An introduction to microlocal analysis with applications to inverse problems, summer 2016

Exercise Problems, all lectures
Return your solutions to Teemu Saksala by 11th of Sebtember at 23:59 by e-mail teemu.saksala@helsinki.fi. In order to get the one credit for the exercises, you should solve atleast 5 of the following problems.

Please let me know, if you find any mistakes etc.

Notations:

- $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-indes. $|\alpha| := \sum_{k=1}^n \alpha_k$
- $\partial_{x_k} := \frac{\partial}{\partial x_k}$ is the k^{th} partial derivative with respect to Cartesian coordinates and $D_k = -i\partial_{x_k}$
- $\partial^\alpha = \prod_{k=1}^n \partial_{x_k}^{\alpha_k}$ and $D^\alpha = \prod_{k=1}^n (-i\partial_{x_k})^{\alpha_k} = (-i)^{|\alpha|} \prod_{k=1}^n \partial^{\alpha_k}$

Problem 1. *Let $w \in \mathbb{R}^n$, $\|w\| = 1$ and $s \in \mathbb{R}$. We denote the Hyperplane*

$$H_{w,s} = \{x \in \mathbb{R}^n : x \cdot w = s\}.$$

Let $dx = d_{x_1} \wedge d_{x_2} \wedge \ldots \wedge d_{x_n}$ be the volume form of $\mathbb{R}^n$. Then hyperplane $H_{w,s}$ has a natural volume form dH given by formula

$$dH = (N \lrcorner dx)|_{H_{w,s}},$$

where N is a unit normal of $H_{s,w}$ and $\lrcorner$ stands for interior multiplication. (See [6]).

Show that the equation

$$dx|_{H_{s,w}} = ds \wedge dH$$

is valid. Here ds should be considered to be the differential of the mapping $x \mapsto x \cdot w$.

Solution 1. *Let $(s, w) \in (\mathbb{R} \times S^{n-1})$. Notice that*

$$H_{s,w} = \{sw + x \in \mathbb{R}^n : x \cdot w = 0\}.$$

Therefore $|s|$ =dist$(H_{s,w}, \{0\})$. Therefore by symmetry we can assume that $s = 0$ and thus the normal vector N of $H_{0,w}$ is w. Then

$$dH = (w \lrcorner dx)|_{H_{w,s}} = dx(w, \cdot, \ldots, \cdot)$$

$$= \sum_{k=1}^n (-1)^{k-1} dx_k(w) dx_1 \wedge \ldots \wedge dx_{k+1} \wedge \widehat{dx_k} \wedge dx_{k+1} \wedge \ldots \wedge dx_n.$$

Here notation $\widehat{dx_k}$ means that one form dx_k is omitted. By changing the coordinates we may assume that $w = x_1$. Then

$$dH = d_{x_2} \wedge \ldots \wedge d_{x_n}.$$

Consider mapping $x \mapsto x_1$ and denote that by s. Then it holds that

$$ds \wedge dH = dx.$$

Problem 2. *Recall that set $U \subset (S^{n-1} \times \mathbb{R})$ is open if and only if for every $p \in U$ there exists a set $p \in (V \times (a, b)) \subset U$, where $V \subset S^{n-1}$ is open.*

Let $f \in C_0^\infty(\mathbb{R}^n)$. We define the Radon transform of f by formula

$$Rf(w, s) = \int_{x \cdot w = s} f(x) dH, \ (w, s) \in (S^{n-1} \times \mathbb{R}).$$

Show that $R : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(S^{n-1} \times \mathbb{R})$ is well defined, linear and continuous.

Solution 2. *Let $f \in C_0^\infty(\mathbb{R}^n)$. Let us first check that Rf has a compact support. Let $R > 0$ be such that supp$f \subset B(0,R)$. Let $|s| \geq R$. Then for any $w \in S^{n-1}$ we have dist$(H_{s,w}, \{0\}) = |s| \geq R$ and thus*

$$Rf(w,s) = \int_{x\cdot w = s} f(x)dH = 0.$$

Thus supp(Rf) is contained in $S^{n-1} \times [R,R]$. This proves that supp(Rf) is compact.

Notice that an equivalent way to compute the Radon transform is

$$Rf(w,s) = \int_{x\cdot w=0} f(ws+x)dH.$$

Since f is compactly supported it is clear that Rf is smooth with respect to s. Since hyper planes $H_{w,0}$ transform smoothly with respect to $w \in S^{n-1}$ and since supp(f) is compact it holds that Rf is also smooth with respect to w. Therefore R is well defined.

Clearly R is linear since integration is linear.

Recall that a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ is said to converge to zero if

- *there exists a compact set $K \subset \mathbb{R}^n$ such that supp$f_k \subset K$ for every $k \in \mathbb{N}$*
- *for every multi-index α*
$$\partial^\alpha f_j \longrightarrow 0 \text{ uniformly}$$

Then it is enough to prove that for every sequence $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ that converges to zero also $(Rf_j)_{j=1}^\infty$ converges to zero in $C_0^\infty(S^{n-1}\times\mathbb{R})$. Let $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ be a sequence that converges to zero. Then there exists $R > 0$ such that supp$(f_j) \subset B(0,R)$ for every $j \in \mathbb{N}$. Let

$$c_n = Vol_{n-1}(B(0,R) \cap \{x \in \mathbb{R}^n : x_n = 0\}).$$

Then

$$|Rf_j(w,s)| = \int_{x\cdot w=s} |f_j(x)|dH \leq c_n \|f_j\|_\infty \overset{j\to\infty}{\longrightarrow} 0.$$

Therefore $\|Rf_j\|_\infty \longrightarrow 0$ as $j \to \infty$.

Not to make things too difficult we consider from now on only the case $n = 2$. Write $w = (\cos(\phi), \sin(\phi))$. Since supp$f_j \subset B(0,R)$ it holds that

$$|\partial_s Rf_j(\phi,s)| = |\int_{x\cdot w=0} \partial_s f(ws+x)dH| \leq \int_{x\cdot w=0} \sum_{k=1}^{2} |\partial_{x_k} f_j\big|_{(sw+x)}|dH$$

$$\leq c_n \sup_{x\in B(0,R)} |\partial_{x_1} f_j + \partial_{x_2} f_j| \overset{j\to\infty}{\longrightarrow} 0\ .$$

Recall the following formula for integrating over moving regions

$$\frac{d}{dt}\int_{U(t)} f(x,t)\ dx = \int_{\partial U(t)} f(x,t)v(x,t)\cdot N\ dS + \int_{U(t)} \partial_t f(x,t)\ dx$$

Here $v(x,t)$ is the velocity of $x \in \partial U(t)$. Since supp$f_j \subset B(0,R)$ that is compact it holds that

$$|\partial_\phi Rf_j(\phi,s)| = |\int_{w\cdot x} \partial_\phi f(sw+x)\ dH| \leq |s| \int_{w\cdot x} |\partial_{x_1} f||_{sw+x} + |\partial_{x_2} f||_{sw+x}\ dH$$

$$\leq Rc_n \sup_{x\in B(0,R)} |\partial_{x_1} f_j + \partial_{x_2} f_j| \stackrel{j\to\infty}{\longrightarrow} 0\ .$$

Therefore we have showed that

$$f_j,\ \partial_s f_j,\ \partial_\phi f_j \longrightarrow 0 \text{ uniformly.}$$

The rest follows with similar arguments.

Problem 3. *Recall that the formal transpose R^t of Radon transform is defined by L^2-duality*

$$(Rf, g)_{L^2(S^{n-1}\times\mathbb{R})} = (f, R^t g)_{L^2(\mathbb{R}^n)},\ \ f \in C_0^\infty(\mathbb{R}^n)\ , g \in C_0^\infty(S^{n-1}\times\mathbb{R}).$$

Then it holds that

$$R^t : C_0^\infty(S^{n-1}\times\mathbb{R}) \to C^\infty(\mathbb{R}^n),\ \ R^t g(x) = \int_{S^{n-1}} g(x\cdot w, w) dw.$$

Compute the normal operator $R^t R$ and show that

$$(R^t R)f = c_n\, \phi * f, \tag{1}$$

where c_n is a dimensional constant and $\phi(x) = \frac{1}{\|x\|}$.

Solution 3. *Let $f \in C_0^\infty(\mathbb{R}^n)$. Then*

$$(R^t R)f(x) = \int_{S^{n-1}} Rf(x\cdot w, w)\, dw = \int_{S^{n-1}}\int_{y\cdot w=0} f((x\cdot w)w + y) dH_y dw.$$

Notice that

$$x = x - (x\cdot w)w + (x\cdot w)w \text{ and } (x - (x\cdot w)w)\cdot w = 0.$$

Therefore

$$(R^t R)f(x) = \int_{S^{n-1}}\int_{y\cdot w=0} f(x+y) dH_y dw.$$

By (VII.2.8 of [10] it holds that)

$$\int_{S^{n-1}}\int_{y\cdot w=0} f(x+y) dH_y dw = Vol(S^{n-2}) \int_{\mathbb{R}^n} \|y\|^{-1} f(x+y) dy$$

$$= Vol(S^{n-2}) \int_{\mathbb{R}^n} \|x-y\|^{-1} f(y) dy.$$

Thus the claim follows and $c_n = Vol(S^{n-2})$ is the area of unit sphere $S^{n-2} \subset \mathbb{R}^{n-1}$.

Problem 4. *Show that*

$$\mathcal{F}(R^t Rf)(\xi) = c_n \frac{\widehat{f}(\xi)}{\|\xi\|^{n-1}}$$

Solution 4. *Since for any $u \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in C_0^\infty(R^n)$ it holds that*

$$\widehat{u * f} = \widehat{u}\widehat{f}.$$

By this and (1) it is enough to show that

$$\widehat{\frac{1}{\|\cdot\|}}(\xi) = \frac{1}{\|\xi\|^{n-1}}$$

We start with simple observation that in $\mathbb{R}^n$, $n \geq 2$ function $x \mapsto \|x\|^{-1}$ is actually in $\mathcal{D}'(\mathbb{R}^n)$ since chancing into polar coordinates yield

$$\int_{\|x\|\leq 1} \|x\|^{-1} dx = \int_0^1 \int_{S^{n-1}} s^{-1} s^{n-1} \, dwds = Vol(S^{n-1}) \int_0^1 s^{n-2} ds = \frac{Vol(S^{n-1})}{n-1}.$$

We say that distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is rotationally invariant if for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for all $A \in O(n)$ holds

$$\langle u, \varphi \circ A \rangle = \langle u, \varphi \rangle.$$

Since for every $A \in O(n)$ holds $\det A = 1$ and $A^T = A^{-1}$ it is easy to show that the Fourier transform of rotationally invariant distribution is also rotationally invariant.

Therefore in order to cumpute $\widehat{1/\|\cdot\|}(\xi)$, it is enough to consider case $\xi = (r, 0, \ldots, 0)$ for some $r > 0$. Then in polar coordinates this yields

$$\widehat{1/\|\cdot\|}(\xi) = c_n \int_{\mathbb{R}^n} \frac{1}{|x|} e^{-ix\cdot\xi} dx = c_n \int_0^\infty \int_{S^{n-1}} s^{n-2} e^{-irs\cos(\theta)} d\omega ds.$$

Since all the other angular variables are free this simplifies (See [10] VII.2 (2.2))

$$c_n \int_0^\infty \int_{S^{n-1}} s^{n-2} e^{-irs\cos(\theta)} d\omega ds = c_n Vol_{n-2}(S^{n-2}) \int_0^\infty \int_0^\pi s^{n-2} e^{-irs\cos(\theta)} \sin^{n-2}(\theta) d\theta ds.$$

Do the change of variables $rs = s'$ to get

$$\int_0^\infty \int_0^\pi s^{n-2} e^{-irs\cos(\theta)} \sin^{n-2}(\theta) d\theta ds = \frac{1}{r^{n-1}} \int_0^\infty \int_0^\pi (s')^{n-2} e^{-is'\cos(\theta)} \sin^{n-2}(\theta) d\theta ds' =: \frac{1}{r^{n-1}} I_n.$$

Thus we have proved the claim modulo a dimensional constant $c = c_n Vol_{n-2}(S^{n-2nnnnnnnnnnnnnnnnnnnnnnnnnnnnnn}$

Problem 5. *Find $f \in C_0^\infty(\mathbb{R} \times S^{n-1})$ such that $R^t f$ is not compactly supported.*

Solution 5. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be a such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi \geq 0$. Define*

$$f(t, w) = \varphi(t), \; (t, w) \in \mathbb{R} \times S^{n-1}.$$

Then $f \in C_0^\infty(\mathbb{R} \times S^{n-1})$. Let $x \in \mathbb{R}^n$ and $A := \{x\}^\perp \cap S^{n-1}$. By continuity of dot product there exists an open neighborhood $V \subset S^{n-1}$ of A such that

$$|w \cdot x| < 1, \; w \in V.$$

Therefore

$$R^t f(x) = \int_{S^{n-1}} f(x \cdot w, w) dw = \int_{S^{n-1}} \varphi(x \cdot w) dw \geq \int_{V \subset S^{n-1}} dw = Vol_{n-1}(V) > 0.$$

This proves the claim.

Problem 6. *Recall the Radon inversion formula (RIF) for test functions is*

$$f = c_n (-\Delta)^{\frac{n-1}{2}} R^t R f, \; f \in C_0^\infty(\mathbb{R}^n),$$

where for $\alpha \in \mathbb{R}$ such that $-n < \alpha$ we define

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \|\xi\|^\alpha \widehat{f}(\xi) d\xi.$$

Show that (RIF) is also valid for any compactly supported distribution. I.e. show

$$u = c_n (-\Delta)^{\frac{n-1}{2}} R^t R u \;\; u \in \mathcal{E}'(\mathbb{R}^n)$$

Solution 6. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\langle (-\Delta)^{\frac{n-1}{2}} R^t R u, \varphi \rangle = \langle u, ((-\Delta)^{\frac{n-1}{2}} R^t R)^t \varphi \rangle.$$

On the other hand

$$(-\Delta)^{\frac{n-1}{2}} R^t R)^t \varphi = (R^t R)^t ((-\Delta)^{\frac{n-1}{2}})^t \varphi = R^t R ((-\Delta)^{\frac{n-1}{2}})^t \varphi$$

Notice that by Parseval identity

$$((-\Delta)^{\frac{n-1}{2}})^t \varphi, \phi)_{L^2(\mathbb{R}^n)} = ((-\Delta)^{\frac{n-1}{2}})\phi, \varphi)_{L^2(\mathbb{R}^n)} = c_n (\mathcal{F}^{-1}(\|\cdot\|^{n-1}\mathcal{F}(\phi), \varphi)_{L^2(\mathbb{R}^n)}$$
$$= c_n(\phi, \mathcal{F}^{-1}(\|\cdot\|^{n-1}\mathcal{F}(\varphi)_{L^2(\mathbb{R}^n)}.$$

Therefore

$$((-\Delta)^{\frac{n-1}{2}})^t = (-\Delta)^{\frac{n-1}{2}}$$

By previous exercises it holds that

$$\mathcal{F}(R^t R((-\Delta)^{\frac{n-1}{2}})^t \varphi) = c_n \mathcal{F}((\frac{1}{\|\cdot\|} * (-\Delta)^{\frac{n-1}{2}})^t \varphi)) = c_n \|\cdot\|^{1-n} \mathcal{F}(\mathcal{F}^{-1}(\|\cdot\|^{n-1}\mathcal{F}(\varphi))) = \mathcal{F}(\varphi).$$

The claim follows from inverse Fourier transform.

Problem 7. *Recall that the wave front set of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is defined by negation as $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ is not in WFu, if there exists $\varphi \in C_0^\infty(U)$ such that $\phi(x_0) \neq 0$, and a neighborhood V of ξ_0 such that for all $\xi \in V$ and $k \in \mathbb{N}$ holds*

$$|\mathcal{F}(\varphi u)(t\xi)| \leq C_k |1+t|^{-k},\ t > 0. \tag{2}$$

Let $n = 2$ and denote by χ the characteristic function of an open unit disc $B(0,1) \subset \mathbb{R}^2$. Prove that

$$WF\chi = \{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\}) : \|x\| = 1,\ \xi \parallel x\}.$$

Solution 7. *Notice first that for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ the distribution φu is compactly supported. Therefore $\mathcal{F}(\varphi u) \in C^\infty(\mathbb{R}^n)$ (see [5]) and thus inequality (2) makes sense.*

We start with computing the wave front set of the characteristic function χ_p of the right half plane $\{z \in \mathbb{C} : Rez > 0\}$. Let $\phi \in C_0^\infty(\mathbb{R})$. We define

$$\Phi(x) := \phi(x_1)\phi(x_2).$$

Then $\Phi \in C_0^\infty(\mathbb{R}^2)$ and

$$\widehat{\Phi\chi_p}(\xi) = \int_{x_1 \geq 0} \Phi(x) e^{-i\xi\cdot x} dx = \int_0^\infty \phi(x_1) e^{-i\xi_1 x_1} dx_1 \int_{-\infty}^\infty \phi(x_2) e^{-i\xi_2 x_2} dx_2$$
$$= \widehat{\phi}(\xi_2) \int_0^\infty \phi(x_1) e^{-i\xi_1 x_1} dx_1.$$

Since

$$\left| \int_0^\infty \phi(x_1) e^{-i\xi_1 x_1} dx_1 \right| \leq \|\phi\|_{L^1(\mathbb{R})},$$

it holds that $\widehat{\Phi\chi_p}(\xi)$ is rapidly decreasing if $\xi_2 \neq 0$. Suppose that $\phi(0) = 1$.

Integrating by parts twice yields

$$\int_0^\infty \phi(s) e^{-its} ds = \frac{i}{t}\Big[\phi(s)e^{-its}\Big]_0^\infty + \frac{1}{t^2}\Big[\phi'(s)e^{-its}\Big]_0^\infty - \frac{1}{t^2}\int_0^\infty \phi''(s) e^{-its} ds$$

$$= \frac{1}{t}\Big(-i - \frac{1}{t}\Big(\phi'(0) + \int_0^\infty \phi''(s) e^{-its} ds \Big)\Big).$$

Therefore

$$t|\widehat{\Phi\chi_p}(t,0)| \geq |\widehat{\phi}(0)|(\Big| i + \frac{1}{t}\phi'(0)\Big| - \frac{1}{t}\|\phi''\|_{L^1(\mathbb{R})}) \overset{t\to\infty}{\longrightarrow} 1.$$

This proves that $t \mapsto \widehat{\Phi\chi_p}(t,0)$ is not rapidly decaying. Thus we have proved that

$$WF\chi_p = \{((0,t);(s,0)) \in \mathbb{R}\times\mathbb{R} : t,s \in \mathbb{R},\ s \neq 0\}.$$

Consider a mapping

$$f(z) = \frac{1+z}{1-z},\ z \in \mathbb{C}\backslash\{1\},$$

Clearly f is a diffeomorphism from open unit disc onto open half plane $\{z \in \mathbb{C} : Rez > 0\}$. By Proposition 11.2.2 of [5] it holds that

$$WF\chi = \{(z, Df(z)^t\xi) : (f(z),\xi) \in WF\chi_p\},$$

here $Df(z)^t$ is the transpose of the Jacobian of f at z. Since

$$f(z) = f(x+iy) = \frac{1-x^2-y^2}{(x-1)^2+y^2} + i\frac{2y}{(x-1)^2+y^2}$$

it holds that

$$Df(i)^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $f(i) = i$ and $Df(i)^t(1,0) = (0,1)$, we conclude that $((0,1);(0,1)) \in WF\chi$. By symmetry we have proved the claim.

An other way to solve this problem is to use the method of stationary phase. See [9].

Notice that in general it holds that the wave front set of a delta function on a smooth hyper surface S is the normal bundle of S.

Problem 8. *Let χ be the characteristic function of unit square $Q := [0,1]\times[0,1] \subset \mathbb{R}^2$ prove that*

$$WF\chi = \{(x,\xi) \in \mathbb{R}^2\times(\mathbb{R}^2\backslash\{0\}) : x \in \partial Q, \xi \parallel x\} \bigcup \{(x,\xi) \in \mathbb{R}^2\times(\mathbb{R}^2\backslash\{0\}) : x_i = \{0,1\},\ \xi \in \mathbb{R}^2\backslash\{0\}\}.$$

I.e. at the corner points every direction is in the wavefront set.

Solution 8. *Let us first consider the corner point $(0,0)$ case. Then χ looks like $x \mapsto H(x_1)H((x_2)$ near origin where H is the Heaviside function. Let ϕ and Φ be as in the previous problem. Then*

$$\widehat{\Phi\chi}(\xi) = \int_0^\infty \phi(s)e^{-is\xi_1}\, ds \int_0^\infty \phi(s)e^{-is\xi_2}\, ds$$

and by computations done in the previous problem this is not rapidly decaying in ξ. Therefore $(\overline{0},x) \in WF\chi$ for all $x \in \mathbb{R}^n\backslash\{\overline{0}\}$.

The rest follows by symmetry of corner points and from computations done in Problem 7.

Problem 9. *Let $F \subset \mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\})$ be closed and conic. Show that there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that*

$$WFu = F.$$

Solution 9. *See Theorem 8.1.4. of [9].*

Problem 10. *Let $k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We define a linear operator.*

$$K : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),\ Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy.$$

Then the adjoint of K with respect to L^2 innerproduct is

$$K^t f(y) = \int_{\mathbb{R}^n} k(x,y)f(x)dx,\ f \in C_0^\infty(\mathbb{R}^n)$$

Prove that for any $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\langle u, K^t\varphi\rangle = \int_{\mathbb{R}^n} \langle u, k(x,\cdot)\rangle \varphi(x)dx$$

Solution 10.

Problem 11. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and $k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that*

$$\langle u, k(x,\cdot)\rangle \in C^\infty(\mathbb{R}^n).$$

Solution 11.

Problem 12. *Recall the Schwartz kernel theorem. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be open sets. Let $A : C_0^\infty(X) \to \mathcal{D}'(Y)$ be linear and continuous. Then there exists a unique $k_A \in \mathcal{D}'(X \times Y)$ such that*

$$\langle A\varphi, \psi\rangle = k_A(\varphi \otimes \psi),\ \varphi \in C_0^\infty(X),\ \psi \in C_0^\infty(Y).$$

Here the tensor product $(\varphi \otimes \psi)(x,y) := \varphi(x)\psi(y)$.

If $a \in C^\infty(X \times Y)$, it determines naturally the operator $A : C_0^\infty(X) \to \mathcal{D}'(Y)$

$$A\varphi(\psi) = \int_X \int_Y a(x,y)\varphi(x)\overline{\psi(y)}dxdy$$

Let $X = Y \subset \mathbb{R}^n$. Consider a partial differential operator

$$A = \sum_{|\alpha|\le k} a_\alpha D^\alpha,\ a_\alpha \in C^\infty(X).$$

Show that the Schwartz kernel of operator A is

$$k_A(x,y) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha \delta(x-y).$$

Solution 12. *Let $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$. By definition*

$$\langle A\varphi, \psi\rangle = \int_{\mathbb{R}^n} (A\varphi)(x)\overline{\psi(x)}\,dx = \int_{\mathbb{R}^n} \overline{\psi(x)} \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha \varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} \overline{\psi(x)} \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha \int_{\mathbb{R}^n} \delta(y-x)\varphi(y)\,dy\,dx$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left[\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha \delta(y-x)\varphi(y)\right]\overline{\psi(x)}\,dy\,dx.$$

This proves the claim.

Problem 13. *Let $m \in \mathbb{N}$ and $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. We define the Schwartz kernel k_p of p as*

$$\langle k_p, \varphi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \frac{p(x,\xi)}{(1+|\xi|^2)^M} (I + \Delta_y)^M \varphi(y)\, dxdyd\xi, \;\; \varphi \in C_0^\infty(\mathbb{R}^n). \tag{3}$$

Show that k_p is well defined and independent of M, if $M \geq \frac{m+n}{2}$

Problem 14. *Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be s.t. $\eta(x) = 1$ when $\|x\| \leq 1$. Show that*

$$k_p = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \eta(\epsilon\xi) e^{i(x-y)\cdot\xi} p(x,\xi), d\xi$$

where $k_p \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by equation (3).

Problem 15. *Let $A, B \in \Psi^m(\mathbb{R}^n)$. show that the Schwartz kernel k_{AB} of composition operator AB satisfies*

$$k_{AB}(x,y) = \int_{\mathbb{R}^n} k_A(x,z) k_B(z,y) dz,$$

when ever right hand side is well defined.

Problem 16. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(x) = 1$, if $\|x\| \leq 1$. Let $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Show that function*

$$F(x,y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \frac{1-\chi(x-y)}{\|x-y\|^{2M}} \Delta_\xi^M p(x,\xi)\, d\xi \in C^k(\mathbb{R}^n \times \mathbb{R}^n),$$

for all $k \in \mathbb{N}$ and is independent of M, if M is large enough.

Show that

$$k_{\widetilde{A}}(x,y) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \chi(x-y) p(x,\xi)\, d\xi$$

is a Schwartz kernel of some $\widetilde{A} \in \Psi^m(\mathbb{R}^n)$.

Problem 17. *Let $A \in \Psi^m(\mathbb{R}^n)$. Show that there is a extension*

$$\widetilde{A} : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

of A that is linear and continuous.

Suppose that $A \in \Psi^m(\mathbb{R}^n)$ is properly supported. Show that there is a linear and continuous extension

$$B : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

of $\widetilde{A}$.

Recall that pseudo differential operator A is properly supported, if the Schwartz kernel k_A is properly supported in $\mathbb{R}^n \times \mathbb{R}^n$ i.e.

$$supp k_A \subset \mathbb{R}^n \times \mathbb{R}^n$$

is proper. A set $X \subset \mathbb{R}^n \times \mathbb{R}^n$ is proper, if for all compact $K \subset \mathbb{R}^n$ the sets

$$\pi_x(\pi_y^{-1} K \cap X) \text{ and } \pi_y(\pi_x^{-1} K \cap X)$$

are compact in $\mathbb{R}^n$. Here $\pi_y(x,y) = y$ and $\pi_x(x,y) = x$.

Problem 18. *Show that for any $A \in \Psi^m(\mathbb{R}^n)$ holds*

$$WF(Au) \subset WFu, \text{ for any } u \in \mathcal{E}'(\mathbb{R}^n).$$

You can use the fact

$$singsupp(Au) \subset singsupp(u), \text{ for any } u \in \mathcal{E}'(\mathbb{R}^n).$$

References

[1] Melrose R & Uhlmann G, An Introduction to Microlocal Analysis

[2] Grigis A & Sjöstrand J, Microlocal Analysis for Differential Operators An Introduction

[3] Saint Raymond X, Elementary Introduction to the Theory of Pseudodiferential Operators

[4] Wong M, An Introduction to Pseudo-Differential Operators

[5] Friedlander F & Joshi M, Introduction to theory of distributions

[6] Lee J, Introduction to smooth manifolds

[7] Helgason S, Radon Transform

[8] Markoe A, Analytic Tomography

[9] Hörmander L, The Analysis of Linear Partial Differential Operators I

[10] Natterer F, Mathematics of computerized tomography