Exercises for the course "Szemerédi's Theorem"

If you would like to get credit for this course, please give your solutions to Hans-Olav Tylli by 31 July 2012.

The cardinality of a set A is denoted by $|A|$. Suppose that N is an integer with $N \geq 2$. As usual, $\mathbb{Z}_N$ denotes the integers modulo N. Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a function. Write $\omega = \exp(2\pi i/N)$. We define the Fourier transform $\hat{f} : \mathbb{Z}_N \to \mathbb{C}$ as follows:

$$\hat{f}(n) = \sum_{j=1}^{N} f(j)\omega^{-jn}.$$

Then the inverse transform is given by

$$f(n) = \frac{1}{N}\sum_{j=1}^{N} \hat{f}(j)\omega^{jn}.$$

We define $||f||_2$ by

$$||f||_2^2 = \sum_{j=1}^{N} |f(j)|^2.$$

Let f, g be complex-valued functions on $\mathbb{Z}_N$. We define

$$(f * g)(s) = \sum_{t \in \mathbb{Z}_N} f(t)\overline{g(t-s)}$$

for all $s \in \mathbb{Z}_N$. In the term $t - s$, we are naturally using addition modulo N.

1. We write $h = f * g$. Prove that

$$\hat{h}(r) = \hat{f}(r)\overline{\hat{g}(r)}$$

for all $r \in \mathbb{Z}_N$.

2. Prove that

$$\sum_{r \in \mathbb{Z}_N} \hat{f}(r)\overline{\hat{g}(r)} = N \sum_{s \in \mathbb{Z}_N} f(s)\overline{g(s)}.$$

If $f : \mathbb{Z}_N \to \mathbb{C}$ is a function, $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and $a_1, a_2, \ldots, a_d, s, k \in \mathbb{Z}_N$, we define

$$\Delta(f; k)(s) = f(s)\overline{f(s-k)}$$

and

$$\Delta(f; a_1, a_2, \ldots, a_d)(s) = \Delta(\Delta(f; a_1, a_2, \ldots, a_{d-1}); a_d)(s).$$

We define, for any $n \geq 1$, the complex conjugation operator $C : \mathbb{C}^n \to \mathbb{C}^n$ by $C(z_1, \ldots, z_n) = (\overline{z_1}, \ldots, \overline{z_n})$. The composition of C with itself m times is denoted by C^m.

3. Write $E(d) = \{0,1\}^d$, where $d \geq 1$. Prove that

$$\Delta(f; a_1, a_2, \ldots, a_d)(s) = \prod_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in E(d)} \left(C^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d} f\right) \left(s - \sum_{i=1}^{d} \varepsilon_i a_i\right).$$

We write $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ for the closed unit disk in the complex plane $\mathbb{C}$. (Note that this is somewhat non-standard notation since usually $\mathbb{D}$ denotes the open unit disk.)

We assume that $f : \mathbb{Z}_N \to \mathbb{D}$ is a function. Suppose that $0 < \alpha \leq 1$ and $d \in \mathbb{N}$. We say that f is $\alpha-$uniform of degree d if

$$\sum_{(a_1, \ldots, a_d) \in \mathbb{Z}_N^d} \left| \sum_{s \in \mathbb{Z}_N} \Delta(f; a_1, \ldots, a_d)(s) \right|^2 \leq \alpha N^{d+2}.$$

For $d = 0$, we define f to be $\alpha-$uniform of degree 0 if

$$\left| \sum_{s \in \mathbb{Z}_N} f(s) \right|^2 \leq \alpha N^2.$$

4. Prove that f is $\alpha-$uniform of degree d if, and only if,

$$\sum_{s\in\mathbb{Z}_N}\ \sum_{(a_1,\dots,a_d,a_{d+1})\in\mathbb{Z}_N^{d+1}} \Delta(f;a_1,\dots,a_d,a_{d+1})(s)$$

$$\leq \alpha N^{d+2}. \tag{0.1}$$

In particular, this requires one to prove that the left hand side of (0.1) is a real number even in f is complex-valued but not necessarily real-valued.

5. Suppose that $d \geq 1$ and that f is $\alpha-$uniform of degree d. Prove that f is $\sqrt{\alpha}-$uniform of degree $d-1$, and hence $\alpha^{1/2^{d-1}}-$uniform of degree 1.

Hint. Use the Cauchy–Schwarz inequality.

Let X and Y be additively written abelian groups. Let Z be a subset of X, and let W be a subset of Y $(Z, W \neq \emptyset)$. Let ϕ be a function of Z into W. Suppose that $k \geq 2$ and that whenever

$$a_1, a_2, \dots, a_{2k} \in Z, \qquad \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{2k} a_i,$$

we have

$$\sum_{i=1}^{k} \phi(a_i) = \sum_{i=k+1}^{2k} \phi(a_i).$$

Then we say that ϕ is a Freiman homomorphism of order k. Note that Z, W are arbitrary subsets, not necessarily subgroups or closed under addition. Thus, while $\sum_{i=1}^{k} a_i$ is a well-defined element of X, it need not lie in Z, and similarly for $\sum_{i=1}^{k} \phi(a_i)$. Also, note that the numbers $a_1, a_2, \ldots, a_{2k}$ need not be distinct.

6. Suppose that $k \geq 3$ and that $\phi : Z \to W$ is a Freiman homomorphism of order k. Prove that ϕ is also a Freiman homomorphism of order $k-1$.

If $Z_1, Z_2 \subset X$, we define

$$Z_1 + Z_2 = \{a + b : a \in Z_1,\, b \in Z_2\},$$

$$Z_1 - Z_2 = \{a - b : a \in Z_1,\, b \in Z_2\}.$$

If $m \in \mathbb{N}$, we write mZ_1 for

$$Z_1 + Z_1 + \cdots Z_1$$

where Z_1 appears m times as a summand.

7. Let $\phi : Z \to W$ be a Freiman homomorphism of order $2k$. Prove that ϕ induces a well-defined function ψ of $kZ - kZ$ into $kW - kW$. How is ψ defined ?

Suppose that $A \subset \mathbb{Z}_N$, and denote the characteristic function of A also by A. If $k \in \mathbb{Z}_N$, we write $A + k = \{a + k : a \in A\}$, where the addition is taken modulo N. Recall that the notation $f * g$ was defined earlier.

8. Prove that

$$||A * A||_2^2 = \sum_{k \in \mathbb{Z}_N} |A \cap (A + k)|^2$$

$$= \left|\{(a, b, c, d) \in A^4 : a - b = c - d\}\right|.$$

9. If $A \subset \mathbb{Z}_N$ is also a genuine arithmetic progression of length $m \geq 1$ as a subset of $\{1, 2, \ldots, \ell\} \subset \mathbb{N}$, where $1 \leq \ell < N/2$, prove that

$$||A*A||_2^2 \leq m^2+2(1^2+2^2+\cdots+(m-1)^2) = \frac{m(2m^2+1)}{3}.$$

10. Suppose that $A \subset \mathbb{N}$ and $|A| = m \geq 1$. We use ordinary addition in $\mathbb{N}$ to define $A + A$. Prove that

$$2m - 1 \leq |A + A| \leq \frac{m(m+1)}{2}$$

and give examples to show that each of the upper and lower bounds is best possible.