proof,

$$B(PA - AD_{x_n})B = BP(AB - I) - (BA - I)D_{x_n}B$$
$$+ BP - D_{x_n}B,$$

Recall $WF(Cu) \subseteq WF'C \circ WF u$.

$Pu = f$

$(y_0, \eta_0) \notin WF(BPu - D_{x_n} Bu)$

Let $v = Bu \in \mathcal{D}'(\mathbb{R}^n)$; $u = Av + (I - AB)u$

$$D_{x_n} v = D_{x_n} Bu = BPu - (BP - D_{x_n} P)u = g$$

$\Rightarrow (y_0, \eta_0) \in WF v \setminus WF g$

By propogat^n of sing's for D_{x_n}, the whole integral curve of $D_{x_n} \subseteq WF v$

But integral curves of $D_{x_n} \longrightarrow$ under cannonical trans^n $\longrightarrow$ curves of H_p

$\Rightarrow$ curves of $H_p \subseteq WF u$.

Parametrices for Operators of Real Principal Type

Model Case: $X = \mathbb{R}^n$

$P = D_{x_1} + A(x, D_x)$ $A \in \Psi^0_{cl}(\mathbb{R}^n)$.

We wish to construct E s.t.

$PE = Id + R$, R smoothing.

Solve first $Pu = \delta$.

When $A = 0$, $u = H(x_1)\delta(x')$ $x = (x_1, x')$

Try sol^n of

$$(D_{x_1} + A)u = \delta_0$$

$$u = \int_0^\infty \int e^{i(x_1 - s)\xi_1 + x'\xi'} a(s, x'; \xi)\, ds\, d\xi'$$

$a \in S^0(\mathbb{R}_s \times \mathbb{R}^{n-1}_{x'} ; \mathbb{R}^n_\xi)$, a compactly supported in s.

Notice:

(1) $\phi(s, x, \xi) = (x_1 - s)\xi_1 + x' \cdot \xi'$
$= x \cdot \xi$ when $s = 0$.

$$\delta_0 = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi}\, d\xi \ , \quad \Lambda_0 = \{(0, \xi)\}$$

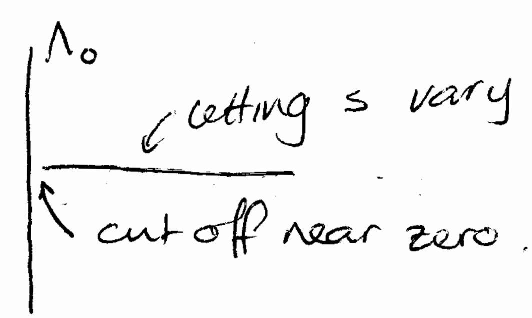
(2) $$v = \iint e^{i((x_1 - s)\xi_1 + x'_1 \cdot \xi')} a(s, x, \xi)\, d\xi\, ds$$
with $a = 0$ for s near 0, & for $s \leq 0$.

Then
$v \in I(\mathbb{R}^n, \Lambda_1)$ ← (later) with
$WF\, v \subseteq \Lambda_1 = \{(x_1, 0, 0, \xi') \ x_1 > 0\}$

$v = \pi_* w(s, x)$ where
$$w(s,x) = \int e^{i((x_1 - s)\xi_1 + x' \cdot \xi')} a(s, x, \xi)\, d\xi.$$

$WF\, w \subset \{(s = x_1, x_1, 0 ; \xi_1, -\xi_1, \xi')\}$

Now



$$(D_{x_1}+A)u = \int_0^\infty \int (\xi_1) e^{i((x_1-s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\,ds\,d\xi$$

$$+ A\int_0^\infty \int e^{i\varphi} a + \int_0^\infty \int e^{i\phi} D_{x_1} a \text{ etc.}$$

$$\int_0^\infty \int (\xi_1) a e^{i\varphi} = \int_0^\infty \int e^{i\phi}(D_s a) + \int e^{ix\cdot\xi} a(0,x,\xi)\,d\xi.$$

$$(D_{x_1}+D_s)a + e^{-ix\cdot\xi} A(e^{ix\cdot\xi}) = 0$$
$$a(0,x,\xi) = \frac{1}{(2\pi)^n}$$

(5/8) (Now we use C for A)

$$(D_{x_1}+D_s)a + e^{-ix\cdot\xi} C(e^{ix\cdot\xi} a) = 0$$
$$a(0,x,\xi) = \frac{1}{(2\pi)^n}$$

$$a \sim \sum_{j\le 0} a_j$$

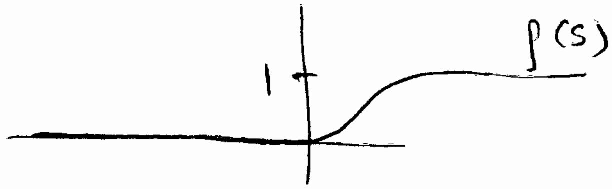
$$(D_{x_1}+D_s)a_0 + \sigma_0(C)\,a_0 = 0$$
$$a_0(0,x,\xi) = \frac{1}{(2\pi)^n}.$$

$$v(s,x) = \int e^{i((x_1-s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\,d\xi$$

$$WF\,v \subseteq \tilde{\Lambda}_1 = \{(s,s,0,\sigma,-\sigma,\xi')\}$$

$$D_{x_1}+D_s = D_{x_1} \text{ on } \tilde{\Lambda}_1$$

Let $w = \iint e^{i((x_1 - s)\xi_1 + x'\cdot\xi')} \rho(s)\, a(s,x,\xi)\, d\xi\, ds$



<u>Claim</u>: $w \in I(\mathbb{R}^n, \Lambda_1)$

$\Lambda_1 = \{(x_1, 0; 0, \xi'),\ x_1 \geq 0\}$

$=$ flow out from $x_1 = 0, \xi_1 = 0$ by $\frac{\partial}{\partial x_1}$

$$w = \int e^{ix'\cdot\xi'} \underbrace{\left(\iint e^{i(x_1 - s)\cdot\xi_1} \rho(s)\, a(s,x,\xi)\, d\xi_1\, ds \right)}_{= b(x,\xi')} d\xi'$$

If we show $b \in S^0(\mathbb{R}^n \times \mathbb{R}^{n-1})$, then

$w \in I(\mathbb{R}^n, \Lambda_1)$

$\phi(x,\xi) = x'\cdot\xi'$

$d_{\xi'}\phi = x' = 0$

$d_{x'}\phi = \xi'$

$d_{x_1}\phi = 0 \Rightarrow \xi_1 = 0$

Do stationary phase in x_1 & s to conclude $b(x,\xi') \in S^0(\mathbb{R}^n \times \mathbb{R}^{n-1})$

Thus $\sigma_0(w) = \rho(x_1)\, a_0(\underset{\uparrow s}{x_1}, \underset{\uparrow x_1}{x_1}, \underset{\uparrow x'}{0}, \underset{\uparrow \xi_1}{0}, \xi')$

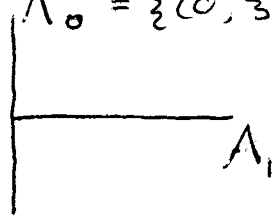
In fact, $w = \pi_* v$ where

$\pi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(s, x) \longmapsto x.$

$$u = \int_0^\infty \int e^{i((x_1 - s)\xi_1 + x' \cdot \xi')} a(s, x, \xi)\, d\xi\, ds.$$

$$\Lambda_0 = \{(0,\xi),\ \xi \neq 0\}$$



Now consider

$$w = \int e^{ix\cdot\xi} \underbrace{\left(\int_0^\infty e^{-is\xi_1} a(s,x,\xi)\, ds \right)}_{d(x,\xi)} d\xi$$

If $d \in S^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, then
$w \in I(\mathbb{R}^n, \Lambda_0)$

$$w = \int e^{ix\cdot\xi} d(x,\xi)\, d\xi$$

$$\Lambda_0 = \{(0,\xi);\ \xi \neq 0\}.$$

$$d(x,\xi) = \int_0^\infty e^{-is\xi_1} a(s,x,\xi)\, ds$$

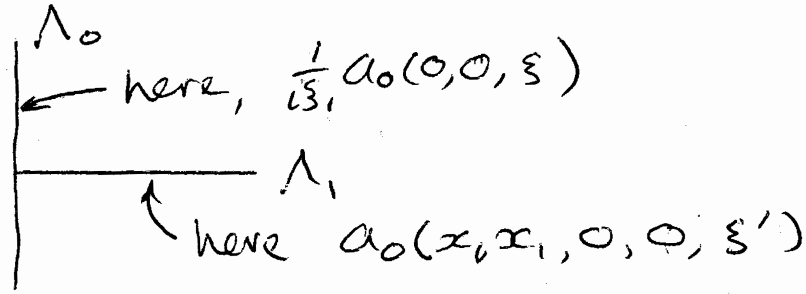
$$= \int_0^\infty \frac{-1}{i\xi_1} \frac{\partial}{\partial s}\left(e^{-is\xi_1}\right) a(s,x,\xi)\, ds$$

$$= \frac{1}{i\xi_1} \int_0^\infty e^{-is\xi_1} \frac{\partial}{\partial s} a(s,x,\xi)\, ds + \frac{1}{i\xi_1} a(0,x,\xi)$$

$$= \left(\frac{1}{i\xi_1}\right)^2 \int_0^\infty e^{-is\xi_1} \left(\frac{\partial^2}{\partial s^2}\right) a(s,x,\xi)\, ds + \frac{1}{i\xi_1} a(0,x,\xi)$$
$$+ \left(\frac{1}{i\xi_1}\right)^2 \partial_s a(0,x,\xi) \quad \text{etc.}$$

<u>Claim.</u> $d \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ for $\xi_1 \neq 0$.
Moreover,
$$d = \frac{1}{i\xi_1} a_0(0, x, \xi) + \text{lower order.}$$

<u>Claim</u> & $u \in I^0(\mathbb{R}^n, \Lambda_0)$ for $\xi_1 \neq 0$.
$$\sigma_0(u) = \left(\frac{1}{i\xi_1}\right) a(0, 0, \xi)$$



$$\Lambda_0 \cap \Lambda_1 = \{x = 0, \ \xi_1 = 0\}.$$

<u>Question</u>: If u is of the form
$$u = \int_0^\infty \int e^{i((x_1 - s)\xi_1 + x' \cdot \xi')} a(s, x, \xi) \, d\xi \, ds$$

What is $\sigma(u)$?

We'll define $\sigma(u) = (a_0, a_1)$ $\quad a_0 \in S^0(\Lambda_0)$
$\quad a_1 \in S^0(\Lambda_1)$
and $(i\xi_1) a_0 = a_1$ on $\Lambda_0 \cap \Lambda_1$

<u>Proposition</u> Let
$$u = \int_0^\infty \int e^{i((x_1-s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\,d\xi\,ds$$
$$a \in S^{m+\frac{1}{2}-n/4}(\mathbb{R}_s \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$$
$a = 0$ for $|s| \geq R$.

Then $WF\,u \subseteq \Lambda_0 \cup \Lambda_1$

where
$$\Lambda_0 = \{(0,\xi);\ \xi \neq 0\}$$
$$\Lambda_1 = \{(x_1,0;0,\xi')\ x_1 \geq 0,\ \xi' \neq 0\}.$$

Let $B \in \Psi^0(\mathbb{R}^n)$ with
$$WF\,B \cap \Lambda_1 = \phi$$
Then
$$Bu \in I^{m-1/2}(\mathbb{R}^n, \Lambda_0)$$
If $WF\,B \cap \Lambda_0 = \phi$

then
$$Bu \in I^m(\mathbb{R}^n, \Lambda_1)$$
Moreover, neglecting half-densities & Maslov contributions,
$$\sigma_{m-1/2}(u) = \frac{1}{i\xi_1} a(0,0,\xi) \text{ on } \Lambda_0 \setminus (\Lambda_0 \cap \Lambda_1)$$

&
$$\sigma_m(u) = a(x_1, x_1, 0, 0, \xi') \text{ on } \Lambda_1 \setminus (\Lambda_0 \cap \Lambda_1)$$

Proof - only need to check $WF\,u \subset \Lambda_0 \cup \Lambda_1$.

$u = \pi_* H(s) V$, where

$$V(s,x) = \int e^{i((x_1 - s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\,d\xi.$$

where $\pi : \mathbb{R}_s \times \mathbb{R}^n_x \to \mathbb{R}^n$

$$WF\,H(s) = \{(0, x, \underset{(\sigma\neq 0)}{\sigma}, 0)\}$$

$$WF\,V \subseteq \{(s, s, 0, -\xi_1, \xi_1, \xi') \quad (\xi_1, \xi') \neq 0\}$$

$$WF(H(s)V) \subseteq WF\,H(s) \cup WF(V) \cup (WF(H(s)) + WF(V))$$

$$WF(\pi_* w) \subseteq \{(x\xi);\ \exists (y,\eta) : \pi(y) = x \ \&\ (d\pi)^t \eta = \xi\}$$

id. in x
0 in s.

$$WF(\pi_* H(s)) = \emptyset \qquad WF(\pi_* V) \subseteq \Lambda_1$$

$$\&\ \ WF(\pi_*(H(s))) + WF(\pi_* V) \subseteq \Lambda_0.$$

(5/14) Denote by $\tilde{\Lambda}_0 = \{(0, \xi)\}$
$\tilde{\Lambda}_1 = \{(x_1, 0; 0, \xi'), x_1 \geq 0\}$.

$$\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1 = \{(0,0;\ 0, \xi')\}$$

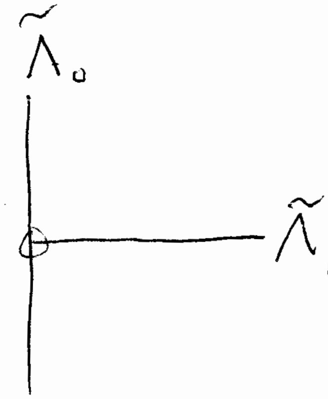
$$u = \int_0^\infty \int e^{i((x_1 - s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\,d\xi\,ds$$

$$WF\,u \subseteq \tilde{\Lambda}_0 \cup \tilde{\Lambda}_1 \qquad a \in S^{m+\frac{1}{2}-\frac{n}{4}}(\mathbb{R}_s \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$$

Constructed u so that

$$(D_{x_1} + C(x, D_{x'}))u = \delta_0$$

$$\uparrow \ \Psi^0_{cl}$$



$u \in I^{m-1/2}(\mathbb{R}^n, \tilde{\Lambda}_0 \setminus (\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1))$
$u \in I^{m}(\mathbb{R}^n, \tilde{\Lambda}_1 \setminus (\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1))$

$\sigma_{m-\frac{1}{2}}(u) = \dfrac{a(0,0,\xi)}{\xi_1}$ modulo lower order on $\tilde{\Lambda}_0 \setminus \tilde{\Lambda}_1$

$\sigma_m(u) = a(x_1, x_1, 0, 0, \xi')$ on $\tilde{\Lambda}_1 \setminus \tilde{\Lambda}_0$

$(s \ x_1 \ x' \ \xi_1 \ \xi')$

$\xi_1 \sigma_{m-\frac{1}{2}}(u) = a(0,0,0,0,\xi')$ on $\tilde{\Lambda}_1 \cap \tilde{\Lambda}_2$

——— " ———

Def^n Let X be a C^∞ manifold.
$\Lambda_0 \subset T^*X \setminus \{0\}$ conic Lagrangian manifold
$\Lambda_1 \subset T^*X \setminus \{0\}$ " " " with boundary.
Λ_0, Λ_1 intersect cleanly if
$T_\lambda(\Lambda_0 \cap \Lambda_1) = T_\lambda(\Lambda_0) \cap T_\lambda(\Lambda_1)$
$\forall \lambda \in \Lambda_0 \cap \Lambda_1$

Def^n Such a pair (Λ_0, Λ_1) will be called an intersecting pair of Lagrangians.

Def^n X: C^∞-manifold, (Λ_0, Λ_1), (Λ_0', Λ_1')
intersecting pairs are locally equivalent at points $\lambda \in \Lambda_0 \cap \Lambda_1$, $\lambda' \in \Lambda_0' \cap \Lambda_1'$
if $\exists$ a conic nbhd V of λ

& a canonical trans^n

$$\chi : V \to T^*X \setminus \{0\}$$

such that $\chi(\lambda) = \lambda'$

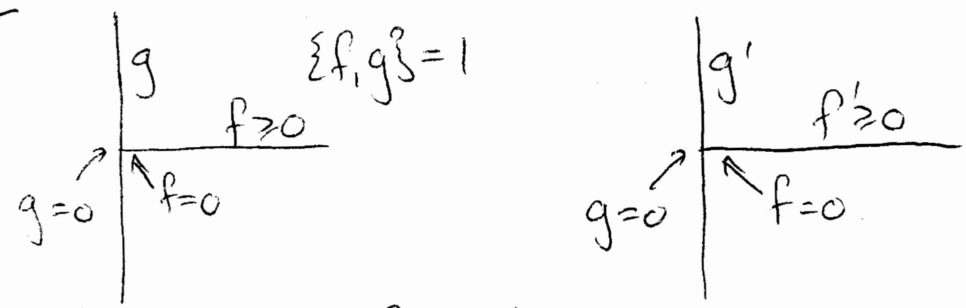
$$\chi(\Lambda_0 \cap V) \subset \Lambda_0'$$

$$\chi(\Lambda_1 \cap V) \subset \Lambda_1'$$

& $\chi(\partial\Lambda_1 \cap V) \subset \partial\Lambda_1'$.

<u>Theorem</u>. All intersecting pairs are locally equivalent.
(We assume that $\operatorname{codim}(\Lambda_0 \cap \Lambda_1)$ in Λ_0 (or Λ_1) is 1)

<u>Idea</u>



(eg. $g = \xi_1$, $f = x_1$)

$$\chi \circ f = f'$$
$$\chi \circ g = g'$$

<u>Def^n</u> Let (Λ_0, Λ_1) be an intersecting pair.
$u \in I^m(X, \Lambda_0, \Lambda_1)$ if

$$u = u_0 + u_1 + \Sigma_j F_j v_j$$

where $u_0 \in I^{m-1/2}(X, \Lambda_0 \setminus \partial\Lambda_1)$

$u_1 \in I^m(X, \Lambda_1 \setminus \partial\Lambda_0)$

F_j is a zero order FIO associated to a canonical transformation χ_j^{-1}: $\chi_j : V_j \to T^*\mathbb{R}^n$, where V_j is a countable covering of

$\Lambda_0 \cap \Lambda_1$, $\quad \chi_j(V_j \cap \Lambda_0) \subset \tilde{\Lambda}_0$

$v_j \in I^m(\mathbb{R}^n, \tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

<u>Remark</u>: For Fourier integral dist^{n}s,

$$u = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$$

An alternative way to have defined such u is to say:

Take $\tilde{\Lambda}_0 = \{(0,\xi) \subset T^*\mathbb{R}^n \setminus \{0\}\}$.

Let $\Lambda \subset T^*X \setminus \{0\}$ be a conic Lagrangian.

Then given $\lambda \in \Lambda$ $\exists$ V a conic nbhd of λ & a canonical transformation so that

$$\chi(V \cap \Lambda) \subset \tilde{\Lambda}_0$$

Then $u \in I^m(X, \Lambda)$

if $u = \sum F_j v_j$ (locally finite sum)

so that F_j is the FIO associated to χ^{-1}

& $v_j = \int e^{ix\cdot\xi} a_j(x,\xi)\, d\xi$, $\quad a_j \in S^{m+\frac{n}{2}-n/4}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$

An advantage: proofs are easier since they reduce to the standard example.

Disadvantage: it is difficult to recognice if $u \in I^m(X, \Lambda)$.

An alternative approach to $I^m(X, \Lambda_0, \Lambda_1)$ is to write

$$u = \sum_j u_j \qquad \text{locally finite}$$

where

$$u_j = \int_0^\infty \int e^{i\phi(x,s,\theta)} a_j(s,x,\theta)\, d\theta\, ds$$

where $\phi(x, 0, \theta)$ parameterizes Λ_0 ($= \Lambda_\phi$ loc.)
and $\phi(x, \frac{s}{|\theta|}, \theta) = \psi(x, (s, \theta))$ ← phase variables
parameterizes Λ_1 for $s > 0$.

In the model example

$$\phi(x, s, \theta) = (x_1 - s)\theta_1 + x' \cdot \theta'$$

$\phi(x, 0, \theta)$ parameterizes $\tilde{\Lambda}_0 = \{(0, \xi)\}$.

Claim: $\phi(x, \frac{s}{|\theta|}, \theta)$ parameterizes $\tilde{\Lambda}_1 = \{(x_1, 0; 0, \xi')\}$.

$$d_s \phi = 0 \Rightarrow \theta_1 = 0$$
$$d_\theta \phi = 0 \Rightarrow x_1 = s,\ x' = 0$$

— " —

Let $u \in I^m(X, \Lambda_0, \Lambda_1)$. What is $\sigma(u)$?
We know

$$\sigma_0(u) \in S^{m - 1/2 + n/4}(\Lambda_0 \setminus \partial\Lambda_1, \Omega^{1/2} \otimes \mathcal{L}_0)$$
$$\sigma_1(u) \in S^{m + n/4}(\Lambda_1 \setminus \partial\Lambda_1, \Omega^{1/2} \otimes \mathcal{L}_1)$$

Can we make sense in general of the relation $\xi_1 \sigma_0(u)\big|_{\partial\tilde{\Lambda}_1} = \sigma_1(u)\big|_{\partial\tilde{\Lambda}_1}$?

Let $\lambda_0 \in \partial\Lambda_1$; choose $n-1$ fc$^{\underline{n}}$s $h_1, \dots, h_{n-1}$
$dh_1(\lambda_0), \dots, dh_{n-1}(\lambda_0)$ linearly independent
and in addition, functions f & g s.t.
$f=0$ on Λ_0, $f>0$ on $\Lambda_1 \setminus \partial\Lambda_1$, $df(\lambda_0) \neq 0$
$g=0$ on Λ_1, $dg(\lambda_0) \neq 0$
and so that $\{f,g\}(\lambda_0) < 0$ (since int$^{\underline{n}}$ is clean). (in example, $f = x_1$, $g = \xi_1$, $\{x_1, \xi_1\} < 0$).

(*) $\begin{cases} \text{Let } a \in C^\infty(\Lambda_0 \setminus \partial\Lambda_1, \Omega^{1/2}) \\ \text{\& } r = ga \in C^\infty(\Lambda_0, \Omega^{1/2}) \quad (\text{i.e. } \xi a \text{ in model}) \end{cases}$
ie $ga = r\,|dh_1 \wedge \dots \wedge dh_{n-1} \wedge dg|^{1/2}$, $r \in C^\infty(\Lambda_0)$
Define $b = r\,|dh_1 \wedge \dots \wedge dh_{n-1} \wedge df|^{1/2} \{g, f\}^{-1/2}$
(makes sense on $\partial\Lambda_1$).

Let's call $b = Ra$ for a's that satisfy (*)

Def$^{\underline{n}}$ Let $\Lambda = \Lambda_0 \cup \Lambda_1$, (Λ_0, Λ_1) an intersecting pair.

$$S^m(\Lambda, \Omega^{1/2}) \subset S^{m+n/4}(\Lambda_1, \Omega^{1/2}) \oplus S^{m-1/2+n/4}(\Lambda_0 \setminus \partial\Lambda_1, \Omega^{1/2})$$

$c \in S^m(\Lambda, \Omega^{1/2})$, $c = (c_1, c_0)$
where

$$c_1|_{\partial\Lambda_1} = Rc_0|_{\partial\Lambda_1}$$

15/15)

Claim: The def^n for $u \in I^m(X, \Lambda_0, \Lambda_1)$ is independent of the choice of F & of χ.

We have defined

$$\sigma : I^m(X; \Lambda_0, \Lambda_1) \to S^{m+\frac{n}{4}}(\Lambda_1 \setminus \partial\Lambda_1, \Omega^{1/2} \otimes \mathcal{L}_1) \oplus S^{m+n/4-1/2}(\Lambda_0 \setminus \partial\Lambda_1, \Omega^{1/2} \otimes \mathcal{L}_0)$$

$$\sigma(u) = (\sigma_1, \sigma_0).$$

$$u \in I^m(X, \Lambda_0, \Lambda_1) \Rightarrow u \in I^m(\Lambda_1 \setminus \partial\Lambda_1)$$
$$u \in I^{m-1/2}(\Lambda_0 - \partial\Lambda_1).$$

Compatibility cond^n on $\Lambda_0 \cap \Lambda_1$ - as on previous page.

Def^{n} Let $u \in I^m(X, \Lambda_0, \Lambda_1)$.
$\sigma(u) = (\sigma_1, \sigma_0)$ as before
$$R\sigma_0|_{\partial\Lambda_1} = \sigma_1|_{\partial\Lambda_1}$$

Prop^n σ is well defined.
Idea:
Suppose $f' = \underset{\alpha>0}{\alpha f}$; $ga = r|dh_1 \wedge \dots \wedge dh_{n-1} \wedge dg|^{1/2}$ as before

$$Ra = b = r|dh_1 \wedge \dots \wedge dh_{n-1} \wedge df'|^{1/2} \{g, f'\}^{-1/2} \text{ on } \partial\Lambda_1$$
$$= r\cancel{|\alpha|^{1/2}}|dh_1 \wedge \dots \wedge dh_{n-1} \wedge df|^{1/2} \cancel{\alpha^{-1/2}} \{g, f\}^{-1/2}$$

Prop^n There is a "naturally" defined Maslov (line) bundle $\mathcal{L}$ over $\Lambda = \Lambda_0 \cup \Lambda_1$, which coincides with $\mathcal{L}_1$ on $\Lambda_1 \setminus \partial\Lambda_1$ & $\mathcal{L}_0$ on $\Lambda_0 \setminus \partial\Lambda_1$.

<u>Def$^{n}_{\#}$</u> $S^m(\Lambda, \Omega^{1/2}\otimes \mathcal{L}) =$

$\{(b,a);\ b \in S^{m+n/4}(\Lambda_1, \Omega^{1/2}\otimes \mathcal{L}_1)$

$a \in S^{m-1/2+n/4}(\Lambda_0, \Omega^{1/2}\otimes \mathcal{L}_0)$

and $R a|_{\partial\Lambda_1} = b|_{\partial\Lambda_1}\}$

= space of symbols of order m associated to an intersecting pair.

<u>Theorem</u> (Symbol Calculus)

$$\frac{I^m(X;\Lambda_0,\Lambda_1)}{I^{m-1}(X;\Lambda_0,\Lambda_1)} \cong \frac{S^m(\Lambda;\Omega^{1/2}\otimes\mathcal{L})}{S^{m-1}(\Lambda;\Omega^{1/2}\otimes\mathcal{L})}$$

To prove this, we can just prove it for the model case.

Idea: $u = \int_0^\infty\int e^{i((x_1-s)\xi_1 + x'\cdot\xi')} a(s,x,\xi)\, d\xi\, ds$

$\sigma(u) = \dfrac{a(0,0,\xi)}{\xi_1}$ on $\Lambda_0 \setminus \partial\Lambda_1$

$\quad\ \ = a(x_1, x_1, 0, 0, \xi')$ on $\Lambda_1\setminus\partial\Lambda_1$

Suppose $a(x_1, x_1, 0, 0, \xi') = 0$ modulo lower order.

Expand in Taylor series to get

$a(s,x,\xi) = 0 + \underbrace{(x_1 - s)}_{=\frac{\partial}{\partial\xi_1}e^{i\cdots}} h_1 + \xi_1 \tilde{h}_1 + x'\cdot\tilde{\tilde{h}}$

integrate by parts $\qquad \frac{\partial}{\partial s}$

get contribution at $s=0$ $\int e^{ix\cdot\xi} a(0,x,\xi)\tilde{h}_1 d\xi$

now develope this in Taylor series.

<u>Prop^n</u> (Λ_0, Λ_1) intersecting pair.

$P \in \Psi^m_{cl}(X)$, $\sigma_m(P) = 0$ on Λ_1

Let $u \in I^{\tilde{m}}(X; \Lambda_0, \Lambda_1)$. $\sigma(u) = (\sigma_1, \sigma_0)$

Then $Pu \in I^{m+\tilde{m}-1}(X; \Lambda_0, \Lambda_1)$.

& $\sigma(Pu) = (b, a)$

$$b = \frac{1}{i} \mathcal{L}_{H_p} \sigma_1 + c\sigma_1 \quad \text{on } \Lambda_1 \setminus \partial\Lambda_1$$

$$a = \sigma_m(p)\, \sigma_0 \quad \text{on } \Lambda_0 \setminus \partial\Lambda_1$$

where c is the subprincipal symbol of p.

<u>Parametrices for operators of Principal Type</u>.

<u>Th^m</u> $P \in \Psi^m_{cl}(X)$, $\Lambda_0 \subset T^*X \setminus \{0\}$, an embedded conic Lagrangian submanifold.

$p = \sigma_m(P)$ $\mathbb{R}$-valued

H_p nowhere tangent to $\Lambda_0 \cap \{p=0\}$.

Let Λ_1 be the forward flow out from $\Lambda_0 \cap \{p=0\}$ by H_p.

Assume Λ_1 is an embedded conic Lagrangian submanifold with boundary.

Given $f \in I^k(X, \Lambda_0)$, $\exists$

$u \in I^{k-m+1/2}(X; \Lambda_0, \Lambda_1)$

such that

$$Pu = f \quad \text{mod. } C^\infty(X)$$

<u>Corollary</u> Take $\Lambda_0 = \Delta$ in $T^*(X \times X) \setminus \{0\}$ & f = Identity. Thus we can find E:

$$PE = I \quad \text{mod. smoothing.}$$

$E \in I(X \times X; \Delta, \Lambda)$, Λ = forward flow out.

Sketch of Proof.

$\sigma(u_0) = (b_0, a_0)$

$\sigma_m(P)\, a_0 = \sigma_k(f)$ on $\Lambda_0 \setminus \partial\Lambda_1$

$a_0 = \dfrac{\sigma_k(f)}{\sigma_m(P)}$ (WLOG can take $m=1$).

$\left(\frac{1}{i}\mathcal{L}_{H_p} b_0 + C b_0\right) = 0$ on $\Lambda_1 \setminus \partial\Lambda_1$

and $b_0|_{\partial\Lambda_1} = R\, a_0|_{\partial\Lambda_1}$

Next write $u = u_0 + u_1$.

We know $Pu_0 = f + r_1$, r_1 one order less. $I(X, \Lambda_0, \Lambda_1)$

$Pu_1 = -r_1$

$\sigma(u_1) = (b_1, a_1)$

$\sigma(a_1) = -\dfrac{\sigma(r_1)}{\sigma_m(P)}$ on $\Lambda_0 \setminus \partial\Lambda_1$

& $\frac{1}{i}\mathcal{L}_{H_p} b_1 + c b_1 = -\sigma(r_1)$ on $\Lambda_1 \setminus \partial\Lambda_1$

& $b_1|_{\partial\Lambda_1} = R a_1|_{\partial\Lambda_1}$.

(Eduardo) <u>ΨDO with complex Principal Symbol.</u>
(§7 FIO II)

<u>$Pu = f$</u>

For local solvability one needs to assume
$$\rho = 0 \Rightarrow \{\rho, \bar{\rho}\} = 0.$$
$$\Leftrightarrow \{\rho, \bar{\rho}\} = a \operatorname{Re}\rho + b \operatorname{Im}\rho.$$

(suppose $\rho = 0$ at $x = \xi = 0$
$\rho = xa + \xi b \quad \bar{\rho} = x\bar{a} + \xi\bar{b}$)

$$\Rightarrow \{\operatorname{Re}\rho, \operatorname{Im}\rho\} = a \operatorname{Re}\rho + b \operatorname{Im}\rho$$
$$\Leftrightarrow [H_{\operatorname{Re}\rho}, H_{\operatorname{Im}\rho}] = a H_{\operatorname{Re}\rho} + b H_{\operatorname{Im}\rho} \text{ on } \rho = 0$$

i.e. $H_{\operatorname{Re}\rho}$ & $H_{\operatorname{Im}\rho}$ are involutive on $\rho = 0$.

Assume $H_{\operatorname{Re}\rho}$ & $H_{\operatorname{Im}\rho}$ linearly independent
$\Rightarrow \{\rho = 0\}$ foliated by integral surfaces.
Call these leaves "Bicharacteristic strips."

<u>First Order Differential Operators w/ $\mathbb{C}$ Symbols.</u>
Consider vectorfields
$L = L_1 + i L_2$ on M (eg. $\{\rho = 0\}$, $L = H_\rho$.)
with 2 cond^{n}s
(IN) $[L_1, L_2]$ a linear comb^n of L_1 & L_2
(LI) L_1 & L_2 linearly independent.

Consequently (foliation) $\exists$ coordinates where

$$L = a_{n-1}\frac{\partial}{\partial x_{n-1}} + a_n \frac{\partial}{\partial x_n} \qquad a_{n-1}, a_n \in C^\infty.$$

L is elliptic within each leaf (in x_{n-1}, x_n) depending smoothly on other variables.
Hence L is locally solvable.

<u>Notice</u>: $[L, \bar{L}]$ is purely imaginary

$$\begin{aligned}[L, \bar{L}] &= [L_1 + iL_2, L_1 - iL_2] \\ &= 2i[L_1, L_2] \\ &= 2i(aL_1 + bL_2) \\ &= \cdots \\ &= (\quad)L - (\quad)\bar{L} \\ &= \lambda L - \bar{\lambda}\bar{L}\end{aligned}$$

Consider $L' = e^{\mu} L$

$[L', \bar{L}'] = \cdots = e^{2\operatorname{Re}\mu}([L, \bar{L}] + (L\bar{\mu})\bar{L} - (\bar{L}\mu)L)$

using above

$= ((\lambda - \bar{L}\mu)L - (\bar{\lambda} - L\bar{\mu})\bar{L})e^{2\operatorname{Re}\mu}.$

Choose μ : $\bar{L}\mu = \lambda$, or $L\bar{\mu} = \bar{\lambda}$;
we can locally solve this.

Theorem

If $L = L_1 + iL_2$ satisfies (IN) & (LI) then $\exists$ local coordinates where

$$L = a\frac{\partial}{\partial \bar z}$$

& $a \in C^\infty$ is never zero.

(since we get $[L, \bar L] = 0$).

(We think of $L = \underset{\substack{\uparrow\\ \text{really forms}}}{a\,\bar\partial}$).

Now $L\bar L = a\bar\partial(\bar a \partial)$

$= a\,\bar\partial\bar a\,\partial + a\bar a\,\bar\partial\partial$

$= \dfrac{a\,\bar\partial\bar a}{\bar a}\bar L + a\bar a\,\bar\partial\partial$

$= \bar b\,\bar L + a\bar a\,\bar\partial\partial.$

& $\bar L L = bL + a\bar a\,\bar\partial\partial$ if a & $\bar a$ anticommute since ∂ & $\bar\partial$ " "

Then

$[L, \bar L] = L\bar L - \bar L L = (-b)L - (\bar b)\bar L.$

$\Rightarrow b = -\lambda.$

$\Rightarrow a\bar a\,\bar\partial\partial = L\bar L + \lambda\bar L = (L + \bar\lambda)\bar L$ "fudge"

this is a Laplacian.

Let φ be defined on a leaf of the foliation.

φ is subharmonic if

$$(L + \bar\lambda)\bar L\varphi \geq 0.$$

& φ is superharmonic if $-\varphi$ is subharmonic.

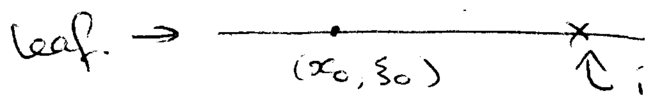
If u is a dist^n, let

$$S_u(x) = \sup\{t : u \in H^t \text{ near } x\}$$

If $Pu = f$, we'll show $f \in C^\infty \Rightarrow S_u$ superharmonic with respect to H_p. $(= L)$.

Aim: to show $WF(u) \setminus WF(f)$ invariant under leaves.

If $(x_0, \xi_0) \notin WF(f)$ & $(x_0, \xi_0) \in WF(u)$



↑ if not in $WF(u)$, then $S_u = \infty$ here & this $\Rightarrow$ ∞ for whole leaf, but $\cancel{\infty}$ at x_0.

Similarly, $S^*_u(x, \xi) = \sup\{t : u \in H^t \text{ "near" } x, \xi\}$

$u \in H^t$ at (x, ξ) iff $u = u_1 + u_2$ where $u_1 \in H^t$ & $(x_0, \xi_0) \notin WF(u_2)$

$\Leftrightarrow \exists A \in \Psi^0_{cl}(X)$ s.t. A is not charact. at (x, ξ) & $Au \in H^t_{loc}$.

Note $u \in H^t$ at $x \Leftrightarrow u \in H^t$ at (x, ξ) $\forall \xi$.

<u>Lemma</u>. If $u \in \mathcal{D}'(M)$, $Lu = f$, and
s is superharmonic is such that $s \leq s_f$
then $\min(s_u, s)$ is superharmonic.

(1) The Lemma is true for $(L+a)u = f$, $a \in C^\infty$.
(2) If $f \in C^\infty$, S_u is superharmonic.

<u>Theorem</u>: $Lu = f$ then
$$\operatorname{supp} u \setminus \operatorname{supp} f$$
is invariant under the bicharacteristic foliation.

<u>Theorem</u>: If $K \subset M$, TFAE:
(a) $LC^\infty(K) = C^\infty(K)$
(b) $(L+a)C^\infty(K) = C^\infty(K)$
(c) $LC^\infty(K)$ & $(L+\bar{\lambda})C^\infty(K)$ are dense in $C^\infty(K)$
(d) $\exists\ \varphi \in C^\infty(K)$ strictly subharmonic in K
(e) no leaf of the L-foliation is contained in K.

<u>Th^{m}</u> (Malgrange).
Assume semi-global solvability (the above)
Then there is global solvability $\Leftrightarrow$ $\forall K \subset\subset M$
& $k \geq 0$, $k \in \mathbb{N}$, $\exists\ K' \subset\subset M$ such that
$v \in \mathcal{E}'(M)$ & ${}^tLv \in \mathcal{E}'^{k}(K)$
$$\Rightarrow v \in \mathcal{E}'(K').$$

Theorem. Assume semi-global solvability. Then TFAE:

(a) $\forall K \Subset M \; \exists K' \Subset M$ s.t. if B is a leaf of the L-foliation & C is a component of $B \cap K^c$ in the leaf topology which is relatively compact in the M topology, then $C \subseteq K'$.

(b) For each $K \Subset M \; \exists K' \Subset M : \forall x \notin K' \; \exists$ $f \in C^\infty(M)$ with $Lf = 0$, and

$$|f(x)| > \sup_{y \in K} |f(y)|$$

(c) $\exists$ a subharmonic fe^n $\varphi \in C^\infty(M)$ s.t.

$$\{x : \varphi(x) \le c\}$$

is compact for every $c \in \mathbb{R}$.

Propagation of Singularities

If $(x, \xi) \in T^*X \setminus \{0\}$,

$$s_u^*(x,\xi) = \sup \{t : \underbrace{u \in H^t \text{ at } (x,\xi)}\}.$$

i.e. $u = u_1 + u_2$ with $u_1 \in H^t$, $(x,\xi) \notin WF(u_2)$.

Now $s_u^*(x, \tau\xi) = s_u^*(x,\xi)$, $\tau > 0$.

If $Lu = f$, & $s_f \geq s$ w/ s superharmonic, then $\min(s_u, s)$ is superharmonic.

Def^n $P \in \Psi^m_{cl}(X)$.

$$N_c = \{(x,\xi) \in T^*X \setminus \{0\} : p(x,\xi) = 0, \; H_{\mathrm{Re}\,p}, H_{\mathrm{Im}\,p}, \xi \tfrac{\partial}{\partial \xi} \text{ linearly independent}\}$$

$$N = \{(x,\xi) \in N_c : \{p, \bar{p}\}(x,\xi) = 0\}^{\circ}.$$ interior relative to N_c

<u>Theorem</u> Let $u \in \mathcal{D}'(X)$, $Pu = f$, s a positively homogeneous of degree 0 fc^{n} which is superharmonic w.r.t. H_p, & such that
$$s \le s_f^* \qquad \text{in } \Omega \subset N$$
Then
$$\min(s_u^*,\ s+m-1)$$
is superharmonic in Ω.

<u>Corollary</u>. $(N \cap WF(u)) \setminus WF(f)$ is invariant under the bicharacteristic foliation in $N \setminus WF(f)$.