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Subject: Introduction to Numerical Methods

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Exercises

1. Find the absolute error we make by computing $y = x_1 x_2^2$, being $x_1 = 2{,}0 \mp 0{,}1$ and $x_2 = 3{,}0 \mp 0{,}2$. Which is the relative error?

2. Let us consider the following system

$$\begin{cases} x + ay = 5, \\ bx + 2y = d, \end{cases}$$

being $a = 1{,}000 \mp 0{,}002$, $b = a^{-1}$ and $d = b - a$. Which is the absolute error of the product xy?

3. The *law of the conjugate points* (used in the optical system) establishes that

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}. \tag{1}$$

This relationship determines the location of the images given a particular focal length (f) and the object distance (p); q is the distance associated with the image. Find:

a) the expression of the absolute error of f in terms of the absolute errors of p and q;

b) the value of the absolute error of f, if $p = (3{,}7 \mp 0{,}2)\ cm$ and $q = (2{,}3 \mp 0{,}3)\ cm$.

Successively, let us put $X = pq$ and $Y = q + p$, and let us express the relationship (1) by beanos of X and Y:

c) find the expression of the absolute error of f in terms of the absolute errors of X and Y;

d) by using item c), find the value of the absolute error of f, if $p = (3{,}7 \mp 0{,}2)\ cm$ and $q = (2{,}3 \mp 0{,}3)\ cm$;

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e) analyze the behavior of the error with relation to the results obtained in b) and d).

4. Let us consider the following nodes $x_i = -1 + hi$ (being $i = 0, 1, ..., 6$ and $h = 0{,}3$) and the following interpolating table (obtaining by approximation of $f(x) = e^{x^2}$ in the same nodes):

x_i	-1	-0.7	-0.4	-0.1	0.2	0.5	0.8
$f(x_i)$	2.7182	1.6323	1.1735	1.0100	1.0408	1.2840	1.8964

a) Find the interpolating polynomial $p_3(x)$ of f in x_0, x_1, x_2 y x_3;

b) find the error we make if we approximate $f(-0{,}21)$ with $p_3(-0{,}21)$;

c) find the interpolating polynomial $p_2(x)$ of f in x_4, x_5 y x_6;

d) find the error we make if we approximate $f(0{,}43)$ with $p_2(0{,}43)$;

e) analyze the behavior of the interpolation in terms of the results obtained in b) and d).

5. Let f be defined in $[0, 1]$; the *Bernstein polynomial of degree n for f* is $B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}$. In particular, the functions $\beta_k^n(x) := \binom{n}{k} x^k(1-x)^{n-k}$ are called the *elementary polynomials of Bernstein*.

a. Find the elementary polynomials of Bernstein of degree 2;

b. find the polynomial of Bernstein of degree 2 for $f(x) = e^{x^2}$;

c. compare the graphs of $f(x)$ and $B_2(f; x)$;

d. develop the previous items for the degree 3 and 4, comparing the graphs of $f(x)$, $B_2(f; x)$, $B_3(f; x)$ and $B_4(f; x)$;

e. analyze the obtained results in terms of the classical **Weierstrass approximation theorem**.

6. Let us consider in $I = [-1, 1]$ the function $f(x) = \log\left(\frac{1}{1+x^2}\right)$; let $\{x_0 = -1, x_1, x_2, ..., x_{21} = 1\}$ be 22 equispaced nodes of I.

a. Find the interpolating polynomial of $f(x)$ by means of a polynomial $p_{21}(x)$, of degree at the most 21;

b. compare the graphs of $f(x)$ and $p_{21}(x)$.

Now let us slightly change the values of $f(x_0), f(x_1), f(x_{20})$ and $f(x_{21})$ of a quantity equal to $9{,}3 \times 10^{-2}$, without changing the value of f on the remaining nodes:

$$\begin{cases} \widetilde{f}(x_i) = f(x_i) + 9{,}3 \times 10^{-2}, & \text{if } i = 0, 1, 20, 21, \\ \widetilde{f}(x_i) = f(x_i) & \text{if, } \; i = 2, 3, .., 19. \end{cases}$$

c. Find the interpolating polynomial $\widetilde{p}_{21}(x)$, of degree at the most 21, on the nodes x_i;

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d. compute $||\widetilde{p}_{21}(x) - p_{21}(x)||_\infty$;

e. compare the graphs of $f(x)$, $p_{21}(x)$ and $\widetilde{p}_{21}(x)$;

f. is the problem stable or instable with respect to the perturbation of the initial data?

7. Let $\{x_0, x_1, ..., x_n\} \subset \mathbb{R}$ with $x_i \neq x_j$ if $i \neq j$ and $f(x) = x^{n+1}$. Find the interpolating polynomial of f in those points by using the error formula for interpolation.

8. Let $f(x) = \frac{20}{1+x^2} - 5e^x$ be a function defined only in the interval $I = [0, 1]$; find the only polynomial of degree $p_2(x) = a_0 + a_1 x + a_2 x^2$ such that

$$p_2(0) = f(0),\ p_2(1) = f(1),\ \int_0^1 p_2(x)dx = \int_0^1 f(x)dx,$$

and compare the graphs of $f(x)$ and $p_2(x)$. Moreover, find a point $x^* \in I$ such that the interpolating polynomial of f in 0, x^* and 1 is equal to $p_2(x)$, previously computed. Finally, write down the interpolation error bound.

9. Let f, $g : [a, b] \mapsto \mathbb{R}$ and $\{x_0, x_1, \ldots, x_n\}$ be two functions and different points of the interval $[a, b]$. Moreover, let p_n and q_n be the interpolating polynomials of f and g in these points, respectively.

   a) Is $\alpha p_n + \beta q_n$, with α, $\beta \in \mathbb{R}$, the interpolating polynomial of $\alpha f + \beta g$ in the points $\{x_0, \ldots, x_n\}$?

   b) Is $p_n q_n$ the interpolating polynomial of fg in these same points?

10. Let us consider the so called *Tchebycehev polynomials* $\{T_n(x)\}_{n=0}^{\infty}$ recursively defined by

$$\begin{cases} T_0(x) = 1,\ T_1(x) = x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n \in \mathbb{N}. \end{cases}$$

Show that the degree of $T_n(x)$ is n and that its director coefficient is 2^{n-1}.

It is also known that for each $n \in \mathbb{N}$ the n roots of $T_n(x)$ belong to the interval $[-1, 1]$ in the points

$$x_k = \cos\left(\frac{(2k+1)\pi}{2n}\right), \quad k = 0, 1, 2, ..., n-1.$$

By using a suitable mapping of $I = [-1, 1]$ into $J = [a, b]$, find the roots in J.