■ 18.7 MECHANICAL VIBRATIONS

An object moves along a straight line. Instead of continuing in one direction, it moves back and forth, oscillating about a central point. Call the central point $x = 0$ and denote by $x(t)$ the displacement of the object at time t. If the acceleration is a constant negative multiple of the displacement,

$$a(t) = -kx(t), \qquad k > 0,$$

then the object is said to be in *simple harmonic motion.*

Since, by definition,

$$a(t) = x''(t),$$

in simple harmonic motion, we have

$$x''(t) = -kx(t),$$

which is the same as

$$x''(t) + kx(t) = 0.$$

To emphasize that k is positive, we set $k = \omega^2$, where $\omega = \sqrt{k} > 0$. The equation of motion then takes the form

(18.7.1)
$$\boxed{x''(t) + \omega^2 x(t) = 0.}$$

This is a second-order, linear differential equation with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0,$$

and the roots are $\pm\omega i$. Therefore, the general solution of Equation (18.7.1) is

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t.$$

A routine calculation shows that the general solution can be written (Exercise 28, Section 18.5)

(18.7.2)
$$\boxed{x(t) = A \sin(\omega t + \phi_0),}$$

where A and ϕ_0 are constants with $A > 0$ and $\phi_0 \in [0, 2\pi)$.

Now let's analyze the motion measuring t in seconds. By adding $2\pi/\omega$ to t we increase $\omega t + \phi_0$ by 2π:

$$\omega\left(t + \frac{2\pi}{\omega}\right) + \phi_0 = \omega t + \phi_0 + 2\pi.$$

This means that the motion is *periodic* with *period* T given by:

$$T = \frac{2\pi}{\omega}.$$

A complete oscillation takes $2\pi/\omega$ seconds. The reciprocal of the period gives the number of complete oscillations per second. This is called the *frequency* f:

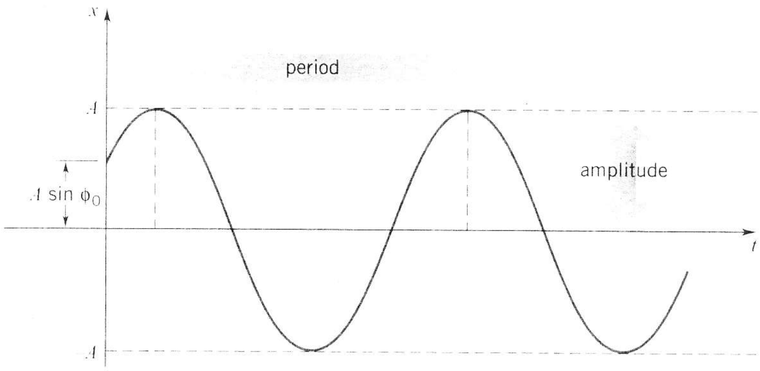
$$f = \frac{\omega}{2\pi}.$$

The number ω is called the *angular frequency*. Since $\sin(\omega t + \phi_0)$ oscillates between -1 and 1,

$$x(t) = A\sin(\omega t + \phi_0)$$

oscillates between $-A$ and A. The number A is called the *amplitude* of the motion.

In Figure 18.7.1 we have plotted x against t. The oscillations along the x-axis are now waves in the xt-plane. The period of the motion, $2\pi/\omega$, is the t distance (the time separation) between consecutive wave crests. The amplitude of the motion, A, is the height of the waves measured in x units from $x = 0$. The number ϕ_0 is known as the *phase constant*, or *phase shift*. The phase constant determines the initial displacement (in the xt-plane the height of the wave at time $t = 0$). If $\phi_0 = 0$, the object starts at the center of the interval of motion (the wave starts at the origin of the xt-plane).



simple harmonic motion

Figure 18.7.1

Example 1 Find an equation for the oscillatory motion of an object, given that the period is $2\pi/3$ and, at time $t = 0$, $x(0) = 1$, $v(0) = x'(0) = 3$.

SOLUTION We begin by setting

$$x(t) = A\sin(\omega t + \phi_0).$$

In general the period is $2\pi/\omega$, so that here

$$\frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{and thus} \quad \omega = 3.$$

The equation of motion takes the form

$$x(t) = A\sin(3t + \phi_0).$$

By differentiation

$$v(t) = 3A\cos(3t + \phi_0).$$

The conditions at $t = 0$ give

$$1 = x(0) = A \sin \phi_0, \qquad 3 = v(0) = 3A \cos \phi_0$$

and therefore

$$1 = A \sin \phi_0, \qquad 1 = A \cos \phi_0.$$

Adding the squares, we have

$$2 = A^2 \sin^2 \phi_0 + A^2 \cos^2 \phi_0 = A^2.$$

Since $A > 0$, $A = \sqrt{2}$.

To find ϕ_0 we note that

$$1 = \sqrt{2} \sin \phi_0, \qquad 1 = \sqrt{2} \cos \phi_0.$$

These equations are satisfied by setting $\phi_0 = \frac{1}{4}\pi$. The equation of motion can be written

$$x(t) = \sqrt{2} \sin(3t + \tfrac{1}{4}\pi). \quad \square$$

Undamped Vibrations

A coil spring hangs naturally to a length l_0. When a bob of mass m is attached to it, the spring stretches l_1 inches. The bob is later pulled down an additional x_0 inches and then released. What is the resulting motion? Throughout we refer to Figure 18.7.2, taking the downward direction as positive.

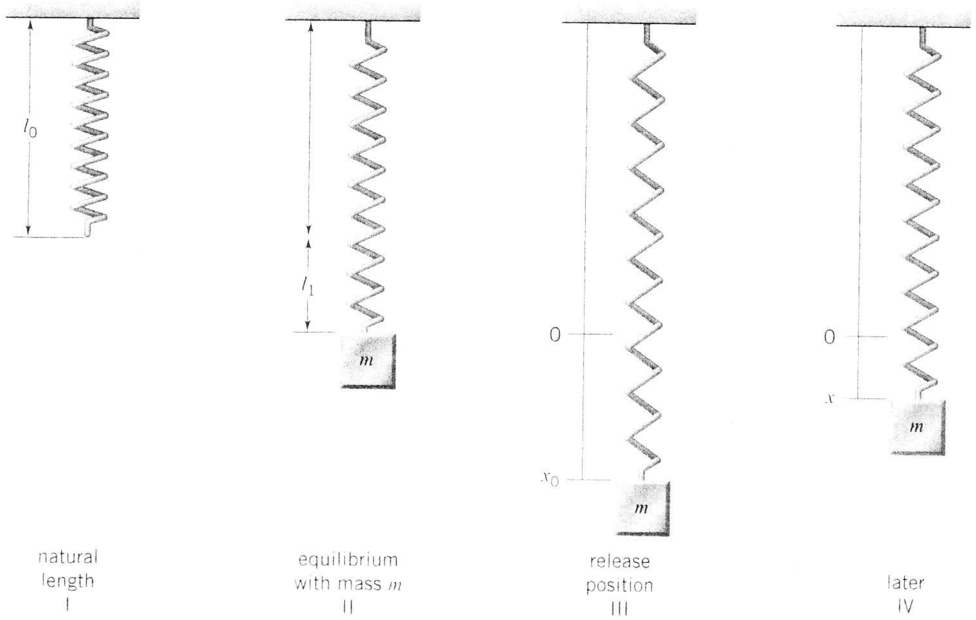


Figure 18.7.2

We begin by analyzing the forces acting on the bob at general position x (stage IV). First there is the weight of the bob:

$$F_1 = mg.$$

This is a downward force, and by our choice of coordinate system, positive. Then there is the restoring force of the spring. This force, by Hooke's law, is proportional to the total displacement $l_1 + x$ and acts in the opposite direction:

$$F_2 = -k(l_1 + x) \qquad \text{with } k > 0.$$

If we neglect resistance, then these are the only forces acting on the bob. Under these conditions the total force is

$$F = F_1 + F_2 = mg - k(l_1 + x),$$

which we rewrite as

$$(1) \qquad F = (mg - kl_1) - kx.$$

At stage II (Figure 18.7.2) there was equilibrium. The force of gravity, mg, plus the force of the spring, $-kl_1$, must have been 0:

$$mg - kl_1 = 0.$$

Equation (1) can therefore be simplified to

$$F = -kx.$$

Using Newton's second law

$$F = ma \qquad \text{(force = mass} \times \text{acceleration)}$$

we have

$$ma = -kx \quad \text{and thus} \quad a = -\frac{k}{m}x.$$

At any time t,

$$x''(t) = -\frac{k}{m}x(t).$$

Since $k/m > 0$, we can set $\omega = \sqrt{k/m}$ and write

$$x''(t) = -\omega^2 x(t).$$

The motion of the bob is simple harmonic motion with period $T = 2\pi/\omega$. ❑

There is something remarkable about harmonic motion that we have not yet specifically pointed out; namely, that the frequency $f = \omega/2\pi$ is completely independent of the amplitude of the motion. The oscillations of the bob occur with frequency

$$f = \frac{\sqrt{k/m}}{2\pi}, \qquad \text{(here } \omega = \sqrt{k/m}\text{).}$$

By adjusting the spring constant k and the mass of the bob m, we can calibrate the spring–bob system so that the oscillations take place exactly once a second (at least almost exactly). We then have a primitive timepiece (a first cousin of the windup clock). With the passing of time, friction and air resistance reduce the amplitude of the oscillations but not their frequency. By giving the bob a little push or pull once in a while (by rewinding our clock), we can restore the amplitude of the oscillations and thus maintain the steady "ticking."

Damped Vibrations

We derived the equation of motion

$$x'' + \frac{k}{m}x = 0$$

from the force equation

$$F = -kx.$$

Unless the spring is frictionless and the motion takes place in a vacuum, there is a resistance to the motion that tends to dampen the vibrations. Experiment shows that the resistance force R is approximately proportional to the velocity x':

$$R = -cx'. \qquad (c > 0)$$

Taking this resistance term into account, the force equation reads

$$F = -kx - cx'.$$

Newton's law $F = ma = mx''$ then gives

$$mx'' = -cx' - kx,$$

which we can write as

(18.7.3)
$$\boxed{x'' + \frac{c}{m}x' + \frac{k}{m}x = 0.}$$

This is the equation of motion in the presence of a *damping* factor To study the motion we analyze this equation.

The characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

There are three possibilities:

$$c^2 - 4km < 0, \qquad c^2 - 4km > 0, \qquad c^2 - 4km = 0.$$

Case 1. $c^2 - 4km < 0$

In this case the characteristic equation has two complex conjugate roots:

$$r_1 = -\frac{c}{2m} + i\omega, \qquad r_2 = -\frac{c}{2m} - i\omega \qquad \text{where} \quad \omega = \frac{\sqrt{4km - c^2}}{2m}.$$

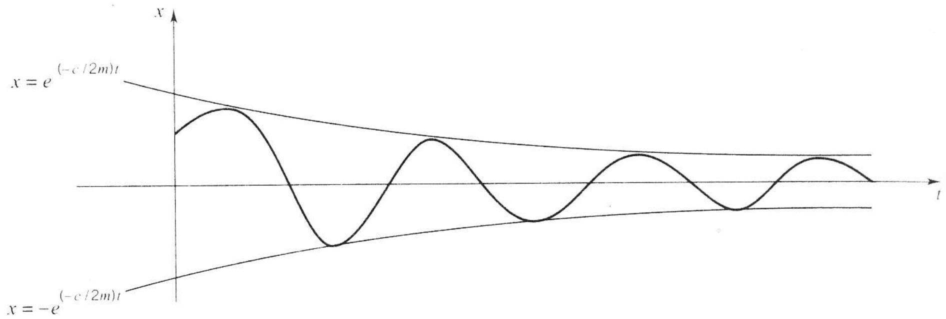
The general solution of (18.7.3),

$$x = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t),$$

can be written

(18.7.4) $$x(t) = Ae^{(-c/2m)t}\sin(\omega t + \phi_0),$$

where, as before, A and ϕ_0 are constants, $A > 0$, $\phi_0 \in [0, 2\pi)$. This is called the *underdamped case.* The motion is similar to simple harmonic motion, except that the damping term $e^{(-c/2m)t}$ ensures that $x \to 0$ as $t \to \infty$. The vibrations continue indefinitely with constant frequency $\omega/2\pi$ but diminishing amplitude $Ae^{(-c/2m)t}$. As $t \to \infty$, the amplitude of the vibrations tends to zero; the vibrations die down. The motion is illustrated in Figure 18.7.3. ❑



damped harmonic motion

Figure 18.7.3

Case 2. $c^2 - 4km > 0$
In this case the characteristic equation has two distinct real roots:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m}, \qquad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

The general solution takes the form

(18.7.5) $$x = C_1e^{r_1t} + C_2e^{r_2t}.$$

This is called the *overdamped case.* The motion is nonoscillatory. Since $\sqrt{c^2 - 4km} < \sqrt{c^2} = c$, both r_1 and r_2 are negative. As $t \to \infty$, $x \to 0$. ❑

Case 3. $c^2 - 4km = 0$
In this case the characteristic equation has only one root

$$r_1 = -\frac{c}{2m}$$

and the general solution takes the form

(18.7.6) $$x = C_1 e^{-(c/2m)t} + C_2 t e^{-(c/2m)t}.$$

This is called the *critically damped case*. Once again the motion is nonoscillatory. Moreover, as $t \to \infty$, $x \to 0$. ❑

In both the overdamped and critically damped cases, the mass moves slowly back to its equilibrium position ($x \to 0$ as $t \to \infty$). Depending upon the side conditions, the mass may move through the equilibrium once, but only once; there is no oscillatory motion. Two typical examples of the motion are shown in Figure 18.7.4.

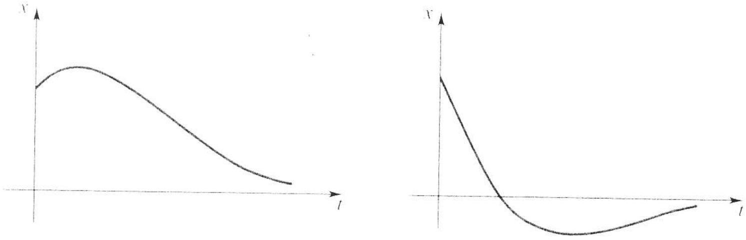


Figure 18.7.4

Forced Vibrations

The vibrations that we have been considering result from the interplay of three forces: the force of gravity, the elastic force of the spring, and the retarding force of the surrounding medium. Such vibrations are called *free vibrations*.

The application of an external force to a freely vibrating system modifies the vibrations and results in what are called *forced vibrations*. In what follows we examine the effect of a pulsating force $F_0 \cos \gamma t$. Without loss of generality we can take both F_0 and γ as positive.

In an undamped system the force equation reads

$$F = -kx + F_0 \cos \gamma t$$

and the equation of motion takes the form

(18.7.7) $$x'' + \frac{k}{m}x = \frac{F_0}{m}\cos \gamma t.$$

As usual we set $\omega = \sqrt{k/m}$ and write

(18.7.8) $$x'' + \omega^2 x = \frac{F_0}{m}\cos \gamma t.$$

As you'll see, the nature of the vibrations depends on the relation between the *applied frequency*, $\gamma/2\pi$, and the *natural frequency* of the system, $\omega/2\pi$.

Case 1. $\gamma \neq \omega$
In this case the method of undetermined coefficients gives the particular solution

$$x_p = \frac{F_0/m}{\omega^2 - \gamma^2}\cos \gamma t.$$

The general equation of motion can thus be written

(18.7.9)
$$x = A\sin(\omega t + \phi_0) + \frac{F_0/m}{\omega^2 - \gamma^2}\cos \gamma t.$$

If ω/γ is rational, the vibrations are periodic. If, on the other hand, ω/γ is not rational, then the vibrations are not periodic and the motion, though bounded by

$$|A| + \left|\frac{F_0/m}{\omega^2 - \gamma^2}\right|,$$

can be highly irregular. ❑

Case 2. $\gamma = \omega$
In this case the method of undetermined coefficients gives

$$x_p = \frac{F_0}{2\omega m} t \sin \omega t$$

and the general solution takes the form

(18.7.10)
$$x = A\sin(\omega t + \phi_0) + \frac{F_0}{2\omega m} t \sin \omega t.$$

The undamped system is said to be in *resonance*. The motion is oscillatory, but, because of the extra t present in the second summand, it is far from periodic. As $t \to \infty$, the amplitude of vibration increases without bound. The motion is illustrated in Figure 18.7.5. ❑

Undamped systems and unbounded vibrations are mathematical fictions. No real mechanical system is totally undamped, and unbounded vibrations do not occur in nature. Nevertheless a form of resonance can occur in a real mechanical system. (See Exercises 24–28.) A periodic external force applied to a mechanical system that is insufficiently damped can set up vibrations of very large amplitude. Such vibrations have caused the destruction of some formidable man-made structures. In 1850 the suspension bridge at Angers, France, was destroyed by vibrations set up by the unified step of a column of marching soldiers. More than two hundred French soldiers were killed in that catastrophe. (Soldiers today are told to break ranks before crossing a

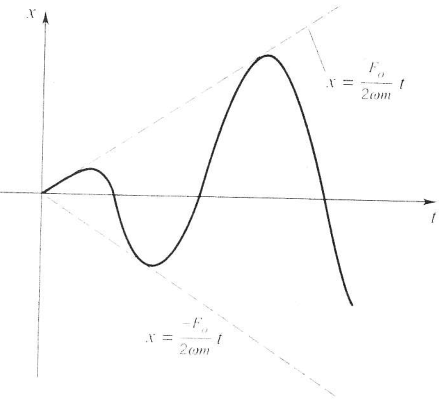


Figure 18.7.5

bridge.) The collapse of the bridge at Tacoma, Washington, is a more recent event. Slender in construction and graceful in design, the Tacoma bridge was opened to traffic on July 1, 1940. The third longest suspension bridge in the world, with a main span of 2800 feet, the bridge attracted many admirers. On November 1 of that same year, after less than five months of service, the main span of the bridge broke loose from its cables and crashed into the water below. (Luckily only one person was on the bridge at the time, and he was able to crawl to safety.) A driving wind had set up vibrations in resonance with the natural vibrations of the roadway, and the stiffening girders of the bridge had not provided sufficient damping to keep the vibrations from reaching destructive magnitude.

EXERCISES 18.7

1. An object is in simple harmonic motion. Find an equation for the motion given that the period is $\frac{1}{4}\pi$ and, at time $t = 0$, $x = 1$ and $v = 0$. What is the amplitude? What is the frequency?

2. An object is in simple harmonic motion. Find an equation for the motion given that the frequency is $1/\pi$ and, at time $t = 0$, $x = 0$ and $v = -2$. What is the amplitude? What is the period?

3. An object is in simple harmonic motion with period T and amplitude A. What is the velocity at the central point $x = 0$?

4. An object is in simple harmonic motion with period T. Find the amplitude given that $v = \pm v_0$ at $x = x_0$.

5. An object in simple harmonic motion passes through the central point $x = 0$ at time $t = 0$ and every 3 seconds thereafter. Find the equation of motion given that $v(0) = 5$.

6. Show that simple harmonic motion $x(t) = A \sin(\omega t + \phi_0)$ can just as well be written: (a) $x(t) = A \cos(\omega t + \phi_1)$; (b) $x(t) = B \sin \omega t + C \cos \omega t$.

Exercises 7–12 are concerned with the motion of the bob depicted in Figure 18.7.2.

7. What is $x(t)$ for the bob of mass m?

8. Find the positions of the bob where the bob attains: (a) maximum speed; (b) zero speed; (c) maximum acceleration; (d) zero acceleration.

9. Where does the bob take on half of its maximum speed?

10. Find the maximal kinetic energy obtained by the bob. (Remember: $KE = \frac{1}{2}mv^2$ where m is the mass of the object and v is the speed.)

11. Find the time average of the kinetic energy of the bob during one period T.

12. Express the velocity of the bob in terms of k, m, x_0, and $x(t)$.

13. Given that $x''(t) = 9 - 4x(t)$ with $x(0) = 0$ and $x'(0) = 0$, show that the motion is simple harmonic motion centered at $x = 2$. Find the amplitude and the period.

14. The figure shows a pendulum of mass m swinging on an arm of length L. The angle θ is measured counterclockwise. Neglecting friction and the weight of the arm, we can describe the motion by the equation

$$mL\theta''(t) = -mg \sin \theta(t)$$