Vektorianalyysi, harj. 7

1. $\gamma(t) = (\sin t, \sin t^2)$, $t > 0$

$\Rightarrow \gamma'(t) = (\cos t, (\cos t^2) \cdot 2t)$

$\Rightarrow$ Polun $\gamma$ tangentti pisteessä $t = \pi$, on

$\gamma'(\pi) = (\cos \pi, (\cos \pi^2) \cdot 2\pi) = \underline{(-1, 2\pi \cos(\pi^2))}$

2. Olk. $\gamma : [0,2] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$

$\Rightarrow \gamma'(t) = (-\sin t, \cos t)$

$\Rightarrow \gamma(t) \cdot \gamma'(t) = (\cos t, \sin t) \cdot (-\sin t, \cos t)$

$= -\cos t \cdot \sin t + \sin t \cos t = 0$

Siis $\gamma(t)$ & $\gamma'(t)$ ovat kohtisuorassa toisiaan vastaan.

$\gamma''(t) = (-\cos t, -\sin t) = -(\cos t, \sin t) = -\gamma(t)$

Siis $\gamma''(t)$ & $\gamma(t)$ ovat vastakkaissuuntaisia.

3. Olk. $\gamma: [0,\infty) \to \mathbb{R}^3$, $t \geq 0$,

(I) $\gamma''(t) = (0,0,-1) \quad \forall t$ &

(II) $\gamma(0) = (0,0,2)$, $\gamma'(0) = (1,1,0)$

$(I) \Rightarrow \gamma'(t) = (c_1, c_2, -t + c_3)$, $c_1, c_2, c_3 \in \mathbb{R}$

$= (0,0,-t) + (c_1, c_2, c_3)$

$\Rightarrow \gamma'(0) = (0,0,0) + (c_1, c_2, c_3) \overset{II}{=} (1,1,0)$

$\Rightarrow \gamma'(t) = (0,0,-t) + (1,1,0) = (1,1,-t)$

$\Rightarrow \gamma(t) = (t + b_1, t + b_2, -\frac{t^2}{2} + b_3)$, $b_1, b_2, b_3 \in \mathbb{R}$

$\Rightarrow \gamma(0) = (b_1, b_2, b_3) \overset{II}{=} (0,0,2)$

$\Rightarrow \gamma(t) = \underline{(t, t, -\frac{t^2}{2} + 2)}$

$\gamma$ leikkaa $xy$-tason, kun $\gamma_3(t) = 0$

$\Leftrightarrow -\frac{t^2}{2} + 2 = 0 \Leftrightarrow t^2 = 4 \Leftrightarrow t = (\pm) 2$

(huom. $t \geq 0$). Siis $\gamma$ leikkaa $xy$-tason ensimmäisen (& ainoan) kerran, kun $t = 2$, pisteessä

$\gamma(2) = (2, 2, -\frac{2^2}{2} + 2) = \underline{(2,2,0)}$

4. Olk. $D \subset \mathbb{R}^2$ avoin & $f \in C^1(D)$.

Pinnan $r(x,y) = (x, y, f(x,y))$ tangenttitason normaali:

$$\partial_1 r(x,y) = (\partial_1 r_1(x,y), \partial_1 r_2(x,y), \partial_1 r_3(x,y))$$

$$= (\partial_1 x, \partial_1 y, \partial_1 f(x,y)) = (1, 0, \partial_1 f(x,y)),$$

$$\partial_2 r(x,y) = (\partial_2 r_1(x,y), \partial_2 r_2(x,y), \partial_2 r_3(x,y))$$

$$= (\partial_2 x, \partial_2 y, \partial_2 f(x,y)) = (0, 1, f(x,y))$$

$\Rightarrow$ Pinnan tangenttitason normaalivektori

$$n(x,y) := \partial_1 r(x,y) \times \partial_2 (r,y) = (1, 0, \partial_1 f(x,y)) \times (0, 1, \partial_2 f(x,y))$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \partial_1 f(x,y) \\ 0 & 1 & \partial_2 f(x,y) \end{vmatrix} = (-\partial_1 f(x,y), -\partial_2 f(x,y), 1)$$

& tangenttitaso on niiden vektorien

$\bar{x} = (x', y', z')$ joukko, joille

$$(\bar{x} - r(x,y)) \cdot n(x,y) = 0$$

$$\Leftrightarrow 0 = (\bar{x} - (x, y, f(x,y))) \cdot (-\partial_1 f(x,y), -\partial_2 f(x,y), 1)$$

$$\Leftrightarrow 0 = (x' - x, y' - y, z' - f(x,y)) (-\partial_1 f(x,y), -\partial_2 f(x,y), 1)$$

$$= -\partial_1 f(x,y)(x' - x) - \partial_2 f(x,y)(y' - y) + z' - f(x,y)$$

$$f(x,y) = xy + x \Rightarrow \partial_1 f(x,y) = y + 1, \quad \partial_2 f(x,y) = x$$

$$\Rightarrow f(0,0) = 0, \quad \partial_1 f(0,0) = 1, \quad \partial_2 f(0,0) = 0$$

$\Rightarrow$ Tang. taso: $0 = -1(x' - 0) - 0(y' - 0) + z' - 0$

$$\Leftrightarrow \underline{x' - z' = 0}$$

5. $f(x,y) = \frac{1}{2}(x^2+y^2)$

$\Rightarrow \partial_1 f(x,y) = x \ , \ \partial_2 f(x,y) = y$

& teht. 4 nojalla tangenttitason yhtälö pisteessä $(x, y, f(x,y))$ on

$$0 = -x(x'-x) - y(y'-y) + z' - \frac{1}{2}(x^2+y^2)$$

$$= -xx' - yy' + z' + \frac{1}{2}(x^2+y^2)$$

6. $f(x,y) = 1/xy, \ xy \neq 0$

$\Rightarrow \partial_1 f(x,y) = -\frac{1}{x^2y} \ , \ \partial_2 f(x,y) = -\frac{1}{xy^2}$

$\Rightarrow$ Pisteen $(2, 1, \frac{1}{2})$ kautta kulkevan tangenttitason yhtälö on

$$0 = -\partial_1 f(2,1)(x'-2) - \partial_2 f(2,1)(y'-1)$$

$$+ z' - f(2,1)$$

$$= +\frac{1}{2^2\cdot 1}(x'-2) + \frac{1}{2\cdot 1^2}(y'-1) + z' - \frac{1}{2\cdot 1}$$

$$= \frac{1}{4}(x'-2) + \frac{1}{2}(y'-1) + z' - \frac{1}{2}$$

$$\Leftrightarrow 0 = x'-2 + 2\cdot(y'-1) + 4z' - 2$$

$$= x' + 2y' + 4z' - 6$$

$$\Leftrightarrow \underline{x' + 2y' + 4z' = 6}$$